UNIVERSITÀ DELLA CALABRIA


**UNIVERSITÀ DELLA CALABRIA**

Dipartimento di Matematica e Informatica


**Dottorato di Ricerca in Matematica e Informatica**

**XXXIII CICLO**

# *On Enriques-Fano threefolds*

**Settore scientifico-disciplinare: MAT/03 Geometria**


**Dottorando:** Dott. Vincenzo Martello





**Supervisori:**
Ch.mo prof. Ciro Ciliberto
Prof.ssa Concettina Galati

**Co-Supervisore:**
Ch.mo prof. Andreas Leopold Knutsen

# Contents

# 0 Introduzione (Italian)

Oggetto di studio della presente tesi sono le *Enriques-Fano threefolds.* Si definisce Enriques-Fano threefold una varietà algebrica normale tridimensionale $W$, dotata di un sistema lineare completo $\mathcal{L}$ di divisori ampi di Cartier, tale che il generico elemento $S \in \mathcal{L}$ sia una superficie di Enriques e tale che $W$ non sia un cono generalizzato su $S$. Il sistema lineare $\mathcal{L}$ induce una mappa razionale $\phi_{\mathcal{L}} : W \dashrightarrow \mathbb{P}^p$, dove $p$ è chiamato *genere* dell'Enriques-Fano threefold $W$ e si ha $2 \leq p \leq 17$ (si vedano [36] and [46]). Anche se impropriamente, gli elementi di $\mathcal{L}$ sono detti *sezioni iperpiane* di $W$, e le intersezioni di due elementi di $\mathcal{L}$ sono dette *curve sezioni* di $W$. La classificazione delle Enriques-Fano threefolds è ancora un problema aperto. Al fine di capire come completarla, analizzeremo le Enriques-Fano threefolds già note, adoperando anche un approccio computazionale con l'ausilio del software *Macaulay2.* Individueremo inoltre la decomposizione isotropica semplice delle curve sezioni delle Enriques-Fano threefolds conosciute. Infine riprenderemo un'idea incompleta di Castelnuovo, adattandola al caso della Enriques-Fano threefold classica di genere 13.

Nello specifico, elencheremo in § 3 gli esempi noti di Enriques-Fano threefolds e le loro proprietà. Ricordiamo che Fano ha trovato Enriques-Fano threefolds di genere $p = 4, 6, 7, 9, 13$ (si veda [23]), Bayle (e anche Sano, con un lavoro simile ma indipendente) ha individuato esempi con $2 \leq p \leq 10$ e $p = 13$ (si vedano [1] e [48]), Prokhorov con $p = 13, 17$ (si veda [46, §3]) e infine Knutsen-Lopez-Muñoz con $p = 9$ (si veda [36, §13]). Denoteremo gli esempi di questi autori, rispettivamente, con $W_F^p$, $W_{BS}^p$, $W_P^p$ e $W_{KLM}^p$.

È noto che ogni Enriques-Fano threefold $(W, \mathcal{L})$ ha singolarità isolate canoniche (si veda [6]). Diremo che due punti singolari distinti di $W$ sono *associati* (o *congiunti*, usando le parole di Fano) se $\phi_{\mathcal{L}} : W \hookrightarrow \mathbb{P}^p$ è un embedding e se la retta che li unisce è contenuta in $W$. Chiameremo *configurazione* dei punti singolari di $W$ il modo in cui essi sono associati. In particolare, se ogni punto singolare è associato allo stesso numero degli altri, diremo che i punti singolari di $W$ sono *simili.* Più in generale i punti singolari di $W$ sono detti simili se "si comportano tutti allo stesso modo". Il concetto di punti singolari associati e simili è importante nella costruzione degli esempi di Fano: in § 4 daremo un'idea del perchè. Tuttavia non forniremo troppi dettagli sul lavoro di Fano: infatti esso probabilmente contiene altre imprecisioni nascoste oltre a quelle individuate da Conte e Murre nel loro articolo [14]. Per questo motivo, esamineremo in § 5 gli esempi razionali $W_F^{p=6,7,9,13}$ trovati da Fano come immagini di sistemi lineari su $\mathbb{P}^3$; usando tecniche di scoppiamenti, verificheremo che essi sono effettivamente Enriques-Fano threefolds e che sono immerse in $\mathbb{P}^p$ con otto punti quadrupli simili aventi come cono tangente il cono sulla Veronese. Ritroveremo anche le configurazioni usate da Fano, dando quindi giustificazione alle sue affermazioni.

Anche le Enriques-Fano threefolds $W_{BS}^p$ hanno otto punti quadrupli con cono tangente il cono sulla Veronese. Sei di queste threefolds, di genere $p = 6, 7, 8, 9, 10, 13$, sono immerse in $\mathbb{P}^p$ e verranno analizzate computazionalmente in § 6. In particolare, mostreremo che i punti singolari di $W_{BS}^{p=6,7,9,13}$ sono simili e che hanno le stesse configurazioni di quelli di $W_F^{p=6,7,9,13}$. Vedremo pure che, per $p = 9, 13$, l'embedding di $W_{BS}^p$ in $\mathbb{P}^p$ è proprio $W_F^p$. Mostreremo infine le Enriques-Fano threefolds $W_{BS}^{p=8,10}$ come immagini di sistemi lineari su $\mathbb{P}^3$ e troveremo che i loro otto punti quadrupli, nonostante siano simili, hanno configurazioni che sono state escluse da Fano: ciò suggerisce che in [23] potrebbero esserci ulteriori problemi nascosti.

In § 7 e § 8 esamineremo le Enriques-Fano threefolds $W_{KLM}^9$, $W_P^{13}$ e $W_P^{17}$. È noto che $W_{KLM}^9$ e $W_P^{17}$ hanno singolarità canoniche non terminali, ma finora non c'erano informazioni sulle loro molteplicità e sui coni tangenti. Con un'analisi computazionale, mostreremo che $W_{KLM}^9$ e $W_P^{17}$ hanno quattro punti quadrupli, il cui cono tangente è un cono sulla Veronese, e un punto sestuplo, il cui cono tangente è un cono su una superficie sestica riducibile nell'unione di quattro piani e di una superficie quadrica. Approfondiremo anche lo studio di $W_P^{13}$, che è stata solamente menzionata da Prokhorov. Mostreremo che $W_P^{13}$ ha quattro punti quadrupli, il cui cono tangente è un cono sulla Veronese, e un punto quintuplo, il cui cono è un cono sull'unione di cinque piani. Quindi $W_{KLM}^9$ e $W_P^{p=13,17}$ hanno punti singolari non simili.

Sia $H$ la classe di una curva sezione su una sezione iperpiana liscia $S$ di una nota Enriques-Fano threefold $W$. In § 9 descriveremo la *decomposizione isotropica semplice* di $H$ (si veda [9, Corollario 4.7] per maggiori dettagli) e individueremo il valore $\phi(H) := \min\{E \cdot H | E \in \mathrm{NS}(S), E^2 = 0, E > 0\}$. Ricordiamo che il valore $\phi$ e le decomposizioni isotropiche semplici permettono in genere di identificare le varie componenti dello spazio dei moduli delle superfici di Enriques polarizzate. Dunque la nostra analisi suggerisce a quali famiglie appartengono le sezioni iperpiane delle Enriques-Fano threefolds.

Infine analizzeremo il sottosistema lineare $\mathcal{L}_\bullet \subset \mathcal{L}$ delle sezioni iperpiane della Enriques-Fano threefold $W_F^{13}$ che sono triple in un punto generico $w \in W_F^{13}$ (si veda § 10). Mostreremo che un generico elemento di questo sistema lineare è birazionale ad una superficie rigata ellittica, e che l'immagine di $W_F^{13}$ tramite la mappa indotta da $\mathcal{L}_\bullet$ è una superficie cubica di Del Pezzo $\Delta \subset \mathbb{P}^3$ con 4 nodi (si veda Theorem 10.25). Questo risultato è interessante perché è legato ad una congettura di Castelnuovo enunciata in [4, pp.187-188]: supponiamo di avere una threefold irriducibile liscia razionale $W$ e un sistema lineare $r$-dimensionale $\mathcal{L}$ su $W$ tale che il suo generico elemento sia una superficie liscia irriducibile $S$ con genere geometrico nullo $p_g(S) = 0$ e genere aritmetico nullo $p_a(S) = 0$. Cosa succede se imponiamo alle superfici di $\mathcal{L}$ di avere un punto triplo in un punto generico $w \in W$? Castelnuovo congettura che si debba ottenere un sottosistema lineare $(r - 10)$-dimensionale $\mathcal{L}_\bullet$ tale che la generica superficie $S_\bullet$ soddisfi una delle tre seguenti proprietà: $S_\bullet$ è una superficie irriducibile con *desingularizzazione irregolare* $\widetilde{S}_\bullet$ tale che $p_g(\widetilde{S}_\bullet) = 0$ e $p_a(\widetilde{S}_\bullet) = -1$; $S_\bullet$ è *riducibile* in due superfici razionali che si intersecano in una curva razionale; $S_\bullet$ *preserva* gli stessi generi geometrico e aritmetico di una generica $S \in \mathcal{L}$. Estenderemo le idee di Castelnuovo a threefolds normali con singolarità isolate e con desingularizzazione regolare, quali le Enriques-Fano threefolds, e troveremo che $W_F^{13}$ e $W_P^{17}$ soddisfano la prima proprietà ipotizzata da Castelnuovo.

Lavoreremo nel campo $\mathbb{C}$ dei numeri complessi. Per le analisi computazionali lavoreremo in un campo finito (sceglieremo $\mathbb{F}_n := \mathbb{Z}/n\mathbb{Z}$ con $n = 10000019$). In Appendix A descriveremo graficamente le configurazioni dei punti singolari di alcune Enriques-Fano threefolds. In Appendix B collezioneremo i codici input usati in Macaulay2.

È in corso la stesura di più articoli tratti dalla presente tesi di Dottorato: attualmente essi sono [38], [39], [40], [41].

## 0.1 Ringraziamenti

Ringrazio i miei tre supervisori per la loro disponibilità e per i loro suggerimenti, fondamentali per la stesura di questa tesi di dottorato: in particolare Concettina Galati per avermi spronato ad intraprendere la strada della ricerca e per avermi aiutato a muovere i primi passi nel mondo della geometria algebrica; Ciro Ciliberto per avermi proposto un argomento interessante e ricco di spunti per nuovi progetti, per essere stato paziente nell'insegnarmi tecniche e concetti a me sconosciuti e per avermi reso partecipe delle sue brillanti idee; Andreas Leopold Knutsen per l'interesse che ha sempre mostrato verso il mio lavoro e per le stimolanti conversazioni durante il mio periodo all'estero in Norvegia. Ringrazio poi la mia collega Martina Anelli con la quale ho condiviso le gioie e i dolori del dottorato. Infine ringrazio chiunque mi sia stato accanto durante questo percorso, soprattutto la mia famiglia, per avermi sostenuto moralmente anche quando credevo di non farcela, e i miei amici, per aver creduto sempre in me.

# 1 Introduction

The research objects of this thesis are the *Enriques-Fano threefolds*. An Enriques-Fano threefold is a normal threefold $W$ endowed with a complete linear system $\mathcal{L}$ of ample Cartier divisors such that the general element $S \in \mathcal{L}$ is an Enriques surface and such that $W$ is not a generalized cone over $S$. The linear system $\mathcal{L}$ defines a rational map $\phi_{\mathcal{L}} : W \dashrightarrow \mathbb{P}^p$ where $p$ is called the *genus* of $W$ and it must be $2 \leq p \leq 17$ (see [36] and [46]). Though improperly, we will refer to the elements of $\mathcal{L}$ as *hyperplane sections* of $W$ and to the curve intersections of two elements of $\mathcal{L}$ as *curve sections* of $W$. The classification of Enriques-Fano threefolds is still an open problem. In order to understand how to complete it, we will analyze known Enriques-Fano threefolds, also using a computational approach thanks to the *Macaulay2* software. We will also identify the simple isotropic decompositions of the curve sections of the known Enriques-Fano threefolds. Finally we will take up an incomplete idea of Castelnuovo, applying it to the case of the classical Enriques-Fano threefold of genus 13.

In particular, we will list the known Enriques-Fano threefolds and their properties in § 3. We recall that Fano found examples of genus $p = 4, 6, 7, 9, 13$ (see [23]), Bayle (and, in a similar and independent way, Sano) found examples with $2 \leq p \leq 10$ and $p = 13$ (see [1] and [48]), Prokhorov with $p = 13, 17$ (see [46, §3]) and finally Knutsen-Lopez-Muñoz with $p = 9$ (see [36, §13]). We will denote the Enriques-Fano threefolds of the above authors, respectively, by $W_F^p$, $W_{BS}^p$, $W_P^p$, $W_{KLM}^p$.

It is known that every Enriques-Fano threefold $(W, \mathcal{L})$ has isolated canonical singularities (see [6]). We will say that two distinct singular points of $W$ are *associated* if $\phi_{\mathcal{L}} : W \hookrightarrow \mathbb{P}^p$ is an embedding and if the line joining them is contained in $W$. The way in which the singular points of $W$ are associated is called the *configuration* of the singular points of $W$. In particular, if each singular point of $W$ is associated with the same number of the others, we will say that the singular points of $W$ are *similar*. More generally, the singular points of $W$ are called similar if they all "behave in the same way". The notions of association and similarity of the singular points of an Enriques-Fano threefold are important in Fano's construction: we will explain why in § 4. However, we will not give too much details of the description of Fano's work, since it probably contains other hidden gaps in addition to those identified by Conte and Murre in [14]. For this reason, in § 5 we will examine the rational examples $W_F^{p=6,7,9,13}$ found by Fano as images of linear systems on $\mathbb{P}^3$; by using blow-ups techniques, we will verify that they actually are Enriques-Fano threefolds and that they are embedded in $\mathbb{P}^p$ with eight similar quadruple points, whose tangent cone is a cone over a Veronese surface. We will also find the configurations used by Fano, thus justifying his statements.

The Enriques-Fano threefolds $W_{BS}^p$ also have eight quadruple points, whose tangent cone is a cone over a Veronese surface. Six of these ones (of genus $p = 6, 7, 8, 9, 10, 13$) are embedded in $\mathbb{P}^p$ and we will computationally study them in § 6. In particular, we will show that the singular points of $W_{BS}^{p=6,7,9,13}$ are similar and that they have the same configurations of the ones of $W_F^{p=6,7,9,13}$. Moreover, we will prove that, for $p = 9, 13$, the embedding of $W_{BS}^p$ in $\mathbb{P}^p$ is the threefold $W_F^p$. We will also show how to construct the Enriques-Fano threefolds $W_{BS}^{p=8,10}$ as images of linear systems on $\mathbb{P}^3$. Finally we will find that the eight quadruple points of $W_{BS}^{p=8,10}$ are similar but they have configurations that were excluded by Fano: this suggests that there may be further hidden gaps in [23].

We will also examine the Enriques-Fano threefolds $W_{KLM}^9$, $W_P^{13}$, $W_P^{17}$ (see § 7, 8). It is known that $W_{KLM}^9$ and $W_P^{17}$ have canonical non-terminal singularities, but so far there was no information about their multiplicities and tangent cones. With a computational analysis, we will show that $W_{KLM}^9$ and $W_P^{17}$ have four quadruple points, whose tangent cone is a cone over a Veronese surface, and one sextuple point, whose tangent cone is a cone over the union of four planes and a quadric surface. We will also deepen the study of $W_P^{13}$, which was mentioned very briefly by Prokhorov. In particular, we will show that it has four quadruple points, whose tangent cone is a cone over a Veronese surface, and a quintuple point, whose tangent cone is a cone over the union of five planes. Anyhow, the threefolds $W_{KLM}^9$, $W_P^{13,17}$ have non-similar singular points.

Let us denote by $H$ the class of a curve section on a smooth hyperplane section $S \in \mathcal{L}$ of a known Enriques-Fano threefold $(W, \mathcal{L})$. In § 9 we will describe the *simple isotropic decomposition* of $H$ (see [9, Corollary 4.7] for the existence) and the value $\phi(H) := \min\{E \cdot H | E \in \mathrm{NS}(S), E^2 = 0, E > 0\}$. We recall that the number $\phi$ and the simple isotropic decompositions allow us to identify the various components of the moduli space of the polarized Enriques surfaces. Thus our analysis suggests which families the hyperplane sections of the Enriques-Fano threefolds belong to.

Finally we will analyze the sublinear system $\mathcal{L}_\bullet \subset \mathcal{L}$ of the hyperplane sections of the Enriques-Fano threefold $W_F^{13}$ having a triple point at a general point $w \in W_F^{13}$ (see § 10). We will show that a general element of this linear system is birational to an elliptic ruled surface and that the image of $W_F^{13}$ via the rational map defined by $\mathcal{L}_\bullet$ is a cubic Del Pezzo surface $\Delta \subset \mathbb{P}^3$ with 4 nodes (see Theorem 10.25). This is interesting because it is related to a Castelnuovo's conjecture stated in [4, pp. 187-188]: let us suppose we have a rational smooth irreducible threefold $W$ and an $r$-dimensional linear system $\mathcal{L}$ on $W$ such that the general element is a smooth irreducible surface $S$ with zero geometric genus $p_g(S) = 0$ and zero arithmetic genus $p_a(S) = 0$. What happens if we force the surfaces of $\mathcal{L}$ to have a triple point at a general point $w \in W$? Castelnuovo thinks that we get an $(r-10)$-dimensional sublinear system $\mathcal{L}_\bullet$ such that the general surface $S_\bullet$ satisfies one of the following three properties: it is an irreducible surface with *irregular desingularization* $\widetilde{S}_\bullet$ which has $p_g(\widetilde{S}_\bullet) = 0$ and $p_a(\widetilde{S}_\bullet) = -1$; it is *reducible* in two rational surfaces intersecting along a rational curve; it has the *same genera* as a general surface $S \in \mathcal{L}$. We will resume the ideas of Castelnuovo adapting them to normal threefolds with isolated singularities and regular desingularization. Examples of such threefolds are the Enriques-Fano threefolds. We will find that $W_F^{13}$ and $W_P^{17}$ satisfy the first property conjectured by Castelnuovo.

We will work over the field $\mathbb{C}$ of the complex numbers. For the computational analysis we will work over a finite field (we will choose $\mathbb{F}_n := \mathbb{Z}/n\mathbb{Z}$ with $n = 10000019$). In Appendix A we will graphically describe the configurations of the singular points of some Enriques-Fano threefolds. In Appendix B we will collect the input codes used in Macaulay2.

Some papers taken from this PhD thesis are currently being written (see [38], [39], [40], [41]).

# 2 Terminology

In this section we gather the basic definitions and the standard conventions that we will use afterwards. We recommend [27], [28], [29], [37] for more details. Let $X$ be a projective variety: we say that $X$ is a *curve*, a *surface* or a *threefold* if $\dim X$ is respectively equal to $1, 2$ or $3$.

We recall that a variety is *normal* if the local ring at every point of the variety is an integrally closed ring. A projective variety $X \subset \mathbb{P}^r$ is said to be *projectively normal* (with respect to the given embedding) if its homogeneous coordinate ring $S(X)$ is integrally closed. It is known that $X \subset \mathbb{P}^r$ is projectively normal if and only if $X$ is normal and for every $k > 0$ the natural map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \to H^0(X, \mathcal{O}_X(k))$ is surjective. If the previous map is surjective for $k = 1$, we say that $X$ is *linearly normal*. A projective variety $X \subset \mathbb{P}^r$ is said to be *arithmetically Cohen-Macaulay* if its homogeneous coordinate ring $S(X)$ is Cohen-Macaulay, which is equivalent to have $H^1(\mathbb{P}^r, \mathcal{I}_{X|\mathbb{P}^r}(k)) = 0$ and $H^i(X, \mathcal{O}_X(k)) = 0$ for all $k > 0$ and for all $0 < i < \dim X$ (see [21, Exercise 18.16.b]). If a projective variety $X \subset \mathbb{P}^r$ is normal and arithmetically Cohen-Macaulay, then it is projectively normal.

Let $D$ be a Cartier divisor on a projective variety $X$. We will use the symbols $\sim$ and $\equiv$ for the linear equivalence and the numerical equivalence, respectively. We will denote by $|\mathcal{O}_X(D)|$, or simply by $|D|$, the complete linear system of divisors linearly equivalent to $D$ on $X$. Linear systems of dimension 1 are called *pencils*. We will say that

(i) $D$ is *big* if $\max_{m\in\mathbb{N}}\{\dim \phi_{|mD|}(X)\} = \dim X$, where $\phi_{|mD|} : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(mD)))$ is the rational map associated with $|mD|$;

(ii) $D$ is *semi-ample* if the linear system $|mD|$ is base point free for some $m > 0$ and so it defines a morphism $\phi_{|mD|} : X \to \mathbb{P}(H^0(X, \mathcal{O}_X(mD)))$;

(iii) $D$ is *very ample* if the linear system $|D|$ is base point free and the associated morphism is a closed embedding $\phi_{|D|} : X \hookrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(D)))$;

(iv) $D$ is *ample* if $mD$ is very ample for some $m > 0$.

Furthermore if $\dim X \geq 2$

(v) $D$ is said to be *nef* if $D \cdot C \geq 0$ for all irreducible curves $C \subset X$.

We have that “semi-ampleness” $\Rightarrow$ “nefness”, and obviously that

“very-ampleness” $\Rightarrow$ “ampleness” $\Rightarrow$ “bigness” and “semi-ampleness”.

We recall now some known results which we will implicitly use in next sections.

**Proposition 2.1.** [37, p. 139] Let $X$ be a normal projective variety and let $D$ be a Cartier divisor on $X$. Then $D$ is big if and only if the rational map $\phi_{|mD|} : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(mD)))$, defined by the linear system $|mD|$, is birational onto its image for some $m > 0$.

**Proposition 2.2.** If $f : Y \to X$ is a birational morphism between two projective varieties and if $D$ is a big divisor on $X$, then $f^*D$ is a big divisor on $Y$.

*Proof.* It follows by the inequality $h^0(Y, \mathcal{O}_Y(f^*D))) \geq h^0(X, \mathcal{O}_X(D))$ and by [37, Lemma 2.2.3]. □

**Remark 2.3.** [37, Example 1.4.4] Let $f : Y \to X$ be a proper mapping. If $D$ is a nef divisor on $X$, then $f^*D$ is a nef divisor on $Y$.

**Proposition 2.4.** [37, Example 1.4.5] Let $|D|$ be a linear system on a projective variety $X$ with the property that $|D|$ is base point free. Then $D$ is nef.

**Theorem of Zariski-Fujita.** [37, Remark 2.1.32] Let $|D|$ be a linear system on a projective variety $X$ with the property that the base locus is a finite set. Then $D$ is semiample.

We will denote by $K_X$ the *canonical divisor* of a smooth projective variety $X$. The numbers $p_g(X) := h^0(X, \mathcal{O}_X(K_X))$ and $P_n(X) := h^0(X, \mathcal{O}_X(nK_X))$ are called respectively the *geometric genus* and the *n-th plurigenus* of $X$, where $n$ is a positive integer. Another important number associated with a variety $X$ is the *arithmetic genus*, denoted by $p_a(X) := (-1)^{\dim X}(\chi(\mathcal{O}_X) - 1)$. We recall that the *irregularity* of a projective variety $X$ is the number $q(X) := h^1(X, \mathcal{O}_X)$ and that $X$ is called *regular* if $q(X) = 0$, otherwise it is said to be *irregular*. If $X$ is a singular projective variety, we say that $X$ has a *regular* (respectively *irregular*) *desingularization* if for each resolution of singularities $f : \widetilde{X} \to X$ we have $q(\widetilde{X}) = 0$ (respectively $q(\widetilde{X}) > 0$). If $p$ is a smooth point of a projective variety $X$, we will denote *the tangent space to $X$ at $p$* by the symbol $T_pX$; if $p$ is a singular point of a projective variety $X$, we will denote *the tangent cone to $X$ at $p$* by the symbol $TC_pX$.

Finally let us recall some fact and some notation about the blow-ups of threefolds. We recommend [27, Chap 4, §6] and [32, Lemma 2.2.14] for more details. Let $X$ be a smooth threefold, let $p \in X$ be a point and let $C \subset X$ be a smooth curve. If $f : \mathrm{Bl}_p X \to X$ is the blow-up of $X$ at $p$ with exceptional divisor $E_p := f^{-1}(p)$, then we have $E_p \cong \mathbb{P}^2$. If $g : \mathrm{Bl}_C X \to X$ is the blow-up of $X$ along $C$ with exceptional divisor $E_C := g^{-1}(C)$, then $E_C$ is a $\mathbb{P}^1$-bundle over $C$ and it is identified with the projectification $\mathbb{P}(\mathcal{N}_{C|X})$ of the normal bundle of $C$ in $X$. We recall that if $C \subset X$ is the complete intersection of two surfaces $S, S' \subset X$, then $\mathcal{N}_{C|X} \cong \mathcal{O}_C(S) \oplus \mathcal{O}_C(S')$ (see [11, Example 10.2]). Let us see an example. Let $l$ be a line of $\mathbb{P}^3$ and let us take

$n$ points $q_1, \dots, q_n$ on $l$. We have $\mathcal{N}_{l|\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Let $X := \mathrm{Bl}_{q_1,\dots,q_n} \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ at the points $q_1, \dots, q_n$ with exceptional divisors $E_i := bl^{-1}(q_i)$, for $1 \le i \le n$. If $C \subset X$ denotes the strict transform of $l \subset \mathbb{P}^3$, then $C$ is the complete intersection of the strict transforms of two hyperplane of $\mathbb{P}^3$ containing $q_1, \dots, q_n$. Thus we have

$$\mathcal{N}_{C|X} \cong \mathcal{O}_C\Big(H - \sum_{i=1}^{n} E_i\Big) \oplus \mathcal{O}_C\Big(H - \sum_{i=1}^{n} E_i\Big) \cong \mathcal{O}_{\mathbb{P}^1}(1-n) \oplus \mathcal{O}_{\mathbb{P}^1}(1-n),$$

where $H$ denotes the pullback of the hyperplane class of $\mathbb{P}^3$.

# 3 Known Enriques-Fano threefolds

## 3.1 Preliminaries on Enriques-Fano threefolds

Let us recall that an *Enriques surface* is a smooth, irreducible surface $S$ with zero irregularity $q(S) = 0$ and non-trivial canonical divisor $K_S$ such that $2K_S \sim 0$.

**Definition 3.1.** A pair $(W, \mathcal{L})$, or simply $W$, is called *Enriques-Fano threefold* if

(i) $W$ is a normal threefold;

(ii) $\mathcal{L}$ is a complete linear system of ample Cartier divisors on $W$ such that the general element $S \in \mathcal{L}$ is an Enriques surface;

(iii) $W$ is not a *generalized cone* over $S$, i.e., $W$ is not obtained by contraction of the negative section on the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_S \oplus \mathcal{O}_S(S))$ over $S$.

We define the *genus* and the *degree* of an Enriques-Fano threefold $(W, \mathcal{L})$ to be respectively the values $p := \frac{S^3}{2} + 1$ and $\deg(W) := S^3$, where $S$ is a general element of $\mathcal{L}$. Hence $\deg(W) = 2p - 2$. The linear system $\mathcal{L}$ defines a rational map $\phi_{\mathcal{L}} : W \dashrightarrow \mathbb{P}^p$, where $\dim \mathcal{L} = p \ge 2$. Furthermore the genus $p$ of an Enriques-Fano threefold $(W, \mathcal{L})$ is at most 17 and the bound is sharp (see [36] and [46]). Though improperly, we will refer to the elements of $\mathcal{L}$ as *hyperplane sections* of $W$ and to the curve intersections of two elements of $\mathcal{L}$ as *curve sections* of $W$.

**Definition 3.2.** Let $W$ be a normal variety such that $K_W$ is $\mathbb{Q}$-Cartier and let $f : \widetilde{W} \to W$ be a resolution of the singularities, with irreducible exceptional divisors $E_i$. Since we have $K_{\widetilde{W}} = f^*(K_W) + \sum a_i E_i$ with $a_i \in \mathbb{Q}$, we say that the singularities of $W$ are *terminal* if $a_i > 0$ for all $i$ and we say that they are *canonical* if $a_i \ge 0$ for all $i$.

It is known that any Enriques-Fano threefold $(W, \mathcal{L})$ is singular with isolated singularities (see [14, Lemma 3.2]): moreover $K_W$ is 2-Cartier and the singularities are canonical (see [6]). Furthermore $W$ has regular desingularization (see [10, Lemma 4.1]).

The classification of Enriques-Fano threefolds $(W, \mathcal{L})$ is still an open problem, but examples have been found by several authors. The first to deal with this problem was Fano, who proposed in [23] an incomplete classification: he claimed that Enriques-Fano threefolds exist only for $p = 4, 6, 7, 9, 13$, but his arguments contain some gaps. Indeed, Conte and Murre proved, under certain assumptions, results that Fano had only stated (see [14]). However, Conte and Murre did not address the classification problem. Under the assumption that the singularities of $W$ are terminal cyclic quotients, Enriques-Fano threefolds were classified by Bayle in [1] (and in a similar and independent way by Sano in [48]). If $W$ is an Enriques-Fano threefold found Bayle and Sano, then it has genus $2 \leq p \leq 10$ or $p = 13$; furthermore $W$ is the quotient of a smooth Fano threefold $X$ via an involution $\sigma$ with 8 fixed points, and $W$ itself has 8 quadruple points, whose tangent cone is a cone over a Veronese surface. More generally, if an Enriques-Fano threefold has terminal singularities, then it admits a $\mathbb{Q}$-*smoothing*, i.e., it appears as central fibre of a small deformation over the 1-parameter unit disk such that a general fibre has only cyclic quotient terminal singularities (see [44, Main Theorem 2]). Hence every Enriques-Fano threefold with only terminal singularities is a limit of someone discovered by Bayle and Sano. Thus, to complete the classification, one has to consider the case of non-terminal canonical singularities. Only a few examples of Enriques-Fano threefolds with non-terminal canonical singularities are known: one of genus $p = 9$ found by Knutsen, Lopez and Muñoz in [36, §13] and another one of genus $p = 17$ found by Prokhorov in [46, §3]. Finally there is an Enriques-Fano threefold of genus $p = 13$, which was mentioned very briefly by Prokhorov (see [46, Remark 3.3]).

## 3.2 List of known Enriques-Fano threefolds

We will list the known Enriques-Fano threefolds, we will talk about their properties and we will give some notation. First we recall two definitions.

**Definition 3.3.** Let $\mathcal{R}$ be a 3-dimensional linear system of quadric sufaces of $\mathbb{P}^3$. Let us suppose that $\mathcal{R}$ is sufficiently general, i.e. $\mathcal{R}$ is base point free and, if $l$ is a double line for $Q \in \mathcal{R}$, then $Q$ is the unique quadric in $\mathcal{R}$ containing $l$. A *Reye congruence* is a surface obtained as the set $\{l \in \mathbb{G}(1,3) | l \text{ is contained in a pencil contained in } \mathcal{R}\}$, where $\mathbb{G}(1,3)$ denotes the Grassmannian variety of lines in $\mathbb{P}^3$.

**Definition 3.4.** A surface in $\mathbb{P}^3$ has *ordinary singularities* if it has at most the following singularities: a curve $\gamma$ of double points (that are generically the transverse intersection of two branches), with at most finitely many pinch points and with $\gamma$ having at most finitely many triple points as singularities, with three independent tangent lines, which are triple points also for the surface.

We will call *F-EF 3-folds* the Enriques-Fano threefolds found by Fano. They are:

(i) the Enriques-Fano threefold $W_F^6 \subset \mathbb{P}^6$ of genus $p = 6$ given by the image of $\mathbb{P}^3$ via the linear system $\mathcal{P}$ of the septic surfaces with double points along three twisted cubics having five points in common (see [23, §3]):

  - this threefold is *rational*;
  - the hyperplane sections of this threefold are Reye congruences (see also [13, Proposition 3]);
  - a general $P \in \mathcal{P}$ has ordinary singularities;

(ii) the Enriques-Fano threefold $W_F^7 \subset \mathbb{P}^7$ of genus $p = 7$ given by the image of $\mathbb{P}^3$ via the linear system $\mathcal{X}$ of the sextic surfaces having double points along the six edges of a tetrahedron and containing a plane cubic curve intersecting each edge at one point (see [23, §4]):

   - this threefold is *rational*;
   - a general $X \in \mathcal{X}$ has ordinary singularities;

(iii) the Enriques-Fano threefold $W_F^9 \subset \mathbb{P}^9$ of genus $p = 9$ given by the image of $\mathbb{P}^3$ by the linear system $\mathcal{K}$ of the septic surfaces having double points along the six edges of two trihedra (see [23, §7]):

   - this threefold is *rational*;
   - a general $K \in \mathcal{K}$ has ordinary singularities;
   - the locus of pairs of trihedra, up to automorphisms of $\mathbb{P}^3$, has dimension $3 = 18 - 15$: indeed the vertex of a trihedron moves in a $\mathbb{P}^3$ and each one of its three faces moves in a $\mathbb{P}^2$; we observe that 3 is the number of moduli of the Enriques-Fano threefolds of genus 9 contained in [1] and [48] (see (XII) below);

(iv) the Enriques-Fano threefold $W_F^{13} \subset \mathbb{P}^{13}$ of genus $p = 13$ given by the image of $\mathbb{P}^3$ via the linear system $\mathcal{S}$ of the sextic surfaces having double points along the six edges of a tetrahedron (see [23, §8]):

   - this threefold is *rational*;
   - a general $\Sigma \in \mathcal{S}$ has ordinary singularities;
   - we will also refer to this threefold as the *classical Enriques-Fano threefold*;

and one "exceptional" case (see § 4.3 to understand better):

(0) the famous *Enriques threefold* $W_F^4 \subset \mathbb{P}^4$, which is a singular sextic hypersurface whose hyperplane section is a sextic surface in $\mathbb{P}^3$ with double points along the six edges of a tetrahedron (see [23, §10]):

   - it has equation

$$x_1x_2x_3x_4(x_0^2 + x_0 \sum_{i=1}^4 a_i x_i + \sum_{i,j=1}^4 b_{ij} x_i x_j) + \\ + c_1 x_2^2 x_3^2 x_4^2 + c_2 x_1^2 x_3^2 x_4^2 + c_3 x_1^2 x_2^2 x_4^2 + c_4 x_1^2 x_2^2 x_3^2 = 0,$$

     where $x_0, \ldots, x_4$ are the homogeneous coordinates of $\mathbb{P}^4$, and $a_i$, $b_{ij}$ and $c_i$ are sufficiently general complex numbers;
   - it has double points along six planes, which are given by the intersections of four spaces $\mathbb{P}^3$ two by two and which all pass through the same point;
   - the general Enriques threefolds $W_F^4$ have been proved to be *non-rational* by Picco-Botta and Verra in [45];
   - it is also contained in [1] and [48] (see (IV) below).

Furthermore, as noted by Conte in [12, p. 225], there is also another "hidden" exceptional case:

(00) the threefold $W^3_F$ given by a quadruple $\mathbb{P}^3$ (see [23, §2]);

  – it is worth mentioning it, because it is also contained in [1] and [48] (see (II) below).

In § 4 we will summarize Fano's approach and Conte-Murre's work. Furthermore in § 5 we will describe the rational F-EF 3-folds in modern language via blow-up techniques.

In order to classify Enriques-Fano threefolds, Bayle assumes the following fact.

**Assumption (B).** Let $(W, \mathcal{L})$ be an Enriques-Fano threefold such that $W$ is the quotient $X/\sigma$ of a smooth Fano threefold $X$ where $\sigma$ is an involution of $X$ with finitely many fixed points.

The number of fixed points of the involution $\sigma$ of Assumption B must be 8 (see [1, §4.1]). Moreover, the images of these 8 points of $X$, via the quotient map $\pi : X \to W$, are eight singular points of $W$ whose tangent cone is a cone over a Veronese surface (see [1, §3]). Bayle's approach to the classification is as follows. By Assumption B, we have that

(i) $b_2(X) + \frac{b_3(X)}{2} \equiv 1 \pmod 2$, where $b_i(X) := \operatorname{rank} H_i(X, \mathbb{R})$ is the $i^{th}$ Betti's number of $X$ (see [1, §4.2]);

(ii) $\deg X := (-K_X)^3 = 4p - 4$ is divisible by 4 (see [1, §4.3]).

In order to classify the Enriques-Fano threefolds $W$ satisfying Assumption B, Bayle considers all the smooth Fano threefolds, classified by Iskovskih in [30] and [31] and by Mori and Mukai in [42], and he eliminates the ones that do not satisfy the above two properties: though a Fano threefold has been erroneously omitted by Mori and Mukai, this has no consequence for Bayle's work, since the degree of this threefold is not divisible by 4 (see [43]). By studying the remaining smooth Fano threefolds, Bayle finds that only 14 of them have an involution with 8 fixed points: thus he finds fourteen Enriques-Fano threefolds, by constructing the quotient map $\pi : X \to W$ as the map defined by the sublinear system of $|-K_X|$ given by the $\sigma$-invariant elements. These threefolds are also contained in [48], so we will refer to them as *BS-EF 3-folds*. They are:

(I) the Enriques-Fano threefold $W^2_{BS}$ of genus $p = 2$ given by the quotient of a double cover of a smooth quadric hypersurface of $\mathbb{P}^4$ branched in an optic surface (see [1, §6.1.6]):

  – in this case $\phi_{\mathcal{L}} : W^2_{BS} \dashrightarrow \mathbb{P}^2$ is a rational map;
  – according to [1, p. 23], these $W^2_{BS}$ depend on 25 moduli;
  – these $W^2_{BS}$ can be also obtained as quotient of the complete intersection of a quadric and quartic in $\mathbb{P}(1^5; 2)$;

- it is also found by Sano (see [48, Theorem 1.1 No.1]);
- Cheltsov *conjectures* that $W^2_{BS}$ is *non-rational* (see [8, Conjecture 19]);

(II) the Enriques-Fano threefold $W^3_{BS}$ of genus $p = 3$ given by the quotient of the complete intersection of three quadric hypersurfaces of $\mathbb{P}^6$ (see [1, §6.1.5]):

- in this case $\phi_{\mathcal{L}} : W^3_{BS} \to \mathbb{P}^3$ is a morphism and it is a quadruple cover of $\mathbb{P}^3$;
- according to [1, p. 22], the number of moduli of $W^3_{BS}$ is 15;
- it is also found by Sano (see [48, Theorem 1.1 No.2]);
- Cheltsov *conjectures* that $W^3_{BS}$ is *non-rational* ([8, Conjecture 19]);

(III) the Enriques-Fano threefold $\overline{W}^3_{BS}$ of genus $p = 3$ given by the quotient of the blow-up of $B_2$ along a curve given by the intersection of two elements of $|-\frac{1}{2}K_{B_2}|$, where $B_2$ is the double cover of $\mathbb{P}^3$ branched in a smooth quartic surface (see [1, §6.2.7]):

- in this case $\phi_{\mathcal{L}} : \overline{W}^3_{BS} \dashrightarrow \mathbb{P}^3$ is a rational map of degree 2;
- according to [1, p. 34], the number of moduli of $\overline{W}^3_{BS}$ is 15;
- these $\overline{W}^3_{BS}$ can also obtained as quotient of the blow-up of a smooth quartic hypersurface of $\mathbb{P}(1^4; 2)$, along a smooth elliptic curve, which is cut out by two hypersurfaces of degree one;
- it is also found by Sano (see [48, Theorem 1.1 No.3]);

(IV) the Enriques-Fano threefold $W^4_{BS}$ of genus $p = 4$ given by the quotient of a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched in a divisor of multidegree $(2, 2, 2)$ (see [1, §6.3.3]):

- in this case $\phi_{\mathcal{L}} : W^4_{BS} \dashrightarrow \mathbb{P}^4$ is a rational map birational onto the image, which is the Enriques threefold $W^4_F \subset \mathbb{P}^4$;
- according to [1, p. 40], the number of moduli of $W^4_{BS}$ is 10;
- it is also found by Sano (see [48, Theorem 1.1 No.5]);
- it is *non-rational* (see [45]);

(V) the Enriques-Fano threefold $\overline{W}^4_{BS}$ of genus $p = 4$ given by the quotient of $\mathbb{P}^1 \times S_2$ where $S_2$ is a double cover of $\mathbb{P}^2$ branched in a quartic curve (see [1, §6.6.2]):

- in this case $\phi_{\mathcal{L}} : \overline{W}^4_{BS} \dashrightarrow \mathbb{P}^4$ is a rational map and it is a double cover of the image, which is a quadric cone;
- according to [1, p. 61], the number of moduli of $\overline{W}^4_{BS}$ is 4;
- it is also found by Sano (see [48, Theorem 1.1 No.4]);
- it is *rational* (see [7, Remark 7.3]);

(VI) the Enriques-Fano threefold $W^5_{BS}$ of genus $p = 5$ given by the quotient of the blow-up of a smooth intersection of two quadric hypersurfaces of $\mathbb{P}^5$, along the elliptic curve given by the intersection of two hyperplane sections (see [1, §6.2.2]):

- in this case $\phi_{\mathcal{L}} : W^5_{BS} \to \mathbb{P}^5$ is a morphism birational onto its image, which has two double planes;
- according to [1, p. 25], the number of moduli of these threefolds is 7;
- it is also found by Sano (see [48, Theorem 1.1 No.7]);
- it was accidentally not listed in [1, Theorem B];
- it is *rational* (see [7, Remark 7.3]);

(VII) the Enriques-Fano threefold $\overline{W}^5_{BS}$ of genus $p = 5$ given by the quotient of a double cover of $\mathbb{P}^3$, branched in a smooth quartic surface (see [1, §6.1.2]):

- in this case $\phi_{\mathcal{L}} : \overline{W}^5_{BS} \to \mathbb{P}^5$ is a morphism and it is a double cover of the image, which is a complete intersection of two quadrics;
- according to [1, p. 18], the number of moduli of these threefolds is 11;
- these $\overline{W}^5_{BS}$ can be also obtained as quotient of a quartic hypersurface of $\mathbb{P}(1^4; 2)$;
- it is also found by Sano (see [48, Theorem 1.1 No.8]);
- it is *rational* (see [8, Theorem 1]);

(VIII) the Enriques-Fano threefold $W^6_{BS}$ of genus $p = 6$ given by the quotient of the complete intersection of three divisors of bidegree $(1, 1)$ on $\mathbb{P}^3 \times \mathbb{P}^3$ (see [1, §6.2.4]):

- in this case $\phi_{\mathcal{L}} : W^6_{BS} \hookrightarrow \mathbb{P}^6$ is an embedding;
- according to [1, p. 29], the number of moduli of these threefolds is 24;
- it is also found by Sano (see [48, Theorem 1.1 No.9]);
- it is *rational* (see [7, Corollary 7.2]);

(IX) the Enriques-Fano threefold $\overline{W}^7_{BS}$ of genus $p = 7$ given by the quotient of $\mathbb{P}^1 \times S_4$, where $S_4$ is a Del Pezzo surface of degree 4 in $\mathbb{P}^4$ (see [1, §6.6.1]):

- in this case $\phi_{\mathcal{L}} : \overline{W}^7_{BS} \to \mathbb{P}^7$ is a morphism birational onto its image;
- according to [1, p. 59], the number of moduli of these threefolds is 2;
- it is also found by Sano (see [48, Theorem 1.1 No.10]);
- it is *rational* (see [7, Corollary 7.2]);

(X) the Enriques-Fano threefold $W^7_{BS}$ of genus $p = 7$ given by the quotient of a smooth divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of multidegree $(1, 1, 1, 1)$ (see [1, §6.4.1]):

- in this case $\phi_{\mathcal{L}} : W^7_{BS} \hookrightarrow \mathbb{P}^7$ is an embedding;
- according to [1, p. 46], the number of moduli of these threefolds is 3;

– it is also found by Sano (see [48, Theorem 1.1 No.11]);

– it is *rational* (see [7, Corollary 7.2]);

(XI) the Enriques-Fano threefold $W^8_{BS}$ of genus $p = 8$ given by the quotient of the blow-up of the cone over a quadric surface $Q \subset \mathbb{P}^3$ along the disjoint union of the vertex and an elliptic curve on $Q$ (see [1, §6.4.2]):

– in this case $\phi_{\mathcal{L}} : W^8_{BS} \hookrightarrow \mathbb{P}^8$ is an embedding;

– according to [1, p. 51], the number of moduli of these threefolds is 2;

– Sano erroneously omits it (see [48, p. 378]);

– it is *rational* (see [7, Corollary 7.2]);

(XII) the Enriques-Fano threefold $W^9_{BS}$ of genus $p = 9$ given by the quotient of the intersection of two quadrics in $\mathbb{P}^5$ (see [1, §6.1.4]):

– in this case $\phi_{\mathcal{L}} : W^9_{BS} \hookrightarrow \mathbb{P}^9$ is an embedding;

– according to [1, p. 21], the number of moduli of these threefolds is 3;

– it is also found by Sano (see [48, Theorem 1.1 No.12]);

– it is *rational* (see [7, Corollary 7.2]);

(XIII) the Enriques-Fano threefold $W^{10}_{BS}$ of genus $p = 10$ given by the quotient of $\mathbb{P}^1 \times S_6$, where $S_6$ is a smooth Del Pezzo surface of degree 6 in $\mathbb{P}^6$ (see [1, §6.5.1]):

– in this case $\phi_{\mathcal{L}} : W^{10}_{BS} \hookrightarrow \mathbb{P}^{10}$ is an embedding;

– this threefold has no moduli (see [1, p. 56]);

– it is also found by Sano (see [48, Theorem 1.1 No.13]);

– it is *rational* (see [7, Corollary 7.2]);

(XIV) the Enriques-Fano threefold $W^{13}_{BS}$ of genus $p = 13$ given by the quotient of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (see [1, §6.3.2]):

– in this case $\phi_{\mathcal{L}} : W^{13}_{BS} \hookrightarrow \mathbb{P}^{13}$ is an embedding;

– this threefold has no moduli (see [1, p. 37]);

– it is also found by Sano (see [48, Theorem 1.1 No.14]);

– it is *rational* (see [7, Corollary 7.2]);

**Remark 3.5.** Sano found another threefold (see [48, Theorem 1.1 No.6]) but Bayle excluded it, by providing a more accurate analysis than Sano's (see [1, §6.2.5]).

**Remark 3.6.** If $(W, \mathcal{L})$ is one of $W^6_{BS}$, $W^7_{BS}$, $W^8_{BS}$, $W^9_{BS}$, $W^{10}_{BS}$ and $W^{13}_{BS}$, then an element $S \in \mathcal{L}$ is very ample (see [1, Theorem A]).

**Remark 3.7.** Since the rational F-EF 3-folds have eight quadruple points whose tangent cone is a cone over a Veronese surface (see [23, p. 44]), then they have only terminal singularities (see [47, Example 1.3]) and therefore they are limits of BS-EF 3-folds (see [44, Main Theorem 2]). In particular, by using Macaulay2, we will find that $W_{BS}^{p=9,13}$ can be obtained exactly as $W_F^{p=9,13}$ (see Theorems 6.11, 6.17).

In the paper of Knutsen-Lopez-Muñoz, the following Enriques-Fano threefold is discovered:

(XV) the Enriques-Fano threefold $W_{KLM}^9 \subset \mathbb{P}^9$ of genus $p = 9$ given by the image of the F-EF 3-fold $W_F^{13} \subset \mathbb{P}^{13}$ via the rational map $\rho_{\langle E_3 \rangle} : \mathbb{P}^{13} \dashrightarrow \mathbb{P}^9$, which is the projection of $\mathbb{P}^{13}$ from the three-dimensional linear subspace $\mathbb{P}^3 \cong \langle E_3 \rangle$ spanned by a smooth irreducible elliptic quartic curve $E_3 \subset W_F^{13}$.

It is known that the Enriques-Fano threefold found by Knutsen-Lopez-Muñoz (shortly *KLM-EF 3-fold*) has canonical non-terminal singularities but so far there was no information about their multiplicities and tangent cones. We will analyze them in § 7 thanks to Macaulay2. The KLM-EF 3-fold is *rational* by construction.

Prokhorov constructed

(XVI) an Enriques-Fano threefold $W_P^{13}$ of genus $p = 13$ given by the quotient of a cone over a smooth Del Pezzo surface of degree 6, under an involution fixing five points (see [46, Remark 3.3]);

(XVII) an Enriques-Fano threefold $W_P^{17}$ of genus $p = 17$ given by the quotient of a cone over the octic Del Pezzo surface obtained by the anticanonical embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, under an involution fixing five points (see [46, Proposition 3.2]).

Thanks to Macaulay2, we will see that the above Enriques-Fano threefolds (shortly *P-EF 3-folds*) are embedded in $\mathbb{P}^{p=13,17}$ (see § 8). We will also find the tangent cones at their singularities. The P-EF 3-folds are (at least) *unirational* by construction.

The rationality of the Enriques-Fano threefolds and the number of their moduli are still open questions, which we will examine in future projects.

## 3.3 Normality and projective normality

Some authors define an *Enriques-Fano threefold* just as a threefold satisfying the following assumption (see for example [25, Definition 1.1] and [36, Definition 1.3]).

**Assumption (*).** Let $W \subset \mathbb{P}^N$ be a non-degenerate threefold whose general hyperplane section $S$ is an Enriques surface and such that $W$ is not a cone over $S$.

If the pair $(W, \mathcal{L} := |\mathcal{O}_W(S)|)$ satisfies Assumption (*), it is enough to take its *normalization* $\nu : W^\nu \to W$ to obtain an Enriques-Fano threefold in the sense of Definition 3.1, that is $(W^\nu, \nu^*\mathcal{L})$. Indeed an element of $\nu^*\mathcal{L}$ is ample, since it is the pullback of a very ample divisor of $\mathcal{L}$ via the finite birational morphism $\nu : W^\nu \to W$

(see [37, Theorem 1.2.13]). Moreover if $(W^\nu, \nu^*\mathcal{L})$ were a (polarized) generalized cone, $W^\nu$ would contain a 3-dimensional family of curves of degree 1 with respect to the given polarization such that they all pass through a point: thus $W \subset \mathbb{P}^N$ would be the union of lines through a point, by contradicting Assumption (*).

An example of "Enriques-Fano threefold" in the sense of Assumption (*) is the KLM-EF 3-fold $W^9_{KLM} \subset \mathbb{P}^9$: instead of proving the normality of this threefold, Knutsen-Lopez-Muñoz study properties of its normalization (see [36, Proposition 13.1]). We will see below that the KLM-EF 3-fold actually is (projectively) normal.

Also the rational F-EF 3-folds $W_F^{p=6,7,9,13} \subset \mathbb{P}^p$ are "Enriques-Fano threefold" in the sense of Assumption (*): indeed their normality is unproved, even if Fano assumed normality at the beginning of his work (see Assumption F1 in § 4.2). The normality of the non-rational F-EF 3-fold $W_F^4$ is unproved too; however it does not exactly satisfy Assumption (*), since its hyperplane sections are not Enriques surfaces, but their minimal desingularization they are (see [16, p.275]). We will see below that the rational F-EF 3-folds of genus 7, 9 and 13 actually are (projectively) normal.

Instead the BS-EF 3-folds and the P-EF 3-folds are normal by construction, since they are quotient of normal threefolds under the action of a finite group defined by a certain involution with a finite number of fixed points (see [19, Proposition 2.15]). In particular, the BS-EF 3-folds with very ample hyperplane sections satisfy Assumption (*) in the projective space in which they are embedded, while the eight BS-EF 3-folds $W_{BS}^{p=2,3,4,5}$ and $\overline{W}_{BS}^{p=3,4,5,7}$ are Enriques-Fano threefolds satisfying exactly the abstract Definition 3.1. Furthermore, as we will (computationally) see in § 8.2 and § 8.3, the P-EF 3-folds $W_P^{p=13,17}$ can be embedded in $\mathbb{P}^p$ and they also satisfy Assumption (*).

**Theorem 3.8.** Let $W \subset \mathbb{P}^N$ be a threefold satisfying Assumption (*). If $S \subset \mathbb{P}^{N-1}$ is linearly normal and if either $N \geq 7$ or $N = 6$ and $S$ is not a Reye congruence, then $h^1(\mathcal{O}_W) = 0$ and $W \subset \mathbb{P}^N$ is projectively normal.

*Proof.* Since the case where $N = 6$ and $S$ is a Reye congruence is excluded, we have that $S \subset \mathbb{P}^{N-1}$ is projectively normal (see [24, Theorem 1.1]). Thus, by using the arguments of [15, Lemmas 1.5,1.6,1.7] (which are inspired by the ones of [20, pp. 10-11]), we obtain that $h^1(\mathcal{O}_W) = 0$ and that $W \subset \mathbb{P}^N$ is projectively normal. □

**Proposition 3.9.** Let $W \subset \mathbb{P}^N$ be a threefold satisfying Assumption (*). If $W \subset \mathbb{P}^N$ is linearly normal and $h^1(\mathcal{O}_W) = 0$, then $S \subset \mathbb{P}^{N-1}$ is linearly normal.

*Proof.* We have to show that $h^0(\mathcal{O}_S(1)) = h^0(\mathcal{O}_{\mathbb{P}^{N-1}}(1)) = N$. This follows by the following exact sequence

$$0 \to \mathcal{O}_W \to \mathcal{O}_W(1) \to \mathcal{O}_S(1) \to 0,$$

since $h^0(\mathcal{O}_W) = 1$, $h^1(\mathcal{O}_W) = 0$ and $h^0(\mathcal{O}_W(1)) = h^0(\mathcal{O}_{\mathbb{P}^N}(1)) = N + 1$ by hypothesis. □

**Corollary 3.10.** Let $W \subset \mathbb{P}^N$ be a threefold satisfying Assumption (*). If $W \subset \mathbb{P}^N$ is linearly normal and $h^1(\mathcal{O}_W) = 0$, then $W \subset \mathbb{P}^N$ is projectively normal (except when $N = 6$ and $S$ is a Reye congruence).

*Proof.* See Theorem 3.8 and Proposition 3.9 $\square$

**Proposition 3.11.** Let $W \subset \mathbb{P}^p$ be a threefold satisfying Assumption (*) such that $p$ is the genus of a curve section of $W$. Then $W \subset \mathbb{P}^p$ and $S \subset \mathbb{P}^{p-1}$ are linearly normal.

*Proof.* By Riemann-Roch on $S$ we obtain $h^0(\mathcal{O}_S(1)) = p$. From $W \subset \mathbb{P}^p$ we have that $h^0(\mathcal{O}_W(1)) \geq p + 1$. On the other hand, from the following exact sequence

$$0 \to \mathcal{O}_W \to \mathcal{O}_W(1) \to \mathcal{O}_S(1) \to 0$$

one gets $h^0(\mathcal{O}_W(1)) \leq p + 1$ and hence equality holds. $\square$

**Corollary 3.12.** The following Enriques-Fano threefolds are projectively normal:

$$W^9_{KLM} \subset \mathbb{P}^9, \quad W_F^{p=7,9,13} \subset \mathbb{P}^p, \quad W_{BS}^{p=7,8,9,10,13} \overset{\phi_{\mathcal{L}}}{\hookrightarrow} \mathbb{P}^p, \quad W_P^{p=13,17} \subset \mathbb{P}^p.$$

*Proof.* See Theorem 3.8 and Proposition 3.11. $\square$

We cannot say the same for the F-EF 3-fold $W_F^6 \subset \mathbb{P}^6$, since its hyperplane sections are Reye congruences (see [13, Proposition 3] and [23, §3]). As for the BS-EF 3-fold $W^6_{BS} \overset{\phi_{\mathcal{L}}}{\hookrightarrow} \mathbb{P}^6$, one can find with Macaulay2 that its hyperplane section $S \subset \mathbb{P}^5$ is not contained in quadric hypersurfaces of $\mathbb{P}^5$ (see Code B.1 of Appendix B): this is equivalent to say that $S \subset \mathbb{P}^5$ is projectively normal (use Riemann-Roch and see [24, Theorem 1.1]), thus we obtain that $W^6_{BS} \subset \mathbb{P}^6$ is projectively normal too (see Theorem 3.8).

# 4 Fano's approach to the classification of Enriques-Fano threefolds

## 4.1 Conte-Murre's work

In order to explain Fano's approach to the classification of Enriques-Fano threefolds, we will first summarize the work of Conte and Murre. In their paper [14], they studied threefolds $W$ satisfying the following assumption, which is a particular case of Assumption (*) of § 3.3.

**Assumption (CM1).** Let $W \subset \mathbb{P}^N$ be a non-degenerate threefold such that

(i) $W$ is projectively normal;

(ii) if $h \cong \mathbb{P}^{N-1}$ is a general hyperplane then $F := W \cap h$ is an Enriques surface;

(iii) $W$ is not a cone over $F$.

By setting $\mathcal{L} := |O_W(F)|$ we observe that a threefold $W$ satisfying Assumption CM1 is an Enriques-Fano threefold, according to Definition 3.1: indeed the projective normality of $W$ implies its normality, and an element of $\mathcal{L}$ is ample since it is very ample. So $W$ has isolated singular points $P_1, \ldots, P_n$ (see also [14, Lemma 3.2]). The genus $p$ of such an Enriques-Fano threefold is equal to the genus of a general curve section $\Gamma := W \cap h \cap h'$: indeed, since $\Gamma$ is a smooth curve on an Enriques surface $F$, by the adjunction formula we have $p = \frac{F^3}{2} + 1 = \frac{\Gamma^2_{|F}}{2} + 1 = \frac{2p_a(\Gamma)-2}{2} + 1 = p_a(\Gamma) = p_g(\Gamma)$. In particular we have $N = p$ (see [14, Corollary 3.6]) and $p > 5$ (see [14, Remark 4.5]).

Conte and Murre also re-proved a result of the paper [26] of Godeaux, useful for the arguments of Fano. Indeed they showed that on a threefold $W$ satisfying Assumption CM1 there exists a linear system $|\varphi|$ of Weil divisors $\varphi$ such that: $\dim |\varphi| = p-1$; for a general $\varphi$ the hyperplane section $\varphi \cap h$ is a canonically embedded curve; $|\varphi|$ has no base points except possibly at the singular points $P_1, \ldots, P_n$ of $W$; $H^1(\varphi, \mathcal{O}_\varphi(n)) = 0, n \geq 0$; $H^2(\varphi, \mathcal{O}_\varphi(n)) = 0, n > 0$; $\dim H^2(\varphi, \mathcal{O}_\varphi) = 1$ (see [14, Lemma 3.7]). We will refer to $|\varphi|$ as the *Godeaux linear system of* $W$ and we will denote by $\lambda_{|\varphi|} : W \dashrightarrow \mathbb{P}^{p-1}$ the rational map defined by $|\varphi|$.

**Assumption (CM2).** Let $W$ be a threefold with isolated singularities $P_1, \ldots, P_n$ such that, if $\pi : \widetilde{W} \to W$ is the blow-up of $W$ in the singular points, then $\widetilde{W}$ is smooth and the exceptional divisors $E_1 := \pi^{-1}(P_1), \ldots, E_n := \pi^{-1}(P_n)$ are smooth too.

Let us consider now a threefold $W$ satisfying Assumptions CM1 and CM2. If $\widetilde{F}$ and $\widetilde{\varphi}$ are respectively the strict transforms of a general hyperplane section of $W$ and of a general element of the Godeaux linear system of $W$, then we have

$$2\widetilde{F} = 2\widetilde{\varphi} + \sum_{i=1}^{n} t_i E_i \quad \text{and} \quad K_{\widetilde{W}} = -\widetilde{\varphi} + \sum_{i=1}^{n} r_i E_i \tag{1}$$

in $\mathrm{Pic}(\widetilde{W})$, where $t_i, r_i \in \mathbb{Z}$ for all $i = 1, \ldots, n$ (see [14, Lemma 3.12]).

**Assumption (CM3).** Let $W$ be a threefold with isolated singularities $P_1, \ldots, P_n$ such that all they "behave in the same way": this means, for example, that if $W$ satisfies Assumptions CM1 and CM2 and if $1 \leq i < j \leq n$, then we have that $t_i = t_j = t$ and $r_i = r_j = r$ in (1); we have that $p_a(C_i) = p_a(C_j)$, where $C_i := \widetilde{\varphi} \cap E_i$, etc.

It follows that if $W$ is a threefold satisfying Assumptions CM1, CM2 and CM3, then all the singular points $P_1, \ldots, P_n$ of $W$ are base points of its Godeaux linear system $|\varphi|$ and furthermore we have that $t_i = t > 0$ in (1) (see [14, Lemma 4.2]).

**Assumption (CM4).** If $W$ is a threefold satisfying Assumptions CM1, CM2 and CM3 and if $\widetilde{\varphi}$ denotes the strict transform of a general element $\varphi$ of its Godeaux linear system, then the linear system $|\widetilde{\varphi}|$ has no base points on $\widetilde{W}$, the curves $C_i := \widetilde{\varphi} \cap E_i$ are smooth and irreducible for all $1 \leq i \leq n$, and $\widetilde{\varphi}$ is smooth.

If $W$ is a threefold satisfying Assumptions CM1, CM2, CM3, CM4 and if $M$ is the image of $W$ via its Godeaux linear system, then we have the following diagram

$$\begin{array}{ccc} \widetilde{W} & & \\ \downarrow \pi & \searrow^{\lambda_{|\widetilde{\varphi}|}} & \\ \mathbb{P}^p \supset W & \overset{\lambda_{|\varphi|}}{\dashrightarrow} & M \subset \mathbb{P}^{p-1}. \end{array}$$

We have all the elements to state the main theorem of Conte-Murre's paper (see [14, Theorem 7.2]), thanks to which they rigorously proved the assertions made by Fano in [23, §1-2].

**Theorem 4.1** (Fano-Conte-Murre Theorem)**.** Let $W$ be a threefold satisfying Assumptions CM1, CM2, CM3 and CM4. Then $W \subset \mathbb{P}^p$ is an Enriques-Fano threefold of genus $p \geq 6$ with $n = 8$ quadruple points $P_1, \dots, P_8$, whose tangent cone is a cone over a Veronese surface. Furthermore $W$ carries a linear system $|\varphi|$ of Weil divisors, the general members of which are K3-surfaces. This system has dimension $(p-1)$, has base points at the points $P_1, \dots, P_8$ and the associated rational map $\lambda_{|\varphi|}$ is birational onto the image. Moreover, the points $P_1, \dots, P_8$ are rational double points on a general $\varphi$. Let $M = \lambda_{|\varphi|}(W) \subset \mathbb{P}^{p-1}$ be the image. Then $M$ has degree $2p-6$ and has K3-surfaces as general hyperplane sections (i.e., $M$ is a *Fano threefold* in the classical sense). Finally $M$ contains 8 planes $\pi_1, \dots, \pi_8$ which are the "images" of the singular points $P_1, \dots, P_8$ of $W$, in the sense that $\pi_i := \lambda_{|\widetilde{\varphi}|}(E_i)$ for $i = 1, \dots, 8$.

**Remark 4.2.** Under the Assumptions CM1, CM2, CM3 and CM4, we have $r_i = r = 0$ and $t_i = t = 1$ in (1) (see [14, Remark 3.14, Lemma 6.3, Corollary 6.5]); hence in $\mathrm{Pic}(\widetilde{W})$ we have $K_{\widetilde{W}} = -\widetilde{\varphi}$ and $2\widetilde{F} = 2\widetilde{\varphi} + \sum_{i=1}^{8} E_i$. The last formula has an important role in Fano's work as we will explain in § 4.2.

**Remark 4.3.** The Enriques-Fano threefolds satisfying Assumptions CM1, CM2, CM3 and CM4 have terminal singularities, since their tangent cone is a cone over a Veronese surface (see [47, Example 1.3]). There is another way to prove it: by Assumption CM2 we can see $\pi : \widetilde{W} \to W$ as the resolution of the singularities $P_1, \dots, P_n$, and so we have $K_{\widetilde{W}} = \pi^* K_W + \sum_{i=1}^{n} a_i E_i$, where $a_i \in \mathbb{Q}$. By fixing $j \in \{1, \dots, n\}$, we have $K_{\widetilde{W}} + E_j = \pi^* K_W + \sum_{i=1}^{n} a_i E_i + E_j$ and by the adjunction formula we have $K_{E_j} = \left(K_{\widetilde{W}} + E_j\right)|_{E_j} = (a_j + 1) E_j|_{E_j}$. Moreover by [14, Corollary 3.15] we have that $K_{E_j} \equiv -(\frac{t_j}{2} + r_j + 1)\widetilde{F}_j|_{E_j}$, where $\widetilde{F}_j$ is the strict transform of a general hyperplane section of $W$ through the point $P_j$. Since $\widetilde{F}_j|_{E_j} \sim (\widetilde{F} - E_j)|_{E_j} \sim -E_j|_{E_j}$, then we have $(\frac{t_j}{2} + r_j + 1) E_j|_{E_j} \equiv K_{E_j} = (a_j + 1) E_j|_{E_j}$. Thus we obtain $a_j = \frac{t_j}{2} + r_j = \frac{1}{2} > 0$.

Now we must recall a notion introduced by Fano in [23, p. 44] and subsequently taken up by Conte and Murre in [14, Remarks 7.3 (iv)].

**Definition 4.4.** Two distinct singular points $P_i$ and $P_j$ of an Enriques-Fano threefold $W$ are said to be *associated* if the line joining them is contained in $W$.

If $W$ is a threefold satisfying Assumptions CM1, CM2, CM3 and CM4, and so we are in the situation described by Theorem 4.1, then the only objects which are contracted by $\lambda_{|\varphi|}$ are the lines $\langle P_i, P_j \rangle$ provided these lines are contained in $W$ (see [14, Remarks 7.3 (ii)]). In this case $P_i$ and $P_j$ are associated and the planes $\pi_i$ and $\pi_j$ have a point in common, that is the contraction of the line $\langle P_i, P_j \rangle$ and which is a double point for $M$. Conte and Murre also observed that, since the only base points of the system $|\varphi|$ are the points $P_i$, then a general $\varphi$ does not contain a line of kind $\langle P_i, P_j \rangle$: hence the general hyperplane section of $M$ is a smooth K3-surface and $M$ has at most isolated singularities.

The singular points of a threefold $W$ satisfying Assumption CM3 are called "similar" by Conte and Murre. However their definition of "similar" takes on a changing meaning in their paper. For this reason, we give the following definition, to which we will refer for the results of this thesis.

**Definition 4.5.** The singular points $P_1, \ldots, P_n$ of an Enriques-Fano threefold $W$ are said to be *similar* if

(i) they have the same multiplicity;

(ii) they have the same tangent cone;

(iii) there is an $m$ such that each $P_i$ is associated with exactly $m$ other singular points.

## 4.2 Fano's work

In order to classify the Enriques-Fano threefolds with $p \geq 6$, Fano used Theorem 4.1, even if he stated it with many gaps and without a real proof. Anyway, let us explain Fano's idea, which is based on the following five assumptions.

**Assumption (F1).** $W \subset \mathbb{P}^{N=p}$ is a normal threefold such that a general hyperplane section $F := W \cap h$ is an Enriques surface, a general curve section $\Gamma := W \cap h \cap h'$ is a smooth curve of genus $p$ and $W$ is not a cone on $F$.

The Assumption CM1 implies the Assumption F1, since the projective normality implies the normalilty.

**Assumption (F2).** The linear system of the curve sections is complete on a hyperplane section $F$ of $W$, i.e. the map $H^0(W, \mathcal{O}_W(1)) \to H^0(F, \mathcal{O}_F(1))$ is surjective.

Conte and Murre proved Assumption F2 as a consequence of Assumption CM1 (see [14, Corollary 3.5]).

**Assumption (F3).** If $p \geq 6$ the Godeaux linear system $|\varphi|$ of $W$ defines a rational map $\lambda : W \dashrightarrow \mathbb{P}^{p-1}$ which is birational onto the image $M$.

Conte and Murre proved Assumption F3, under the Assumptions CM1, CM2, CM3 and CM4 (see [14, §5.24]). Fano showed that a threefold $W$ satisfying Assumptions F1, F2 and F3 has eight quadruple points, whose tangent cone is a cone over a Veronese surface (see [23, §2]). Conte and Murre observed that Fano's arguments are inaccurate (see [14, footnote (2) p.54]). However they found the same result of Fano, under the Assumptions CM1, CM2, CM3 and CM4, as we have said by stating Theorem 4.1.

**Assumption (F4).** Each of the planes $\pi_1, \dots, \pi_8$ contained in $M$ intersects the other seven planes at most at *distinct* points (see also Figure 1).

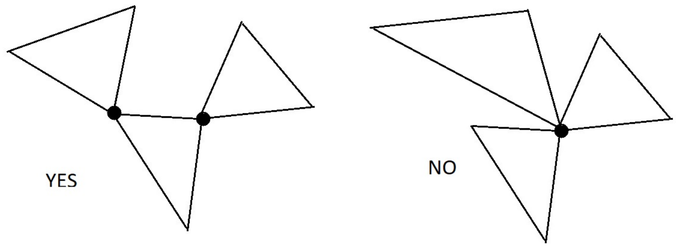


*Figure 1: If two planes $\pi_i$ and $\pi_j$ in $M$ intersect a third plane $\pi_k$, for $1 \leq i < j < k \leq 8$, the situation on the left is admitted by Assumption F4, while the situation on the right is not.*

**Assumption (F5).** Each singular point $P_{i=1,\dots,8}$ of $W$ is associated with the same number $0 \leq m \leq 7$ of the other singular points. The corresponding plane $\pi_i \subset M$ intersects the corresponding $m$ planes.

Conte and Murre observed that Assumption CM3 implies Assumption F5 (see [14, Remarks 7.3 (iv)]).

Fano's approach to the classification of Enriques-Fano threefolds of genus $p \geq 6$ is essentially based on three steps:

(step 1) search for a Fano threefold $M \subset \mathbb{P}^{p-1}$ containing 8 planes $\pi_1, \dots, \pi_8$ satisfying Assumptions F4 ans F5;

(step 2) search for a $p$-dimensional linear system on $M$ whose general element is an Enriques surface $f$ such that $2f \sim 2\phi + \sum_{i=1}^{8} \pi_i$, where $\phi$ is a general hyperplane section of $M$;

**Remark 4.6.** The relation $2f \sim 2\phi + \sum_{i=1}^{8} \pi_i$ in $M$ corresponds to the relation in Remark 4.2, by setting $f := \lambda_{|\widetilde{\varphi}|}(\widetilde{F})$ and $\phi := \lambda_{|\widetilde{\varphi}|}(\widetilde{\varphi})$.

(step 3) the image of the rational map defined by $|f|$ is the desired Enriques-Fano threefold $W$.

**Remark 4.7.** In simple words, Fano used a sort of inverse of Theorem 4.1, giving importance to the similarity and the association of the singular points of $W$. By using this method, Fano constructed the F-EF 3-folds of genus $p \geq 6$, whose search can be summarized in the following way:

(i) $p = 6 \Rightarrow m = 7 \Rightarrow P_1, \dots, P_8$ must be associated as in Figure 21 of Appendix A $\Rightarrow \exists$ F-EF 3-fold $W_F^6 \subset \mathbb{P}^6$;

(ii) $p \geq 7 \Rightarrow M$ is intersection of quadrics in $\mathbb{P}^{p-1}$;

(iii) $p = 7 \Rightarrow m = 6 \Rightarrow P_1, \ldots, P_8$ must be associated as in Figure 22 of Appendix A $\Rightarrow \exists$ F-EF 3-fold $W_F^7 \subset \mathbb{P}^7$;

(iv) $p > 7 \Rightarrow$ there are no three mutually associated points $\Rightarrow m \leq 4$;

(v) $m = 4 \Rightarrow P_1, \ldots, P_8$ must be associated as in Figure 24 of Appendix A $\Rightarrow p = 9 \Rightarrow \exists$ F-EF 3-fold $W_F^9 \subset \mathbb{P}^9$;

(vi) $m \leq 3 \Rightarrow m = 3 \Rightarrow P_1, \ldots, P_8$ must be associated as in Figure 26 of Appendix A $\Rightarrow p = 13 \Rightarrow \exists$ F-EF 3-fold $W_F^{13} \subset \mathbb{P}^{13}$.

In [14, §8] and [23] one can find the description of the Fano threefolds $M$ associated with the F-EF 3-folds of genus $p \geq 6$.

## 4.3 Exceptional cases and possible generalizations

Fano also found an Enriques-Fano threefold $W_F^4 \subset \mathbb{P}^4$ of genus $p = 4$, which behaves differently from the F-EF 3-folds of genus $p \geq 6$. Indeed $W_F^4$ is a sextic hypersurface of $\mathbb{P}^4$ with six double planes, four triple lines and a quadruple point. Its hyperplane section $F := W_F^4 \cap h$ is a sextic surface of $h \cong \mathbb{P}^3$ double along the six edges of a tetrahedron and triple at its four vertices. So $F$ is not a (smooth) Enriques surface as required by Assumption F1 (and CM1), but its minimal desingularization is (see [16, p.275]). Furthermore in this case the Godeaux linear system $|\varphi|$ defines a double cover of $\mathbb{P}^3$ (see [23, §10]). Hence $W_F^4$ is a kind of *exception* in the analysis of Fano and Conte-Murre.

We have already said that the rational F-EF 3-folds $W_F^{p=6,7,9,13}$ are linked to the BS-EF 3-folds $W_{BS}^{p=6,7,9,13}$ with very ample hyperplane sections (see Remark 3.7). We also recall that the F-EF 3-fold $W_F^4$ is the birational image of the BS-EF 3-fold $(W_{BS}^4, \mathcal{L})$ via the rational map $\phi_{\mathcal{L}} : W_{BS}^4 \dashrightarrow \mathbb{P}^4$ (see [1, §6.33]). This suggests that one could obtain the BS-EF 3-folds with ample (but not very ample) hyperplane sections, by using a weaker form of Assumption F1 (and CM1) and by resuming Fano-Conte-Murre techniques: indeed another link between BS-EF 3-folds and F-EF 3-folds is given by the hidden presence of the BS-EF 3-fold $W_{BS}^3$ in Fano's paper (as we said in § 3.2 (00)). Re-examining the brilliant ideas of Fano with the techniques of Conte and Murre would be very interesting, even if no one has yet shown interest in the problem.

However, one must be careful of hidden mistakes in reviewing Fano's paper. For example, the BS-EF 3-folds $W_{BS}^8$ and $W_{BS}^{10}$ do not appear in the description of Fano (for some strange reason), although they behave like the other BS-EF 3-folds $W_{BS}^{p>6}$ with very ample hyperplane sections: they are projectively normal (see § 3.3) and their eight quadruple points are similar (see Remarks 6.6, 6.13). One of the reasons why they don't appear in Fano's paper could be the fact they seem to be in contradiction with Remark 4.7 (iv) (see Remark 6.6, 6.13). It is a situation that should be understood better.

Fano-Conte-Murre's techniques might also be useful to include the P-EF 3-folds. In these cases Assumption CM1 is satisfied (see § 3.3) while Assumptions CM2 and CM3 are not (see Remarks 8.5, 8.12, 8.9, 8.16). So one should eventually weaken these two assumptions. Anyhow, by weakening Assumptions CM2 and CM3 we could obtain information on Enriques-Fano threefolds with non-terminal canonical singularities: indeed as we have seen in Remark 4.3, it seems that they have an important role for the terminality of the singularities. Finally the nature of the KLM-EF 3-fold $W^9_{KLM}$ and of the F-EF 3-fold $W^7_F$ suggests that some Enriques-Fano 3-folds could be obtained via projection techniques.

# 5 Modern analysis of the rational F-EF 3-folds

## 5.1 Abstract

We recall that Fano found five Enriques-Fano threefolds (see [23]): one of genus 4, which is non-rational (see [45]) and four of genus $p = 6, 7, 9, 13$, which are rational. However, in his paper there are many hidden gaps, as Conte and Murre showed in [14] and as we will see in Remarks 6.6, 6.13. By using blow-ups techniques, we will verify that the images of the following linear systems on $\mathbb{P}^3$ actually are rational Enriques-Fano threefolds with eight quadruple points, as Fano said: the linear system $\mathcal{S}$ of the sextic surfaces double along the six edges of a tetrahedron; the linear system $\mathcal{K}$ of the septic surfaces double along the six edges of two trihedra; the linear system $\mathcal{X}$ of the sextic surfaces double along the six edges of a tetrahedron and containing a plane cubic curve intersecting each edge at one point; the linear system $\mathcal{P}$ of the septic surfaces double along three twisted cubics having five points in common. We will start with the classical case (see § 5.2), in order to have a model to refer to, and then we will continue with the lesser known ones (see § 5.3, 5.4, 5.5). Furthermore we will find that the singular points of the F-EF 3-folds $W_F^{p=6,7,9,13}$ are associated in the way imposed by Fano (see respectively Figures 21, 22, 24, 26 of Appendix A). For some results we will also use Macaulay2.

## 5.2 F-EF 3-fold of genus 13

### 5.2.1 Construction of $W^{13}_F$

Let us take a tetrahedron $T \subset \mathbb{P}^3$ with vertices $v_0$, $v_1$, $v_2$, $v_3$ as in Figure 2. Let $f_i$ be the face of $T$ opposite to the vertex $v_i$ and let us denote the edges of $T$ by $l_{ij} := f_i \cap f_j$, for $0 \leq i < j \leq 3$. Let $\mathcal{S}$ be the linear system of the sextic surfaces of $\mathbb{P}^3$ double along the six edges of $T$. Up to a change of coordinates, we can consider in $\mathbb{P}^3_{[s_0:s_1:s_2:s_3]}$ the tetrahedron $T = \{s_0s_1s_2s_3 = 0\}$ with faces $f_i = \{s_i = 0\}$, for $0 \leq i \leq 3$. The linear system $\mathcal{S}$ is defined by the zero locus of the following homogeneous polynomial

$$\lambda_0 s_1^2 s_2^2 s_3^2 + \lambda_1 s_0^2 s_2^2 s_3^2 + \lambda_2 s_0^2 s_1^2 s_3^2 + \lambda_3 s_0^2 s_1^2 s_2^2 + s_0 s_1 s_2 s_3 Q(s_0, s_1, s_2, s_3),$$

where $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ and $Q(s_0, s_1, s_2, s_3) = \sum_{i \leq j} q_{ij} s_i s_j$ is a quadratic form (see [27, p.635]). Since $\dim H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = \binom{3+2}{2}$, then $\dim \mathcal{S} = 13$.

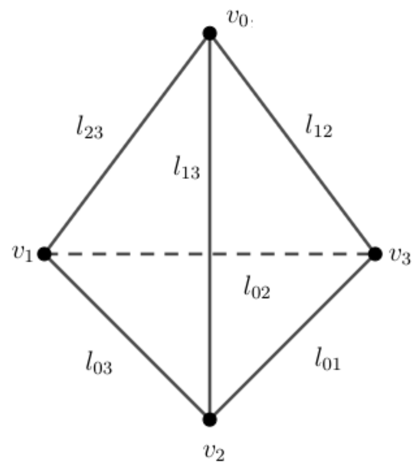


*Figure 2:* *Tetrahedron* $T \subset \mathbb{P}^3$.

**Remark 5.1.** Let $\Sigma$ be a general element of $\mathcal{S}$. By looking locally at the equation of $\mathcal{S}$, then we obtain the following two assertions, for distinct indices $i, j, k, h \in \{0, 1, 2, 3\}$:

(i) $\Sigma$ has triple points at the vertices of $T$ and $TC_{v_i}\Sigma = f_j \cup f_k \cup f_h$;

(ii) if $p \in l_{ij}$ with $p \neq v_k$ and $p \neq v_h$, then $TC_p\Sigma$ is the union of two variable planes containing $l_{ij}$, depending on the choice of the point $p$ and of the surface $\Sigma$, and coinciding for finitely many points $p$.

**Lemma 5.2.** The rational map $\nu_{\mathcal{S}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{13}$ defined by $\mathcal{S}$ is birational onto the image.

*Proof.* It is sufficient to verify that the map defined by $\mathcal{S}$ on a general $\Sigma \in \mathcal{S}$ is birational onto the image, and this actually happens because $\mathcal{S}|_\Sigma$ contains a sublinear system that defines a birational map. Indeed $\mathcal{S}$ contains a sublinear system $\overline{\mathcal{S}} \subset \mathcal{S}$ whose fixed part is given by the tetrahedron $T$ and such that $\overline{\mathcal{S}}|_\Sigma$ coincides with the linear system on $\Sigma$ cut out by the quadric surfaces of $\mathbb{P}^3$. □

**Remark 5.3.** The proof of Lemma 5.2 tells us that the linear system $\mathcal{S}$ is very ample outside the tetrahedron $T$. So $\nu_{\mathcal{S}} : \mathbb{P}^3 \dashrightarrow \nu_{\mathcal{S}}(\mathbb{P}^3) \subset \mathbb{P}^{13}$ is an isomorphism outside $T$.

**Theorem 5.4.** [23, §8] Let $W_F^{13}$ be the image of the map $\nu_{\mathcal{S}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{13}$. Then $W_F^{13}$ is an Enriques-Fano threefold of genus $p = 13$.

*Proof.* The idea of the proof is to blow-up $\mathbb{P}^3$ along the base locus of $\mathcal{S}$, until we obtain a smooth rational threefold $Y$ and a base point free linear system $\widetilde{\mathcal{S}}$ on $Y$. By Lemma 5.2, the new linear system $\widetilde{\mathcal{S}}$ will define a birational morphism $\nu_{\widetilde{\mathcal{S}}} : Y \to W_F^{13} \subset \mathbb{P}^{13}$. To obtain that $W_F^{13}$ is an Enriques-Fano threefold, it will be sufficient to verify that the general hyperplane section $S$ is an Enriques surface and that $W_F^{13}$ is not a cone on $S$ (see § 3.3). Furthermore, to obtain the genus $p = 13$ of $W_F^{13}$ we will compute the degree of the threefold, which is $24 = \widetilde{\Sigma}^3 = \deg W_F^{13} = 2p - 2$ for $\widetilde{\Sigma} \in \widetilde{\mathcal{S}}$. The proof is divided into several steps, given by the Remarks 5.5,..., 5.14 and the Theorem 5.15 below.

We blow-up first $\mathbb{P}^3$ at the vertices of $T$, obtaining a smooth threefold $Y'$ and a birational morphism $bl' : Y' \to \mathbb{P}^3$ with exceptional divisors $E_i := (bl')^{-1}(v_i)$, for $0 \le i \le 3$. Let $\mathcal{S}'$ be the strict transform of $\mathcal{S}$ and let us denote by $H$ the pullback on $Y'$ of the hyperplane class on $\mathbb{P}^3$. Then an element of $\mathcal{S}'$ is linearly equivalent to $6H - 3\sum_{i=0}^3 E_i$. Let $\widetilde{f}_i$ be the strict transform of the face $f_i$, for $0 \le i \le 3$. We denote by $\gamma_{ij} := E_i \cap \widetilde{f}_j$ the line cut out by $\widetilde{f}_j$ on $E_i$, for $0 \le i < j \le 3$. We have that $\gamma_{ij}$ is a $(-1)$-curve on $\widetilde{f}_j$. If $\Sigma'$ is the strict transform of a general $\Sigma \in \mathcal{S}$, then $\Sigma' \cap E_i = \bigcup_{\substack{j=0\\ j\neq i}}^3 \gamma_{ij}$, for all $0 \le i \le 3$, and $\Sigma'$ is smooth at a general point of $\gamma_{ij}$ (see Remark 5.1). The base locus of $\mathcal{S}'$ is now given by the union of the strict transforms $\widetilde{l}_{ij}$ of the six edges of $T$ (along which a general $\Sigma' \in \mathcal{S}'$ has double points) and the 12 lines $\gamma_{ij}$ (see Remark 5.1). Let us blow-up the strict transforms of the edges of $T$: we obtain a smooth threefold $Y''$ and a birational morphism $bl'' : Y'' \to Y'$ with exceptional divisors

$$(bl'')^{-1}(\widetilde{l}_{ij}) =: F_{ij} \cong \mathbb{P}(\mathcal{N}_{\widetilde{l}_{ij}|Y'}) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{F}_0,$$

for $0 \le i < j \le 3$. This blow-up has no effect on $\widetilde{f}_i$, for $0 \le i \le 3$, so, by abuse of notation, we will use the same symbol to indicate its strict transform on $Y''$.

**Remark 5.5.** Let $\widetilde{E}_i$ be the strict transform of $E_i$ and let us consider the curve $\alpha_{kij} := \widetilde{E}_k \cap F_{ij}$, where $i, j, k$ are distinct indices in $\{0, 1, 2, 3\}$ and $i < j$ (see Figure 3). Since $\alpha_{kij}$ is a $(-1)$-curve on $\widetilde{E}_k$ and it is a fibre on $F_{ij}$, then we have that $F_{ij}^2 \cdot \widetilde{E}_k = \alpha_{kij}^2|_{\widetilde{E}_k} = -1$ and $\widetilde{E}_k^2 \cdot F_{ij} = \alpha_{kij}^2|_{F_{ij}} = 0$.

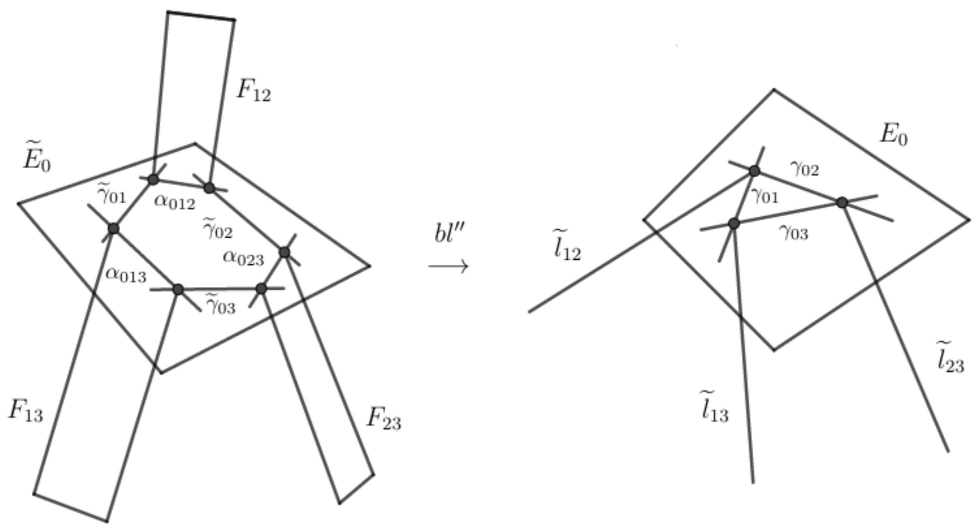


*Figure 3: Description of $bl''|_{\widetilde{E}_0} : \widetilde{E}_0 \to E_0$. The same happens on $\widetilde{E}_k$, for $1 \le k \le 3$.*

Let $\mathcal{S}''$ be the strict transform of $\mathcal{S}'$: an element of $\mathcal{S}''$ is linearly equivalent to $6H - 3\sum_{i=0}^3 \widetilde{E}_i - 2\sum_{0\le i<j\le 3} F_{ij}$, where $H$ denotes the pullback $bl''^*H$, by abuse of notation. The base locus of $\mathcal{S}''$ is given by the disjoint union of the strict transforms $\widetilde{\gamma}_{ij}$ of the 12 lines $\gamma_{ij}$, for $i, j \in \{0, 1, 2, 3\}$ and $i \neq j$ (see Remark 5.1).

**Remark 5.6.** Let $l_k$ be the linear equivalence class of the lines of $E_k \cong \mathbb{P}^2$: then $E_k|_{E_k} \sim -l_k$ (see [27, Chap 4, §6] and [32, Lemma 2.2.14]). Let $L_k$ be the strict transform of $l_k$ via $bl''|_{\widetilde{E}_k} : \widetilde{E}_k \to E_k$. Since $bl''^*(E_k) = \widetilde{E}_k$, then $\widetilde{E}_k|_{\widetilde{E}_k} \sim -L_k$ and $\widetilde{E}_k^3 = 1$.

**Remark 5.7.** By construction we have that $\widetilde{\gamma}_{ij}^2|_{\widetilde{E}_i} = -1$ and $\widetilde{\gamma}_{ij}^2|_{\widetilde{f}_j} = -1$, for $i, j \in \{0,1,2,3\}$ and $i \neq j$. We also have that $\widetilde{\gamma}_{ij}|_{\Sigma''} = -1$, where $\Sigma''$ is the strict transform on $Y''$ of a general element $\Sigma \in \mathcal{S}$. Indeed, since these twelve curves are disjoint, then $(\Sigma'' \cap \widetilde{E}_i)^2|_{\Sigma''} = \sum_{\substack{j=0 \\ j\neq i}}^3 \widetilde{\gamma}_{ij}^2|_{\Sigma''}$, for all $0 \leq i \leq 3$. On the other hand we have that $(\Sigma'' \cap \widetilde{E}_i)^2|_{\Sigma''} = \widetilde{E}_i^2 \cdot \Sigma'' = -3$ (see Remarks 5.5, 5.6). Thus $(\widetilde{\gamma}_{ij})^2|_{\Sigma''} = -1$, since the curves $\widetilde{\gamma}_{ij}$ behave in the same way.

Finally let us consider $bl''' : Y \to Y''$ the blow-up of $Y''$ along the twelve curves $\widetilde{\gamma}_{ij}$, for $i, j \in \{0,1,2,3\}$ and $i \neq j$, with exceptional divisors $\Gamma_{ij} := bl'''^{-1}(\widetilde{\gamma}_{ij})$. We denote by $\mathcal{E}_i$ the strict transform of $\widetilde{E}_i$, by $\mathcal{F}_{ij}$ the strict transform of $F_{ij}$ and by $\mathcal{H}$ the pullback of $H$, for $0 \leq i < j \leq 3$.

**Remark 5.8.** We have that

$$\Gamma_{ij} = \mathbb{P}(\mathcal{N}_{\widetilde{\gamma}_{ij}|Y''}) \cong \mathbb{P}(\mathcal{O}_{\widetilde{\gamma}_{ij}}(E_i) \oplus \mathcal{O}_{\widetilde{\gamma}_{ij}}(\widetilde{f}_j)) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{F}_0$$

and that $\Gamma_{ij}^3 = -\deg(\mathcal{N}_{\widetilde{\gamma}_{ij}|Y''}) = 2$ (see [27, Chap 4, §6] and [32, Lemma 2.2.14]).

**Remark 5.9.** Let us take three distinct indices $i, j, k \in \{0,1,2,3\}$: if $j < k$, then $\Gamma_{ij}$ intersects $\mathcal{F}_{jk}$ along a $\mathbb{P}^1$, which is a fibre on $\Gamma_{ij}$ and a $(-1)$-curve on $\mathcal{F}_{jk}$. Similarly $\mathcal{F}_{kj}^2 \cdot \Gamma_{ij} = 0$ and $\Gamma_{ij}^2 \cdot \mathcal{F}_{kj} = -1$ if $k < j$. We also observe that $\Gamma_{ij}$ intersects $\mathcal{E}_i$ along a $\mathbb{P}^1$ belonging to the other ruling of $\Gamma_i$, so we have $\mathcal{E}_i^2 \cdot \Gamma_{ij} = 0$. Furthermore we still have $\Gamma_{ij}^2 \cdot \mathcal{E}_i = -1$, since $bl''' : Y \to Y''$ has no effect on $\widetilde{E}_i$. For this reason we will denote $\Gamma_{ij} \cap \mathcal{E}_i$ by $\widetilde{\gamma}_{ij}$, by abuse of notation. Let us suppose now $i < j$ and let us consider the strict transforms $\widetilde{\alpha}_{kij}$ of the curves $\alpha_{kij}$ defined in Remark 5.5. Then we have that $\mathcal{F}_{ij}^2 \cdot \mathcal{E}_k = \widetilde{\alpha}_{kij}^2|_{\mathcal{E}_k} = -1$ and $\mathcal{E}_k^2 \cdot \mathcal{F}_{ij} = \widetilde{\alpha}_{kij}^2|_{\mathcal{F}_{ij}} = -2$. Finally we recall that a general line of $\mathbb{P}^3$ does not intersect the edges of $T$ and that a general plane of $\mathbb{P}^3$ intersects each one of them at one point. Hence we have that $\mathcal{H}^2 \cdot \mathcal{F}_{ij} = 0$ and $\mathcal{F}_{ij}^2 \cdot \mathcal{H} = -1$.

**Remark 5.10.** By construction we have $bl'''^*(\widetilde{E}_k) = \mathcal{E}_k + \sum_{\substack{i=0 \\ i\neq k}}^3 \Gamma_{ki}$, for $0 \leq k \leq 3$. If $\mathcal{L}_k$ is the strict transform of $L_k$ via $bl'''|_{\mathcal{E}_k} : \mathcal{E}_k \to \widetilde{E}_k$, then we have that $-\mathcal{E}_k|_{\mathcal{E}_k} \sim \mathcal{L}_k + \sum_{\substack{i=0 \\ i\neq k}}^3 \widetilde{\gamma}_{ki} \sim 4\mathcal{L}_k - 2\sum_{\substack{0\leq i<j\leq 3 \\ i,j\neq k}} \widetilde{\alpha}_{kij}$ and $\mathcal{E}_k^3 = 4$ (see Remark 5.6).

**Remark 5.11.** Let us fix four distinct indices $i, j, k, h \in \{0,1,2,3\}$ with $i < j$. By [32, Lemma 2.2.14] we have that $F_{ij}^3 = -\deg(\mathcal{N}_{\widetilde{l}_{ij}|Y'}) = 2$ (see also [27, Chap 4, §6]). Since $bl'''^*(F_{ij}) = \mathcal{F}_{ij}$, then we still have $\mathcal{F}_{ij}^3 = 2$.

Let $\widetilde{\Sigma}$ be the strict transform on $Y$ of an element of $\mathcal{S}''$: then

$$\widetilde{\Sigma} \sim 6\mathcal{H} - \sum_{i=0}^{3} 3\mathcal{E}_k - \sum_{0\leq i<j\leq 3} 2\mathcal{F}_{ij} - \sum_{\substack{i,j=0 \\ i\neq j}}^{3} 4\Gamma_{ij}.$$

Let us take the linear system $\widetilde{\mathcal{S}} := |\mathcal{O}_Y(\widetilde{\Sigma})|$ on $Y$. It is base point free and it defines a morphism $\nu_{\widetilde{\mathcal{S}}} : Y \to \mathbb{P}^{13}$ birational onto the image $W_F^{13} := \nu_{\widetilde{\mathcal{S}}}(Y)$, which is a threefold of degree $\deg W_F^{13} = 24$. This follows by Lemma 5.2 and by the fact that $\widetilde{\Sigma}^3 = 24$: indeed by Remarks 5.8, 5.9, 5.10 5.11 we have

$$\widetilde{\Sigma}^3 = 216\mathcal{H}^3 - 27\sum_{i=0}^{3}\mathcal{E}_k^3 - 8\sum_{0\le i<j\le 3}\mathcal{F}_{ij}^3 - 64\sum_{\substack{i,j=0\\ i\neq j}}^{3}\Gamma_{ij}^3 - 3(36\mathcal{H}^2)\cdot\Big(2\sum_{0\le i<j\le 3}\mathcal{F}_{ij}\Big)+$$

$$-3\Big(9\sum_{i=0}^{3}\mathcal{E}_k^2\Big)\cdot\Big(2\sum_{0\le i<j\le 3}\mathcal{F}_{ij}\Big) - 3\Big(9\sum_{i=0}^{3}\mathcal{E}_k^2\Big)\cdot\Big(4\sum_{\substack{i,j=0\\ i\neq j}}^{3}\Gamma_{ij}\Big) + 3\Big(4\sum_{0\le i<j\le 3}\mathcal{F}_{ij}^2\Big)\cdot(6\mathcal{H})+$$

$$-3\Big(4\sum_{0\le i<j\le 3}\mathcal{F}_{ij}^2\Big)\cdot\Big(3\sum_{i=0}^{3}\mathcal{E}_k\Big)-3\Big(4\sum_{0\le i<j\le 3}\mathcal{F}_{ij}^2\Big)\cdot\Big(4\sum_{\substack{i,j=0\\ i\neq j}}^{3}\Gamma_{ij}\Big)-3\Big(16\sum_{\substack{i,j=0\\ i\neq j}}^{3}\Gamma_{ij}^2\Big)\cdot\Big(3\sum_{i=0}^{3}\mathcal{E}_k\Big)+$$

$$-3\Big(16\sum_{\substack{i,j=0\\ i\neq j}}^{3}\Gamma_{ij}^2\Big)\cdot\Big(2\sum_{0\le i<j\le 3}\mathcal{F}_{ij}\Big) - 6\Big(3\sum_{i=0}^{3}\mathcal{E}_k\Big)\cdot\Big(2\sum_{0\le i<j\le 3}\mathcal{F}_{ij}\Big)\cdot\Big(4\sum_{\substack{i,j=0\\ i\neq j}}^{3}\Gamma_{ij}\Big) =$$

$$= 216 - 27\cdot 4\cdot 4 - 8\cdot 6\cdot 2 - 64\cdot 12\cdot 2 + 0 - 3\cdot 9\cdot 2\cdot 4\cdot 3\cdot(-2) + 0 + 3\cdot 4\cdot 6\cdot 6\cdot(-1)+$$

$$-3\cdot 4\cdot 3\cdot 6\cdot 2\cdot(-1) + 0 - 3\cdot 16\cdot 3\cdot 12\cdot(-1) - 3\cdot 16\cdot 2\cdot 12\cdot 2\cdot(-1) - 6\cdot 3\cdot 2\cdot 4\cdot 4\cdot 3\cdot 2 =$$

$$= 216 - 432 - 96 - 1536 + 0 + 1296 - 432 + 432 + 0 + 1728 + 2304 - 3456 = 24.$$

Then we have the following diagram:

$$\begin{array}{ccccccc}
Y & & & & & \overset{\nu_{\widetilde{\mathcal{S}}}}{\searrow} & \\
\downarrow{\scriptstyle bl'''} & & & & & & \\
Y'' & \xrightarrow{bl''} & Y' & \xrightarrow{bl'} & \mathbb{P}^3 & \overset{\nu_{\mathcal{S}}}{\dashrightarrow} & W_F^{13} \subset \mathbb{P}^{13}.
\end{array}$$

**Remark 5.12.** Since $bl''' : Y \to Y''$ has no effect on the divisor $\widetilde{f}_i$, for $0 \le i \le 3$, we continue to use the same notation to denote its strict transform. The eight divisors $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\widetilde{f}_0$, $\widetilde{f}_1$, $\widetilde{f}_2$, $\widetilde{f}_3$ are contracted by $\nu_{\widetilde{\mathcal{S}}} : Y \to W_F^{13} \subset \mathbb{P}^{13}$ to points of $W_F^{13}$. Indeed, if $\widetilde{\Sigma}$ is a general element of $\widetilde{\mathcal{S}}$, then by construction we have $\widetilde{\Sigma}\cdot\mathcal{E}_i = 0 = \widetilde{\Sigma}\cdot\widetilde{f}_i$ for all $0 \le i \le 3$.

**Remark 5.13.** The morphism $\nu_{\widetilde{\mathcal{S}}} : Y \to W_F^{13} \subset \mathbb{P}^{13}$ blows-down the twelve exceptional divisors $\Gamma_{ij}$ to twelve curves of $W_F^{13}$. This follows by the fact that $\widetilde{\Sigma}\cdot\Gamma_{ij} \neq 0$ and $\widetilde{\Sigma}^2\cdot\Gamma_{ij} = 0$ for a general element $\widetilde{\Sigma} \in \widetilde{\mathcal{S}}$ and for all $i, j \in \{0,1,2,3\}$ with $i \neq j$. Indeed by Remarks 5.8, 5.9 we obtain

$$\widetilde{\Sigma}^2\cdot\Gamma_{ij} = \widetilde{\Sigma}\cdot\Bigg( - 3\mathcal{E}_i\cdot\Gamma_{ij} - \sum_{\substack{0\le x<y\le 3\\ i\notin\{x,y\},\, j\in\{x,y\}}} 2(\mathcal{F}_{xy}\cdot\Gamma_{ij}) - 4\Gamma_{ij}^2\Bigg) =$$

$$= 9\mathcal{E}_i^2 \cdot \Gamma_{ij} + \sum_{\substack{0\le x<y\le 3\\ i\notin\{x,y\},\, j\in\{x,y\}}} 6(\mathcal{E}_i \cdot \mathcal{F}_{xy} \cdot \Gamma_{ij}) + 12\mathcal{E}_i \cdot \Gamma_{ij}^2 + \sum_{\substack{0\le x<y\le 3\\ i\notin\{x,y\},\, j\in\{x,y\}}} 6(\mathcal{E}_i \cdot \mathcal{F}_{xy} \cdot \Gamma_{ij}) +$$

$$+ \sum_{\substack{0\le x<y\le 3\\ i\notin\{x,y\},\, j\in\{x,y\}}} 4(\mathcal{F}_{xy}^2 \cdot \Gamma_{ij}) + \sum_{\substack{0\le x<y\le 3\\ i\notin\{x,y\},\, j\in\{x,y\}}} 8(\mathcal{F}_{xy} \cdot \Gamma_{ij}^2) + 12\mathcal{E}_i \cdot \Gamma_{ij}^2 + \sum_{\substack{0\le x<y\le 3\\ i\notin\{x,y\},\, j\in\{x,y\}}} 8(\mathcal{F}_{xy} \cdot \Gamma_{ij}^2) + 16\Gamma_{ij}^3 =$$

$$= 0 + 6\cdot 2\cdot 1 + 12\cdot(-1) + 6\cdot 2\cdot 1 + 0 + 8\cdot 2\cdot(-1) + 12\cdot(-1) + 8\cdot 2\cdot(-1) + 16\cdot 2 = 0.$$

**Remark 5.14.** Let $i, j, k, h$ be four distinct indices in $\{0, 1, 2, 3\}$ such that $i < j$ and let $\widetilde{\Sigma}$ be a general element of $\widetilde{\mathcal{S}}$. By Remarks 5.9, 5.11 we obtain

$$\widetilde{\Sigma}^2 \cdot \mathcal{F}_{ij} = \widetilde{\Sigma} \cdot (6\mathcal{H} \cdot \mathcal{F}_{ij} - 3\mathcal{E}_k \cdot \mathcal{F}_{ij} - 3\mathcal{E}_h \cdot \mathcal{F}_{ij} - 2\mathcal{F}_{ij}^2 - 4\Gamma_{ki} \cdot \mathcal{F}_{ij} - 4\Gamma_{kj} \cdot \mathcal{F}_{ij} - 4\Gamma_{hi} \cdot \mathcal{F}_{ij} - 4\Gamma_{hj} \cdot \mathcal{F}_{ij}) =$$

$$= 36\mathcal{H}^2 \cdot \mathcal{F}_{ij} - 12\mathcal{H} \cdot \mathcal{F}_{ij}^2 + 9\mathcal{E}_k^2 \cdot \mathcal{F}_{ij} + 6\mathcal{E}_k \cdot \mathcal{F}_{ij}^2 + 12\mathcal{E}_k \cdot \Gamma_{ki} \cdot \mathcal{F}_{ij} + 12\mathcal{E}_k \cdot \Gamma_{kj} \cdot \mathcal{F}_{ij} +$$

$$+ 9\mathcal{E}_h^2 \cdot \mathcal{F}_{ij} + 6\mathcal{E}_h \cdot \mathcal{F}_{ij}^2 + 12\mathcal{E}_h \cdot \Gamma_{hi} \cdot \mathcal{F}_{ij} + 12\mathcal{E}_h \cdot \Gamma_{hj} \cdot \mathcal{F}_{ij} - 12\mathcal{H} \cdot \mathcal{F}_{ij}^2 + 6\mathcal{E}_k \cdot \mathcal{F}_{ij}^2 + 6\mathcal{E}_h \cdot \mathcal{F}_{ij}^2 + 4\mathcal{F}_{ij}^3 +$$

$$+ 8\Gamma_{ki} \cdot \mathcal{F}_{ij}^2 + 8\Gamma_{kj} \cdot \mathcal{F}_{ij}^2 + 8\Gamma_{hi} \cdot \mathcal{F}_{ij}^2 + 8\Gamma_{hj} \cdot \mathcal{F}_{ij}^2 + 12\Gamma_{ki} \cdot \mathcal{E}_k \cdot \mathcal{F}_{ij} + 12\Gamma_{kj} \cdot \mathcal{E}_k \cdot \mathcal{F}_{ij} +$$

$$+ 12\Gamma_{hi} \cdot \mathcal{E}_h \cdot \mathcal{F}_{ij} + 12\Gamma_{hj} \cdot \mathcal{E}_h \cdot \mathcal{F}_{ij} + 8\Gamma_{ki} \cdot \mathcal{F}_{ij}^2 + 8\Gamma_{kj} \cdot \mathcal{F}_{ij}^2 + 8\Gamma_{hi} \cdot \mathcal{F}_{ij}^2 + 8\Gamma_{hj} \cdot \mathcal{F}_{ij}^2 +$$

$$+ 16\Gamma_{ki}^2 \cdot \mathcal{F}_{ij} + 16\Gamma_{kj}^2 \cdot \mathcal{F}_{ij} + 16\Gamma_{hi}^2 \cdot \mathcal{F}_{ij} + 16\Gamma_{hj}^2 \cdot \mathcal{F}_{ij} = 0 + 12 - 18 - 6 + 12 + 12 - 18 - 6 + 12 + 12 +$$

$$+ 12 - 6 - 6 + 8 + 0 + 0 + 0 + 0 + 12 + 12 + 12 + 12 + 0 + 0 + 0 + 0 - 16 - 16 - 16 - 16 = 4 > 0.$$

Thus the curve $\widetilde{\Sigma} \cap \mathcal{F}_{ij}$ is not contracted by the rational map defined by $\widetilde{\mathcal{S}}|_{\widetilde{\Sigma}}$.

**Theorem 5.15.** Let $S$ be a general hyperplane section of the threefold $W_F^{13} \subset \mathbb{P}^{13}$. Then $S$ is an Enriques surface and $W_F^{13}$ is not a cone over $S$.

*Proof.* A general hyperplane section $S$ of $W_F^{13}$ is the image of a general element $\widetilde{\Sigma} \in \widetilde{\mathcal{S}}$ via the morphism $\nu_{\widetilde{\mathcal{S}}} : Y \to W_F^{13} \subset \mathbb{P}^{13}$. Let us take $\Sigma'' := bl'''(\widetilde{\Sigma}) \in \mathcal{S}''$. Since $bl''' : Y \to Y''$ has no effect on $\Sigma''$, then $\widetilde{\Sigma} \cap \Gamma_{ij}$ is still a $(-1)$-curve on $\widetilde{\Sigma}$, for all $i, j \in \{0, 1, 2, 3\}$ and $i \neq j$ (see Remark 5.7). Since $\nu_{\widetilde{\mathcal{S}}}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to S$ is the blow-down of these twelve $(-1)$-curves (see Remarks 5.3, 5.12, 5.13, 5.14), then $S$ is the minimal desingularization of the corresponding $\Sigma := bl'(bl''(bl'''(\widetilde{\Sigma}))) \in \mathcal{S}$ (see [27, p.621]). It is known that the minimal desingularization of a sextic surface $\Sigma \in \mathcal{S}$ is an Enriques surface (see [16, p.275]). It remains to show that $W_F^{13}$ is not a cone over $S$. Since $Y$ is rational by construction, then $W_F^{13}$ is rational too. If $W_F^{13}$ were a cone, then it would be birational to $S \times \mathbb{P}^1$, for a general hyperplane section $S$ of $W_F^{13}$. Thus, $S$ would be unirational, which is a contradiction because $S$ is an Enriques surface. □

By Theorem 5.15 we have that $W_F^{13} \subset \mathbb{P}^{13}$ satisfies the Assumption (*) of § 3.3. Let $p$ be the genus of a curve section of $W_F^{13}$: by the adjunction formula we have that $24 = 2p - 2$. Then $W_F^{13}$ is an Enriques-Fano threefold of genus $p = 13$, since $W_F^{13} \subset \mathbb{P}^{13}$ is (projectively) normal by Theorem 3.8 and Proposition 3.11.

□

### 5.2.2 Singularities of $W_F^{13}$

**Proposition 5.16.** The points $P_{i+1} := \nu_{\widetilde{\mathcal{S}}}(\mathcal{E}_i)$ and $P'_{i+1} := \nu_{\widetilde{\mathcal{S}}}(\widetilde{f}_i)$, $0 \leq i \leq 3$, are quadruple points of $W_F^{13}$ whose tangent cone is a cone over a Veronese surface.

*Proof.* First we recall that $\nu_{\widetilde{\mathcal{S}}}(\mathcal{E}_i)$ and $\nu_{\widetilde{\mathcal{S}}}(\widetilde{f}_i)$ actually are points of $W_F^{13}$, for $0 \leq i \leq 3$ (see Remark 5.12). Let us consider the sublinear system $(\mathcal{S} - \mathcal{E}_k) \subset \widetilde{\mathcal{S}}$ for a fixed $0 \leq k \leq 3$. It corresponds to taking the hyperplane sections of $W_F^{13} \subset \mathbb{P}^{13}$ passing through the point $P_{k+1}$. The linear system $(\mathcal{S}-\mathcal{E}_k)|_{\mathcal{E}_k}$ coincides with $|\mathcal{O}_{\mathcal{E}_k}(-\mathcal{E}_k)|$, which is isomorphic to the linear system of the quartic plane curves on $E_k$ with nodes at the three points $E_k \cap \widetilde{l}_{ij}$ for $0 \leq i < j \leq 3$ and $i, j \neq k$ (see Remark 5.10). By applying a quadratic transformation, we obtain the linear system of the conics, whose image is the Veronese surface. Let us consider now the hyperplane sections of $W_F^{13} \subset \mathbb{P}^{13}$ passing through $P'_{i+1}$, for a fixed $0 \leq i \leq 3$. It corresponds to taking the sublinear system $\mathcal{S}_i$ of the sextic surfaces of $\mathcal{S}$ containing the face $f_i$. The movable part of $\mathcal{S}_i$ is given by the quintic surfaces $Q_i$ of $\mathbb{P}^3$ containing the three edges of $T$ contained in $f_i$ and with double points along the other three edges of $T$. Such a surface $Q_i$ cuts on $f_i$ a quintic curve given by the three edges of $T$ contained in $f_i$ and a variable conic. Let us denote by $\widetilde{\mathcal{S}}_i$ the strict transform on $Y$ of $\mathcal{S}_i$ and let $\widetilde{Q}_i$ be the strict transform on $Y$ of $Q_i$. Then $\widetilde{\mathcal{S}}_i|_{\widetilde{f}_i} \cong |\mathcal{O}_{\widetilde{f}_i}(\widetilde{Q}_i)| \cong |\mathcal{O}_{\mathbb{P}^2}(2)|$, whose image is the Veronese surface. □

Since $\nu_{\mathcal{S}} : \mathbb{P}^3 \dashrightarrow W_F^{13} \subset \mathbb{P}^{13}$ is an isomorphism outside $T$ (see Remark 5.3), then $P_1$, $P_2$, $P_3$, $P_4$, $P'_1$, $P'_2$, $P'_3$ and $P'_4$ are the only singular points of $W_F^{13}$ (see Remarks 5.12, 5.13, 5.14). By recalling Definition 4.4 we have the following result.

**Theorem 5.17.** Each singular point of $W_F^{13}$ is associated with *at least* $m = 3$ of the other singular points.

*Proof.* We know that the twelve exceptional divisors of $bl''' : Y \to Y''$ are mapped by $\nu_{\widetilde{\mathcal{S}}} : Y \to W_F^{13} \subset \mathbb{P}^{13}$ to curves of $W_F^{13}$ (see Remark 5.13). In particular they are mapped to twelve lines joining the points $P_1$, $P_2$, $P_3$, $P_4$, $P'_1$, $P'_2$, $P'_3$, $P'_4$ as in Figure 26 of Appendix A. Let us show it, by fixing two indices $i, j \in \{0, 1, 2, 3\}$ with $j \neq i$. Let $\widetilde{\Sigma}$ be a general element of $\widetilde{\mathcal{S}}$: by construction we have that $\widetilde{\Sigma} \cap \Gamma_{ij}$ belongs to one of the two rulings of $\Gamma_{ij} \cong \mathbb{F}_0$. Then $\widetilde{\mathcal{S}}|_{\Gamma_{ij}} \cong \mathbb{P}^1$ and so $\nu_{\widetilde{\mathcal{S}}}(\Gamma_{ij}) \subset W_F^{13}$ is a line. In particular $\nu_{\widetilde{\mathcal{S}}}(\Gamma_{ij})$ joins the points $P_{i+1} = \nu_{\widetilde{\mathcal{S}}}(\mathcal{E}_i)$ and $P'_{j+1} = \nu_{\widetilde{\mathcal{S}}}(\widetilde{f}_j)$, since $\Gamma_{ij} \cap \mathcal{E}_i \neq \emptyset$ and $\Gamma_{ij} \cap \widetilde{f}_j \neq \emptyset$. □

**Remark 5.18.** Thanks to a computational anaylisis with Macaulay2, we see that each singular point of $W_F^{13}$ is associated with *exactly* $m = 3$ of the other singular points, as in Figure 26 of Appendix A. This follows by Remark 6.16, since the embedding of the BS-EF 3-fold $W_{BS}^{13}$ in $\mathbb{P}^{13}$ is the F-EF 3-fold $W_F^{13}$ (see Theorem 6.17).

## 5.3 F-EF 3-fold of genus 9

### 5.3.1 Construction of $W_F^9$

We take two trihedra $T$ and $T'$ in $\mathbb{P}^3$ as in Figure 4: the trihedron $T$ with vertex $v$, faces $f_i$ and edges $l_{ij} := f_i \cap f_j$ and the trihedron $T'$ with vertex $v'$, faces $f'_i$ and edges

$l'_{ij} := f'_i \cap f'_j$, for $1 \leq i < j \leq 3$. Let us consider the linear system $\mathcal{K}$ of the septic surfaces of $\mathbb{P}^3$ double along the six edges of the two trihedra $T$ and $T'$.

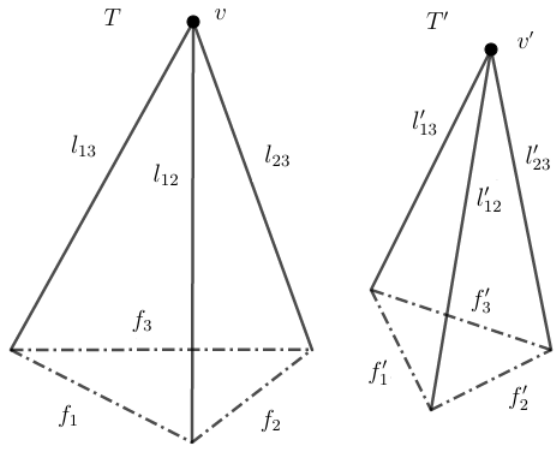


*Figure 4:* Trihedra $T$ and $T'$ in $\mathbb{P}^3$.

**Remark 5.19.** A septic surface $K \in \mathcal{K}$ contains the nine lines $r_{ij} := f_i \cap f'_j$, for $i, j \in \{1, 2, 3\}$. Assume the contrary: then, by Bezout's Theorem, $K \cap r_{ij}$ is given by 7 points. Furthermore, each line $r_{ij}$ intersects two edges of $T$ contained in $f_i$ and two edges of $T'$ contained in $f'_j$. Hence $r_{ij}$ is a line through four double points of $K$. We obtain that $K \cap r_{ij}$ contains at least 8 points, counted with multiplicity, which is a contradiction. Thus it must be $r_{ij} \subset K$.

**Proposition 5.20.** The linear system $\mathcal{K}$ is defined by the zero locus of the following homogeneous polynomial of degree seven

$$F(s_0, s_1, s_2, s_3) = f_1 f_2 f_3 f'_1 f'_2 f'_3 (\lambda_0 s_0 + \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3) +$$

$$+ f'_1 f'_2 f'_3 (\lambda_4 f_3^2 f_2^2 + \lambda_5 f_1^2 f_3^2 + \lambda_6 f_1^2 f_2^2) + f_1 f_2 f_3 (\lambda_7 {f'_3}^2 {f'_2}^2 + \lambda_8 {f'_1}^2 {f'_3}^2 + \lambda_9 {f'_1}^2 {f'_2}^2),$$

where $\lambda_0, \ldots, \lambda_9 \in \mathbb{C}$ and where $f_i$ and $f'_i$ denote, by abuse of notation, the linear homogeneous polynomials defining, respectively, the faces $f_i$ and $f'_i$, for $1 \leq i \leq 3$. The linear system $\mathcal{K}$ therefore has $\dim \mathcal{K} = 9$.

*Proof.* Let $F \in \mathbb{C}[s_0 : s_1 : s_2 : s_3]$ be the homogeneous polynomial of degree 7 defining a general element $K$ of $\mathcal{K}$. We recall that the intersection of an irreducible septic surface of $\mathbb{P}^3$ with a plane is a septic curve: in particular, $K$ intersects each face $f_i$ of $T$ along the septic curve given by the two double edges contained in that face plus the three lines $r_{ij}$, for $1 \leq j \leq 3$. The same happens with the faces of $T'$. This implies that it must be $K \cap f_i = \{f'_1 f'_2 f'_3 f_k^2 f_h^2 = 0,\ f_i = 0\} = 2l_{ik} + 2l_{ih} + \sum_{j=1}^3 r_{ij}$ and $K \cap f'_i = \{f_1 f_2 f_3 {f'_k}^2 {f'_h}^2 = 0,\ f'_i = 0\} = 2l'_{ik} + 2l'_{ih} + \sum_{j=1}^3 r_{ji}$, for distinct indices $i, k, h \in \{1, 2, 3\}$. Then it must be

$$F(s_0, s_1, s_2, s_3) = f_1 g_6(s_0, s_1, s_2, s_3) + \lambda_4 f'_1 f'_2 f'_3 f_3^2 f_2^2,$$

where $\lambda_4 \in \mathbb{C}$ and $g_6$ is a homogeneous polynomial of degree 6 such that

$$g_6(s_0, s_1, s_2, s_3) = f_2 g_5(s_0, s_1, s_2, s_3) + \lambda_5 f_1' f_2' f_3' f_1 f_3^2,$$

where $\lambda_5 \in \mathbb{C}$ and $g_5$ is a homogeneous polynomial of degree 5 such that

$$g_5(s_0, s_1, s_2, s_3) = f_3 g_4(s_0, s_1, s_2, s_3) + \lambda_6 f_1' f_2' f_3' f_1 f_2,$$

where $\lambda_6 \in \mathbb{C}$ and $g_4$ is a homogeneous polynomial of degree 4 such that

$$g_4(s_0, s_1, s_2, s_3) = f_1' g_3(s_0, s_1, s_2, s_3) + \lambda_7 f_2'^2 f_3'^2,$$

where $\lambda_7 \in \mathbb{C}$ and $g_3$ is a homogeneous polynomial of degree 3 such that

$$g_3(s_0, s_1, s_2, s_3) = f_2' g_3(s_0, s_1, s_2, s_3) + \lambda_8 f_1' f_3'^2,$$

where $\lambda_8 \in \mathbb{C}$ and $g_2$ is a homogeneous polynomial of degree 2 such that

$$g_2(s_0, s_1, s_2, s_3) = f_3'(\lambda_0 s_0 + \lambda_1 s_1 + \lambda_2 s_2 + \lambda_3 s_3) + \lambda_9 f_1' f_2',$$

where $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_9 \in \mathbb{C}$. So $F$ has the expression of the statement. Since $\{K \in \mathcal{K} | K \supset f_1\} = \{F = 0 | \lambda_4 = 0\}$, then $\operatorname{codim}(\{K \in \mathcal{K} | K \supset f_1\}, \mathcal{K}) = 1$. Let us see that containing the six faces $f_1$, $f_2$, $f_3$, $f_1'$, $f_2'$, $f_3'$ imposes independent conditions: there exists a septic surface in $\mathcal{K}$ containing $f_1$ but not $f_2$, that is $\{F = 0 | \lambda_4 = 0, \lambda_5 \neq 0\}$; there exists a septic surface in $\mathcal{K}$ containing $f_1$ and $f_2$ but not $f_3$, that is $\{F = 0 | \lambda_4 = \lambda_5 = 0, \lambda_6 \neq 0\}$; there exists a septic surface in $\mathcal{K}$ containing $f_1$, $f_2$ and $f_3$ but not $f_1'$, that is $\{F = 0 | \lambda_4 = \lambda_5 = \lambda_6 = 0, \lambda_7 \neq 0\}$ there exists a septic surface in $\mathcal{K}$ containing $f_1$, $f_2$, $f_3$ and $f_1'$ but not $f_2'$, that is $\{F = 0 | \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = 0, \lambda_8 \neq 0\}$ there exists a septic surface in $\mathcal{K}$ containing $f_1$, $f_2$, $f_3$, $f_1'$, and $f_2'$ but not $f_3'$, that is $\{F = 0 | \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = 0, \lambda_9 \neq 0\}$. Thus we obtain $\operatorname{codim}(\{K \in \mathcal{K} | K \supset T \cup T'\}, \mathcal{K}) = 6$. Furthermore each element of $\{K \in \mathcal{K} | K \supset T \cup T'\}$ is of the form $T \cup T' \cup \pi$, where $\pi$ is a general plane of $\mathbb{P}^3$. Thus we have $\dim\{K \in \mathcal{K} | K \supset T \cup T'\} = \dim |\mathcal{O}_{\mathbb{P}^3}(1)| = 3$ and finally $\dim \mathcal{K} = 3 + 6 = 9$. □

Let us consider the points mentioned in Remark 5.19: they are $q_{ijk} := l_{ij} \cap r_{ik} = l_{ij} \cap r_{jk}$ and $q'_{ijk} := l'_{ij} \cap r_{ki} = l'_{ij} \cap r_{kj}$ for $i, j, k \in \{1, 2, 3\}$ with $i < j$. These points also represent the intersection points between the faces of a trihedron and the edges of the other trihedron. Indeed we have that $q_{ijk} = l_{ij} \cap f'_k$ and $q'_{ijk} = l'_{ij} \cap f_k$ (see Figure 5).

**Remark 5.21.** Let $K$ be a general element of $\mathcal{K}$. By looking locally at the equation of $\mathcal{K}$ (see Proposition 5.20), then we find that:

(i) $K$ has triple points at the vertices of $T$ and $T'$ and $TC_v K = \bigcup_{i=1}^3 f_i$ and $TC_{v'} K = \bigcup_{i=1}^3 f'_i$;

(ii) $TC_{q_{ijk}} K = f_i \cup f_j$ and $TC_{q'_{ijk}} = f'_i \cup f'_j$, for $i, j, k \in \{1, 2, 3\}$ with $i < j$;

(iii) if $p \in l_{ij}$, with $p \neq v$ and $p \neq q_{ijk}$, then $TC_p K$ is the union of two variable planes containing $l_{ij}$, depending on the choice of the point $p$ and of the surface $K$, and coinciding for finitely many points $p$. Similarly if $p \in l'_{ij}$, with $p \neq v'$ and $p \neq q'_{ijk}$, then $TC_p K$ is the union of two elements of $|\mathcal{I}_{l'_{ij}|\mathbb{P}^3}(1)|$ that depend on the choice of $p$ and $K$ and that can also coincide for finitely many points $p$;

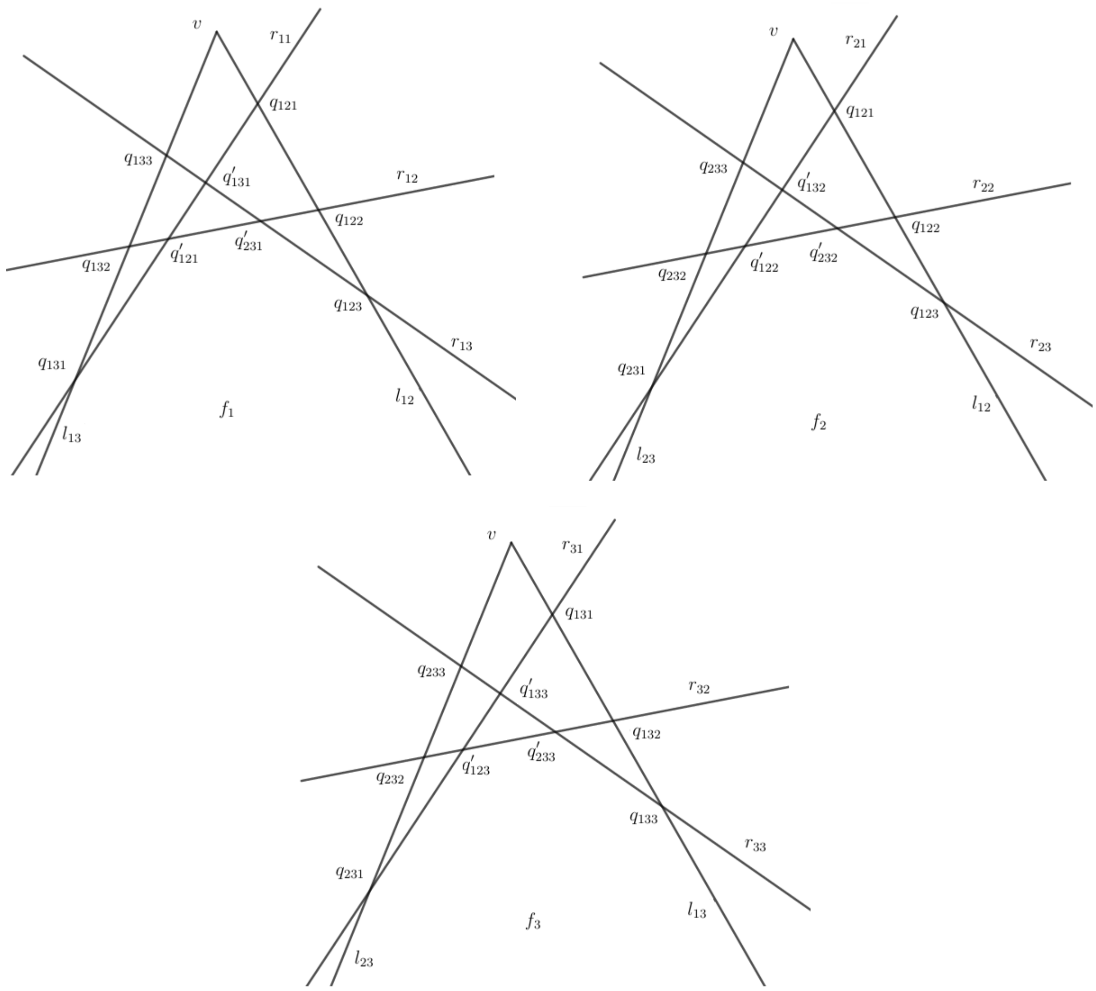


*Figure 5: Description of faces of $T$. The same happens on $T'$ by taking $v$ instead of $v'$; $f'_i$ instead of $f'_i$; $r_{ki}$ instead of $r_{ik}$; $l'_{ij}$ instead of $l_{ij}$; $q_{ijk}$ instead of $q'_{ijk}$ and $q'_{ijk}$ instead of $q_{ijk}$.*

(iv) $K$ is smooth along $r_{ik}$, except at the points contained in the edges of the two trihedra.

**Lemma 5.22.** The rational map $\nu_{\mathcal{K}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^9$ defined by $\mathcal{K}$ is birational onto the image.

*Proof.* It is sufficient to prove that the map defined by $\mathcal{K}$ on a general $K \in \mathcal{K}$ is birational onto the image. This actually happens because $\mathcal{K}|_K$ contains a sublinear system that defines a birational map. Indeed $\mathcal{K}$ contains a sublinear system $\overline{\mathcal{K}} \subset \mathcal{K}$ whose fixed part is given by the two trihedra $T$ and $T'$ and such that $\overline{\mathcal{K}}|_K$ coincides with the linear system on $K$ cut out by the planes of $\mathbb{P}^3$. □

**Remark 5.23.** The proof of Lemma 5.22 tells us that the linear system $\mathcal{K}$ is very ample outside the two trihedra $T$ and $T'$. So $\nu_{\mathcal{K}} : \mathbb{P}^3 \dashrightarrow \nu_{\mathcal{K}}(\mathbb{P}^3) \subset \mathbb{P}^9$ is an isomorphism outside $T \cup T'$.

**Theorem 5.24.** [23, §7] The image of $\nu_{\mathcal{K}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^9$ is an Enriques-Fano threefold $W^9_F$ of genus $p = 9$.

*Proof.* We will prove the theorem by using the approaches of the proof of Theorem 5.4. In particular the proof is divided into several steps, given by the Remark 5.25, the Proposition 5.26, the Remarks 5.27,…, 5.37 and the Theorem 5.38 below.

We blow-up first the vertices of the trihedra and the 18 points $q_{ijk}$ and $q'_{ijk}$ for $i, j, k \in \{1, 2, 3\}$ and $i < j$. We obtain a smooth threefold $Y'$ and a birational morphism $bl' : Y' \to \mathbb{P}^3$ with exceptional divisors $E := (bl')^{-1}(v)$, $E' := (bl')^{-1}(v')$, $E_{ijk} := (bl')^{-1}(q_{ijk})$, $E'_{ijk} := (bl')^{-1}(q'_{ijk})$. Let $\mathcal{K}'$ be the strict transform of $\mathcal{K}$ and let us denote by $H$ the pullback on $Y'$ of the hyperplane class on $\mathbb{P}^3$. Then an element of $\mathcal{K}'$ is linearly equivalent to $7H - 3E - 3E' - 2\sum_{\substack{i,j,k=1\\ i<j}}^{3}(E_{ijk} + E'_{ijk})$. Let $\widetilde{f}_i$ and $\widetilde{f}'_i$ be the strict transforms of the faces $f_i$ and $f'_i$, for $1 \le i \le 3$. We denote by $\gamma_i := E \cap \widetilde{f}_i$ the line cut out by $\widetilde{f}_i$ on $E$ and by $\gamma'_i := E' \cap \widetilde{f}'_i$ the one cut out by $\widetilde{f}'_i$ on $E'$. By construction, the curves $\gamma_i$ and $\gamma'_i$ are $(-1)$-curves respectively on $\widetilde{f}_i$ and $\widetilde{f}'_i$. If $K'$ is the strict transform of a general $K \in \mathcal{K}$, then $K' \cap E = \bigcup_{i=0}^{3}\gamma_i$ and $K' \cap E' = \bigcup_{i=0}^{3}\gamma'_i$ and $K'$ is smooth at a general point of $\gamma_i$ and of $\gamma'_i$ (see Remark 5.21). We also consider the lines $\lambda_{ijk,h} := E_{ijk} \cap \widetilde{f}_h$ and $\lambda'_{ijk,h} := E'_{ijk} \cap \widetilde{f}'_h$, where $i, j, k \in \{1, 2, 3\}$ with $i < j$ and $h \in \{i, j\}$. They are $(-1)$-curves on the strict transforms of the faces containing them. Furthermore we have that $K' \cap E_{ijk} = \bigcup_{h=i,j}\lambda_{ijk,h}$ and $K' \cap E'_{ijk} = \bigcup_{h=i,j}\lambda'_{ijk,h}$ (see Remark 5.21). Let us consider the strict transforms $\widetilde{l}_{ij}$, $\widetilde{l}'_{ij}$ and $\widetilde{r}_{ij}$ of the lines $l_{ij}$, $l'_{ij}$ and $r_{ik}$, for $i, j, k \in \{1, 2, 3\}$ and $i < j$. Then the base locus of $\mathcal{K}'$ is given by the union of the six curves $\widetilde{l}_{ij}$, $\widetilde{l}'_{ij}$ (along which a general $K' \in \mathcal{K}'$ has double points), of the nine curves $\widetilde{r}_{ik}$, of the six lines $\gamma_i$, $\gamma'_i$, and of the 36 lines $\lambda_{ijk,h}$, $\lambda'_{ijk,h}$ (see Remark 5.21). Let us blow-up $Y'$ along the strict transforms of the edges of the trihedra and of the nine lines $r_{ij}$. We obtain a smooth threefold $Y''$ and a birational morphism $bl'' : Y'' \to Y'$ with exceptional divisors

$$(bl'')^{-1}(\widetilde{l}_{ij}) =: F_{ij} \cong \mathbb{P}(\mathcal{N}_{\widetilde{l}_{ij}|Y'}) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)) \cong \mathbb{F}_0,$$

$$(bl'')^{-1}(\widetilde{l}'_{ij}) =: F'_{ij} \cong \mathbb{P}(\mathcal{N}_{\widetilde{l}'_{ij}|Y'}) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)) \cong \mathbb{F}_0,$$

$$(bl'')^{-1}(\widetilde{r}_{ij}) =: R_{ij} \cong \mathbb{P}(\mathcal{N}_{\widetilde{r}_{ij}|Y'}) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)) \cong \mathbb{F}_0.$$

This blow-up has no effect on $\widetilde{f}_i$ and $\widetilde{f}'_i$, for $1 \le i \le 3$, so, by abuse of notation, we use the same symbols to indicate their strict transforms on $Y''$. Let us denote by $\widetilde{E}$, $\widetilde{E}'$, $\widetilde{E}_{ijk}$ and $\widetilde{E}'_{ijk}$ respectively the strict transforms of $E$, $E'$, $E_{ijk}$ and $E'_{ijk}$.

**Remark 5.25.** Let us take the curves $\alpha_{ij} := \widetilde{E} \cap F_{ij}$, $\alpha'_{ij} := \widetilde{E}' \cap F'_{ij}$, $\alpha_{ijk} := \widetilde{E}_{ijk} \cap F_{ij}$, $\alpha'_{ijk} := \widetilde{E}'_{ijk} \cap F'_{ij}$, $\alpha_{ijk,h} := \widetilde{E}_{ijk} \cap R_{hk}$, $\alpha'_{ijk,h} := \widetilde{E}_{ijk} \cap R_{kh}$, where $i, j, k \in \{1, 2, 3\}$ with $i < j$ and $h \in \{i, j\}$. By construction, $\alpha_{ij}$ and $\alpha'_{ij}$ are $(-1)$-curves respectively on $\widetilde{E}$ and $\widetilde{E}'$; $\alpha_{ijk}$ and $\alpha_{ijk,h}$ are $(-1)$-curves on $\widetilde{E}_{ijk}$ (see Figure 6); $\alpha'_{ijk}$ and $\alpha'_{ijk,h}$ are $(-1)$-curves on $\widetilde{E}'_{ijk}$; $\alpha_{ij}$ and $\alpha_{ijk}$ are fibres on $F_{ij}$; $\alpha'_{ij}$ and $\alpha'_{ijk}$ are fibres on $F'_{ij}$; $\alpha_{ijk,h}$ and $\alpha'_{ijk,h}$ are fibres respectively on $R_{hk}$ and $R_{kh}$.

Let $\mathcal{K}''$ be the strict transform of $\mathcal{K}'$: an element of $\mathcal{K}''$ is linearly equivalent to $7H - 3\widetilde{E} - 3\widetilde{E}' - 2\sum_{\substack{i,j,k=1\\ i<j}}^{3}(\widetilde{E}_{ijk} + \widetilde{E}'_{ijk}) - 2\sum_{1\le i<j\le 3}(F_{ij} + F'_{ij}) - \sum_{i,j=1}^{3} R_{ij}$, where, by abuse of notation, $H$ also denotes the pullback $bl''^{*}H$.

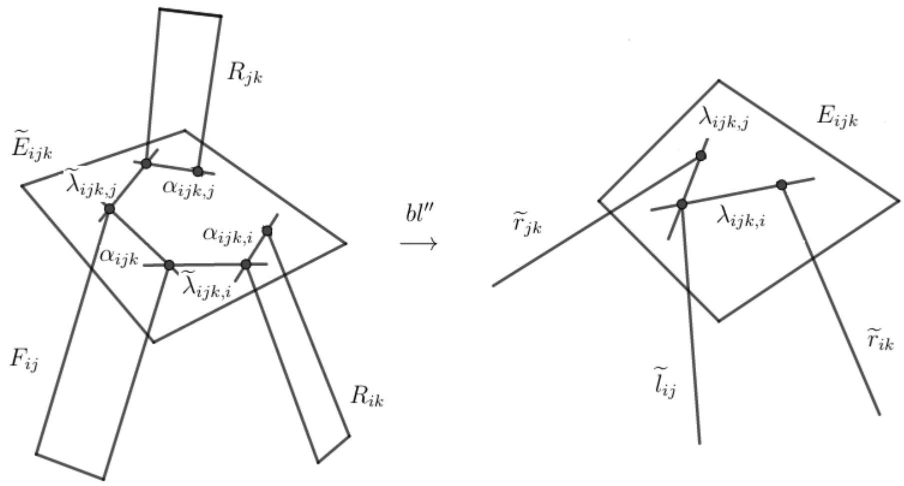


*Figure 6:* Description of $bl'''|_{\widetilde{E}_{ijk}} : \widetilde{E}_{ijk} \to E_{ijk}$.

**Proposition 5.26.** A general element $K'' \in \mathcal{K}''$ is a smooth surface with zero arithmetic genus $p_a(K'') = 0$.

*Proof.* The smoothness of $K''$ is shown in [27, p.620-621], since $K''$ is the blow-up of a surface $K \in \mathcal{K}$ with ordinary singularities along its singular curves (see Definition 3.4 and Remark 5.21). We have to compute the arithmetic genus $p_a(K'') = \chi(\mathcal{O}_{K''}) - 1$. By Serre Duality, we have that $p_a(K'') = \chi(\mathcal{O}_{K''}(K_{K''})) - 1$. By the adjunction formula, we have the following exact sequence

$$0 \to \mathcal{O}_{Y''}(K_{Y''}) \to \mathcal{O}_{Y''}(K_{Y''} + K'') \to \mathcal{O}_{K''}(K_{K''}) \to 0.$$

Since $Y''$ is a smooth rational threefold, then we have that $h^0(Y'', \mathcal{O}_{Y''}(K_{Y''})) = p_g(Y'') = 0$. By Serre Duality, we have that $h^i(Y'', \mathcal{O}_{Y''}(K_{Y''})) = h^{3-i}(Y'', \mathcal{O}_{Y''}) = 0$ for $i = 1, 2$, and $h^3(Y'', \mathcal{O}_{Y''}(K_{Y''})) = h^0(Y'', \mathcal{O}_{Y''}) = 1$. Hence $\chi(\mathcal{O}_{Y''}(K_{Y''})) = -1$ and

$$p_a(K'') = \chi(\mathcal{O}_{Y''}(K_{Y''} + K'')) - \chi(\mathcal{O}_{Y''}(K_{Y''})) - 1 = \chi(\mathcal{O}_{Y''}(K_{Y''} + K'')).$$

Since the canonical divisor of $Y''$ is linearly equivalent to

$$-4H + 2\widetilde{E} + 2\widetilde{E}' + 2 \sum_{\substack{i,j,k=1 \\ i<j}}^{3} (\widetilde{E}_{ijk} + \widetilde{E}'_{ijk}) + \sum_{1 \le i < j \le 3} (F_{ij} + F'_{ij}) + \sum_{i,j=1}^{3} R_{ij}$$

(see [27, p.187]), then we have $K_{Y''} + K'' \sim 3H - \widetilde{E} - \widetilde{E}' - \sum_{1 \le i < j \le 3} (F_{ij} + F'_{ij})$. Let us denote by $f_{ij}$ and $f'_{ij}$ respectively the fibre class of $F_{ij}$ and $F'_{ij}$. Then we have the following two exact sequences

$$0 \to \mathcal{O}_{Y''}(3H - \widetilde{E} - \widetilde{E}') \to \mathcal{O}_{Y''}(3H) \to \mathcal{O}_{\widetilde{E}} \oplus \mathcal{O}_{\widetilde{E}'} \to 0,$$

$$0 \to \mathcal{O}_{Y''}(K_{Y''} + K'') \to \mathcal{O}_{Y''}(3H - \widetilde{E} - \widetilde{E}') \to \oplus_{1 \le i < j \le 3} \mathcal{O}_{F_{ij}}(2f_{ij}) \oplus_{1 \le i < j \le 3} \mathcal{O}_{F'_{ij}}(2f'_{ij}) \to 0,$$

and we obtain $\chi(\mathcal{O}_{Y''}(K_{Y''} + K'')) = \binom{3+3}{3} - 2 - 6 \cdot 3 = 0$. $\square$

By Remark 5.21 we have that the base locus of $\mathcal{K}''$ is given by the disjoint union of the strict transforms $\widetilde{\gamma}_i$, $\widetilde{\gamma}'_i$, $\widetilde{\lambda}_{ijk,h}$, $\widetilde{\lambda}'_{ijk,h}$ of the 42 lines defined as above.

**Remark 5.27.** We observe that $\widetilde{\gamma}_i^2|_{\widetilde{E}} = \widetilde{\gamma}_i^2|_{\widetilde{f}_i} = -1$, $\widetilde{\gamma}_i'^2|_{\widetilde{E}'} = \widetilde{\gamma}_i'^2|_{\widetilde{f}'_i} = -1$, $\widetilde{\lambda}^2_{ijk,h}|_{\widetilde{E}_{ijk}} = \widetilde{\lambda}^2_{ijk,h}|_{\widetilde{f}_h} = -1$, $\widetilde{\lambda}'^2_{ijk,h}|_{\widetilde{E}'_{ijk}} = \widetilde{\lambda}^2_{ijk,h}|_{\widetilde{f}'_h} = -1$. Furthermore, by using similar arguments to the ones in Remark 5.7 we also have that $\widetilde{\gamma}_i$, $\widetilde{\gamma}'_i$, $\widetilde{\lambda}_{ijk,h}$, $\widetilde{\lambda}'_{ijk,h}$ are $(-1)$-curves on the strict transform $K''$ of a general $K' \in \mathcal{K}'$.

Finally let us consider the blow-up of $Y''$ along the above 42 curves, which is the map $bl''' : Y \to Y''$ with exceptional divisors $\Gamma_i := bl'''^{-1}(\widetilde{\gamma}_i)$, $\Gamma'_i := bl'''^{-1}(\widetilde{\gamma}'_i)$, $\Lambda_{ijk,h} := bl'''^{-1}(\widetilde{\lambda}_{ijk,h})$, $\Lambda'_{ijk,h} := bl'''^{-1}(\widetilde{\lambda}'_{ijk,h})$. We denote by $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{E}_{ijk}$, $\mathcal{E}'_{ijk}$, respectively, the strict transform of $\widetilde{E}$, $\widetilde{E}'$, $\widetilde{E}_{ijk}$, $\widetilde{E}'_{ijk}$; by $\mathcal{F}_{ij}$ the strict transform of $F_{ij}$; by $\mathcal{R}_{ik}$ the strict transform of $R_{ik}$; by $\mathcal{H}$ the pullback of $H$, for $i, j, k \in \{1, 2, 3\}$ with $i < j$ and $h \in \{i, j\}$.

**Remark 5.28.** We have that

$$\Gamma_i = \mathbb{P}(\mathcal{N}_{\widetilde{\gamma}_i|Y''}) \cong \mathbb{P}(\mathcal{O}_{\widetilde{\gamma}_i}(\widetilde{E}) \oplus \mathcal{O}_{\widetilde{\gamma}_i}(\widetilde{f}_i)) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{F}_0,$$

$$\Gamma'_i = \mathbb{P}(\mathcal{N}_{\widetilde{\gamma}'_i|Y''}) \cong \mathbb{P}(\mathcal{O}_{\widetilde{\gamma}'_i}(\widetilde{E}') \oplus \mathcal{O}_{\widetilde{\gamma}'_i}(\widetilde{f}'_i)) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{F}_0,$$

$$\Lambda_{ijk,h} = \mathbb{P}(\mathcal{N}_{\widetilde{\lambda}_{ijk,h}|Y''}) \cong \mathbb{P}(\mathcal{O}_{\widetilde{\lambda}_{ijk,h}}(\widetilde{E}_{ijk}) \oplus \mathcal{O}_{\widetilde{\lambda}_{ijk,h}}(\widetilde{f}_h)) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{F}_0,$$

$$\Lambda'_{ijk,h} = \mathbb{P}(\mathcal{N}_{\widetilde{\lambda}'_{ijk,h}|Y''}) \cong \mathbb{P}(\mathcal{O}_{\widetilde{\lambda}'_{ijk,h}}(\widetilde{E}'_{ijk}) \oplus \mathcal{O}_{\widetilde{\lambda}'_{ijk,h}}(\widetilde{f}'_h)) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{F}_0.$$

Furthermore we have $\Gamma_i^3 = -\deg(\mathcal{N}_{\widetilde{\gamma}_i|Y''}) = 2$, $\Gamma_i'^3 = -\deg(\mathcal{N}_{\widetilde{\gamma}'_i|Y''}) = 2$, $\Lambda^3_{ijk,h} = -\deg(\mathcal{N}_{\widetilde{\lambda}_{ijk,h}|Y''}) = 2$, $\Lambda'^3_{ijk,h} = -\deg(\mathcal{N}_{\widetilde{\lambda}'_{ijk,h}|Y''}) = 2$ (see [27, Chap 4, §6] and [32, Lemma 2.2.14]).

**Remark 5.29.** Let us take $i, j, k \in \{1, 2, 3\}$ with $i < j$ and $h \in \{i, j\}$. The divisor $\mathcal{F}_{ij}$ intersects $\Gamma_i$, $\Gamma_j$, $\Lambda_{ijk,h}$ each along a $\mathbb{P}^1$, which is a $(-1)$-curve on $\mathcal{F}_{ij}$ and a fibre on $\Gamma_i$, $\Gamma_j$, $\Lambda_{ijk,h}$. The same happens with $\mathcal{F}'_{ij}$ and $\Gamma'_i$, $\Gamma'_j$, $\Lambda'_{ijk,h}$. Similarly we have $\Lambda^2_{ijk,h} \cdot \mathcal{R}_{hk} = \Lambda'^2_{ijk,h} \cdot \mathcal{R}_{kh} = -1$ and $\Lambda_{ijk,h} \cdot \mathcal{R}^2_{hk} = \Lambda'_{ijk,h} \cdot \mathcal{R}^2_{kh} = 0$. Let us consider the strict transforms $\widetilde{\alpha}_{ij}$, $\widetilde{\alpha}'_{ij}$, $\widetilde{\alpha}_{ijk}$, $\widetilde{\alpha}'_{ijk}$, $\widetilde{\alpha}_{ijk,h}$, $\widetilde{\alpha}'_{ijk,h}$ of the curves defined in Remark 5.25. Then we have

$$\widetilde{\alpha}^2_{ij}|_{\mathcal{E}} = \mathcal{F}^2_{ij} \cdot \mathcal{E} = -1, \quad \widetilde{\alpha}^2_{ij}|_{\mathcal{F}_{ij}} = \mathcal{E}^2 \cdot \mathcal{F}_{ij} = -2,$$

$$\widetilde{\alpha}'^2_{ij}|_{\mathcal{E}'} = {\mathcal{F}'_{ij}}^2 \cdot \mathcal{E}' = -1, \quad \widetilde{\alpha}'^2_{ij}|_{\mathcal{F}_{ij}} = \mathcal{E}'^2 \cdot \mathcal{F}'_{ij} = -2,$$

$$\widetilde{\alpha}^2_{ijk}|_{\mathcal{E}_{ijk}} = \mathcal{F}^2_{ij} \cdot \mathcal{E}_{ijk} = -1, \quad \widetilde{\alpha}^2_{ijk}|_{\mathcal{F}^2_{ij}} = \mathcal{E}^2_{ijk} \cdot \mathcal{F}_{ij} = -2,$$

$$\widetilde{\alpha}'^2_{ijk}|_{\mathcal{E}'_{ijk}} = \mathcal{F}^2_{ij} \cdot \mathcal{E}'_{ijk} = -1, \quad \widetilde{\alpha}'^2_{ijk}|_{\mathcal{F}_{ij}} = \mathcal{E}'^2_{ijk} \cdot \mathcal{F}_{ij} = -2,$$

$$\widetilde{\alpha}^2_{ijk,h}|_{\mathcal{E}_{ijk}} = \mathcal{R}^2_{hk} \cdot \mathcal{E}_{ijk} = -1, \quad \widetilde{\alpha}^2_{ijk,h}|_{\mathcal{R}_{hk}} = \mathcal{E}^2_{ijk} \cdot \mathcal{R}_{hk} = -1,$$

$$\widetilde{\alpha}'^2_{ijk,h}|_{\mathcal{E}'_{ijk}} = \mathcal{R}^2_{kh} \cdot \mathcal{E}'_{ijk} = -1, \quad \widetilde{\alpha}'^2_{ijk,h}|_{\mathcal{R}_{kh}} = \mathcal{E}'^2_{ijk} \cdot \mathcal{R}_{kh} = -1.$$

Finally we recall that a general line of $\mathbb{P}^3$ does not intersect the edges of the trihedra and the nine lines $r_{ij}$, while a general plane of $\mathbb{P}^3$ intersects each of these lines at one point. Hence we have that $\mathcal{H}^2 \cdot \mathcal{F}_{ij} = \mathcal{H}^2 \cdot \mathcal{F}'_{ij} = \mathcal{H}^2 \cdot \mathcal{R}_{ik} = 0$ and $\mathcal{F}^2_{ij} \cdot \mathcal{H} = \mathcal{F}'^2_{ij} \cdot \mathcal{H} = \mathcal{R}^2_{ik} \cdot \mathcal{H} = -1$.

**Remark 5.30.** By construction we have that

$$bl'''^*(\widetilde{E}) = \mathcal{E} + \sum_{1\leq x<y\leq 3} \Gamma_{xy}, \quad bl'''^*(E_{ijk}) = \mathcal{E}_{ijk} + \Lambda_{ijk,i} + \Lambda_{ijk,j},$$

$$bl'''^*(\widetilde{E}') = \mathcal{E}' + \sum_{1\leq x<y\leq 3} \Gamma'_{xy}, \quad bl'''^*(E'_{ijk}) = \mathcal{E}'_{ijk} + \Lambda'_{ijk,i} + \Lambda'_{ijk,j},$$

where $i, j, k \in \{1, 2, 3\}$ and $i < j$. By abuse of notation, we denote $\mathcal{E} \cap \Gamma_{ij}$, $\mathcal{E}' \cap \Gamma'_{ij}$, $\mathcal{E}_{ijk} \cap \Lambda_{ijk,h}$, $\mathcal{E}'_{ijk} \cap \Lambda'_{ijk,h}$, respectively, by $\widetilde{\gamma}_{ij}$, $\widetilde{\gamma}'_{ij}$, $\widetilde{\lambda}_{ijk,h}$, $\widetilde{\lambda}'_{ijk,h}$, where $h \in \{i, j\}$. Let $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{L}_{ijk}$, $\mathcal{L}'_{ijk}$ be the strict transforms on $Y$ of a general line respectively of $E$, $E'$, $E_{ijk}$, $E'_{ijk}$. By using similar arguments to the ones in Remark 5.10 we obtain

$$\mathcal{E}|_{\mathcal{E}} \sim -(\mathcal{L} + \sum_{t=1}^{3} \widetilde{\gamma}_i) \sim -(4\mathcal{L} - 2 \sum_{0\leq x<y\leq 3} \widetilde{\alpha}_{xy}),$$

$$\mathcal{E}'|_{\mathcal{E}'} \sim -(\mathcal{L}' + \sum_{t=1}^{3} \widetilde{\gamma}'_i) \sim -(4\mathcal{L}' - 2 \sum_{0\leq x<y\leq 3} \widetilde{\alpha}'_{xy}),$$

$$\mathcal{E}_{ijk}|_{\mathcal{E}_{ijk}} \sim -(\mathcal{L}_{ijk} + \widetilde{\lambda}_{ijk,i} + \widetilde{\lambda}_{ijk,j}) \sim -(3\mathcal{L}_{ijk} - 2\widetilde{\alpha}_{ijk} - \widetilde{\alpha}_{ijk,i} - \widetilde{\alpha}_{ijk,j}),$$

$$\mathcal{E}'_{ijk}|_{\mathcal{E}'_{ijk}} \sim -(\mathcal{L}'_{ijk} + \widetilde{\lambda}'_{ijk,i} + \widetilde{\lambda}'_{ijk,j}) \sim -(3\mathcal{L}'_{ijk} - 2\widetilde{\alpha}'_{ijk} - \widetilde{\alpha}'_{ijk,i} - \widetilde{\alpha}'_{ijk,j}),$$

so we have $\mathcal{E}^3 = 4$, $\mathcal{E}'^3 = 4$, $\mathcal{E}^3_{ijk} = 3$ and $\mathcal{E}'^3_{ijk} = 3$.

**Remark 5.31.** With similar arguments to the ones in Remark 5.11, we have $\mathcal{F}^3_{ij} = -\deg(\mathcal{N}_{\widetilde{l}_{ij}|Y'}) = 6$, $\mathcal{F}'^3_{ij} = -\deg(\mathcal{N}_{\widetilde{l}'_{ij}|Y'}) = 6$, $\mathcal{R}^3_{ki} = -\deg(\mathcal{N}_{\widetilde{r}_{ki}|Y'}) = 6$, for $i, j, k \in \{1, 2, 3\}$ with $i < j$.

Let $\widetilde{K}$ be the strict transform on $Y$ of an element of $\mathcal{K}''$: then

$$\widetilde{K} \sim 7\mathcal{H} - 3\mathcal{E} - 3\mathcal{E}' - 2 \sum_{\substack{i,j,k=1 \\ i<j}}^{3} (\mathcal{E}_{ijk} + \mathcal{E}'_{ijk}) - 2 \sum_{1\leq i<j\leq 3} (\mathcal{F}_{ij} + \mathcal{F}'_{ij}) - \sum_{i,j=1}^{3} \mathcal{R}_{ij} +$$

$$-4 \sum_{i=1}^{3} (\Gamma_i + \Gamma'_i) - 3 \sum_{\substack{i,j,k=1 \\ i<j,\, h=i,j}}^{3} (\Lambda_{ijk,h} + \Lambda'_{ijk,h}).$$

Let us take the linear system $\widetilde{\mathcal{K}} := |\mathcal{O}_Y(\widetilde{K})|$ on $Y$. It is base point free and it defines a morphism $\nu_{\widetilde{\mathcal{K}}} : Y \to \mathbb{P}^9$ birational onto the image $W^9_F := \nu_{\widetilde{\mathcal{K}}}(Y)$, which is a threefold of degree $\deg W^9_F = 16$. This follows by Lemma 5.22 and by the fact that $\widetilde{K}^3 = 16$: indeed by Remarks 5.28, 5.29, 5.30, 5.31) we have

$$\widetilde{K}^3 = (7\mathcal{H})^3 - 27\mathcal{E}^3 - 27\mathcal{E}'^3 - 8 \sum_{\substack{i,j,k=1 \\ i<j}}^{3} (\mathcal{E}^3_{ijk} + \mathcal{E}'^3_{ijk}) - 8 \sum_{1\leq i<j\leq 3} (\mathcal{F}^3_{ij} + \mathcal{F}'^3_{ij}) - \sum_{i,j=1}^{3} \mathcal{R}^3_{ij} +$$

$$-64\sum_{i=1}^{3}(\Gamma_i^3+\Gamma_i'^3)-27\sum_{\substack{i,j,k=1\\ i<j,\,h=i,j}}^{3}(\Lambda_{ijk,h}^3+\Lambda_{ijk,h}'^3)-(3\cdot 49\cdot 2)\sum_{1\le i<j\le 3}\mathcal{H}^2\cdot(\mathcal{F}_{ij}+\mathcal{F}'_{ij})+$$

$$-(3\cdot9\cdot2)\sum_{1\le i<j\le 3}\mathcal{E}^2\cdot\mathcal{F}_{ij}-(3\cdot9\cdot4)\sum_{i=1}^{3}\mathcal{E}^2\cdot\Gamma_i-(3\cdot9\cdot2)\sum_{1\le i<j\le 3}\mathcal{E}'^2\cdot\mathcal{F}'_{ij}-(3\cdot9\cdot4)\sum_{i=1}^{3}\mathcal{E}'^2\cdot\Gamma'_i+$$

$$-(3\cdot4\cdot2)\Big(\sum_{\substack{i,j,k=1\\ i<j}}^{3}\mathcal{E}_{ijk}^2\Big)\cdot\Big(\sum_{1\le i<j\le 3}\mathcal{F}_{ij}\Big)-(3\cdot4\cdot1)\Big(\sum_{\substack{i,j,k=1\\ i<j}}^{3}\mathcal{E}_{ijk}^2\Big)\cdot\Big(\sum_{i,j=1}^{3}\mathcal{R}_{ij}\Big)+$$

$$-(3\cdot4\cdot2)\Big(\sum_{\substack{i,j,k=1\\ i<j}}^{3}\mathcal{E}_{ijk}'^2\Big)\cdot\Big(\sum_{1\le i<j\le 3}\mathcal{F}'_{ij}\Big)-(3\cdot4\cdot1)\Big(\sum_{\substack{i,j,k=1\\ i<j}}^{3}\mathcal{E}_{ijk}'^2\Big)\cdot\Big(\sum_{i,j=1}^{3}\mathcal{R}_{ij}\Big)+$$

$$-(3\cdot4\cdot3)\Big(\sum_{\substack{i,j,k=1\\ i<j}}^{3}\mathcal{E}_{ijk}^2\Big)\cdot\Big(\sum_{\substack{i,j,k=1\\ i<j,\,h=i,j}}^{3}\Lambda_{ijk,h}\Big)-(3\cdot4\cdot3)\Big(\sum_{\substack{i,j,k=1\\ i<j}}^{3}\mathcal{E}_{ijk}'^2\Big)\cdot\Big(\sum_{\substack{i,j,k=1\\ i<j,\,h=i,j}}^{3}\Lambda'_{ijk,h}\Big)+$$

$$+(3\cdot4\cdot7)\sum_{1\le i<j\le 3}(\mathcal{F}_{ij}^2+\mathcal{F}_{ij}'^2)\cdot\mathcal{H}-(3\cdot4\cdot3)\sum_{1\le i<j\le 3}(\mathcal{F}_{ij}^2\cdot\mathcal{E}+\mathcal{F}_{ij}'^2\cdot\mathcal{E}')+$$

$$-(3\cdot4\cdot2)\Big(\sum_{1\le i<j\le 3}\mathcal{F}_{ij}^2\Big)\cdot\Big(\sum_{\substack{i,j,k=1\\ i<j}}^{3}\mathcal{E}_{ijk}\Big)-(3\cdot4\cdot2)\Big(\sum_{1\le i<j\le 3}\mathcal{F}_{ij}'^2\Big)\cdot\Big(\sum_{\substack{i,j,k=1\\ i<j}}^{3}\mathcal{E}'_{ijk}\Big)+$$

$$-(3\cdot4\cdot4)\Big(\sum_{1\le i<j\le 3}\mathcal{F}_{ij}^2\Big)\cdot\Big(\sum_{i=1}^{3}\Gamma_i\Big)-(3\cdot4\cdot4)\Big(\sum_{1\le i<j\le 3}\mathcal{F}_{ij}'^2\Big)\cdot\Big(\sum_{i=1}^{3}\Gamma'_i\Big)+$$

$$-(3\cdot4\cdot3)\Big(\sum_{1\le i<j\le 3}\mathcal{F}_{ij}^2\Big)\cdot\Big(\sum_{\substack{i,j,k=1\\ i<j,\,h=i,j}}^{3}\Lambda_{ijk,h}\Big)-(3\cdot4\cdot3)\Big(\sum_{1\le i<j\le 3}\mathcal{F}_{ij}'^2\Big)\cdot\Big(\sum_{\substack{i,j,k=1\\ i<j,\,h=i,j}}^{3}\Lambda'_{ijk,h}\Big)+$$

$$+(3\cdot1\cdot7)\sum_{i,j=1}^{3}\mathcal{R}_{ij}^2\cdot\mathcal{H}-(3\cdot1\cdot2)\Big(\sum_{i,j=1}^{3}\mathcal{R}_{ij}^2\Big)\cdot\Big(\sum_{\substack{i,j,k=1\\ i<j}}^{3}\mathcal{E}_{ijk}+\mathcal{E}'_{ijk}\Big)+$$

$$-(3\cdot1\cdot3)\Big(\sum_{i,j=1}^{3}\mathcal{R}_{ij}^2\Big)\cdot\Big(\sum_{\substack{i,j,k=1\\ i<j,\,h=i,j}}^{3}\Lambda_{ijk,h}+\Lambda'_{ijk,h}\Big)-(3\cdot16\cdot3)\sum_{i=1}^{3}(\Gamma_i^2\cdot\mathcal{E}+\Gamma_i'^2\cdot\mathcal{E}')+$$

$$-(3\cdot16\cdot2)\Big(\sum_{i=1}^{3}\Gamma_i^2\Big)\cdot\Big(\sum_{1\le i<j\le 3}\mathcal{F}_{ij}\Big)-(3\cdot16\cdot2)\Big(\sum_{i=1}^{3}\Gamma_i'^2\Big)\cdot\Big(\sum_{1\le i<j\le 3}\mathcal{F}'_{ij}\Big)+$$

$$-(3\cdot9\cdot2)\Big(\sum_{\substack{i,j,k=1\\ i<j,\,h=i,j}}^{3}\Lambda_{ijk,h}^2\Big)\cdot\Big(\sum_{\substack{i,j,k=1\\ i<j}}^{3}\mathcal{E}_{ijk}\Big)-(3\cdot9\cdot2)\Big(\sum_{\substack{i,j,k=1\\ i<j,\,h=i,j}}^{3}\Lambda_{ijk,h}'^2\Big)\cdot\Big(\sum_{\substack{i,j,k=1\\ i<j}}^{3}\mathcal{E}'_{ijk}\Big)+$$

$$-(3\cdot 9\cdot 2)\Big(\sum_{\substack{i,j,k=1\\ i<j,\,h=i,j}}^{3}\Lambda^{2}_{ijk,h}\Big)\cdot\Big(\sum_{1\le i<j\le 3}\mathcal{F}_{ij}\Big)-(3\cdot 9\cdot 2)\Big(\sum_{\substack{i,j,k=1\\ i<j,\,h=i,j}}^{3}\Lambda'^{2}_{ijk,h}\Big)\cdot\Big(\sum_{1\le i<j\le 3}\mathcal{F}'_{ij}\Big)+$$

$$-(3\cdot 9\cdot 1)\Big(\sum_{\substack{i,j,k=1\\ i<j,\,h=i,j}}^{3}\Lambda^{2}_{ijk,h}+\Lambda'^{2}_{ijk,h}\Big)\cdot\Big(\sum_{i,j=1}^{3}\mathcal{R}_{ij}\Big)+$$

$$-(6\cdot 3\cdot 2\cdot 4)\mathcal{E}\cdot\Big(\sum_{1\le i<j\le 3}\mathcal{F}_{ij}\Big)\cdot\Big(\sum_{i=1}^{3}\Gamma_i\Big)-(6\cdot 3\cdot 2\cdot 4)\mathcal{E}'\cdot\Big(\sum_{1\le i<j\le 3}\mathcal{F}'_{ij}\Big)\cdot\Big(\sum_{i=1}^{3}\Gamma'_i\Big)+$$

$$-(6\cdot 2\cdot 1\cdot 3)\Big(\sum_{\substack{i,j,k=1\\ i<j}}^{3}\mathcal{E}_{ijk}\Big)\cdot\Big(\sum_{i,j=1}^{3}\mathcal{R}_{ij}\Big)\cdot\Big(\sum_{\substack{i,j,k=1\\ i<j,\,h=i,j}}^{3}\Lambda_{ijk,h}\Big)+$$

$$-(6\cdot 2\cdot 1\cdot 3)\Big(\sum_{\substack{i,j,k=1\\ i<j}}^{3}\mathcal{E}'_{ijk}\Big)\cdot\Big(\sum_{i,j=1}^{3}\mathcal{R}_{ij}\Big)\cdot\Big(\sum_{\substack{i,j,k=1\\ i<j,\,h=i,j}}^{3}\Lambda'_{ijk,h}\Big)+$$

$$-(6\cdot 2\cdot 2\cdot 3)\Big(\sum_{\substack{i,j,k=1\\ i<j}}^{3}\mathcal{E}_{ijk}\Big)\cdot\Big(\sum_{1\le i<j\le 3}\mathcal{F}_{ij}\Big)\cdot\Big(\sum_{\substack{i,j,k=1\\ i<j,\,h=i,j}}^{3}\Lambda_{ijk,h}\Big)+$$

$$-(6\cdot 2\cdot 2\cdot 3)\Big(\sum_{\substack{i,j,k=1\\ i<j}}^{3}\mathcal{E}'_{ijk}\Big)\cdot\Big(\sum_{1\le i<j\le 3}\mathcal{F}'_{ij}\Big)\cdot\Big(\sum_{\substack{i,j,k=1\\ i<j,\,h=i,j}}^{3}\Lambda'_{ijk,h}\Big)=$$

$$=343-108-108-432-288-54-768-1944+0+324+0+324+0+432+216+432+216+$$
$$+0+0-504+108+108+216+216+0+0+0+0-189+108+108+0+0+432+432+$$
$$+576+576+972+972+972+972+486+486-864-864-648-648-1296-1296=16.$$

Then we have the following diagram:

$$\begin{array}{ccccccc}
Y & & & & & \overset{\nu_{\widetilde{\mathcal{K}}}}{\searrow} & \\
\downarrow{\scriptstyle bl'''} & & & & & & \\
Y'' & \xrightarrow{bl''} & Y' & \xrightarrow{bl'} & \mathbb{P}^3 & \overset{\nu_{\mathcal{K}}}{\dashrightarrow} & W^9_F\subset\mathbb{P}^9.
\end{array}$$

It remains to show that the general hyperplane section of the threefold $W^9_F$ is an Enriques surface.

**Remark 5.32.** By construction we have $\widetilde{K}\cdot\mathcal{E}=\widetilde{K}\cdot\mathcal{E}=\widetilde{K}\cdot\mathcal{E}_{ijk}=\widetilde{K}\cdot\mathcal{E}'_{ijk}=0$, for all $i,j,k\in\{1,2,3\}$ with $i<j$.

**Remark 5.33.** Since $bl''':Y\to Y''$ has no effect on the divisors $\widetilde{f}_i$ and $\widetilde{f}'_i$ for $1\le i\le 3$, we will continue to use the same notations to denote their strict transforms. By construction we have $\widetilde{K}\cdot\widetilde{f}_i=\widetilde{K}\cdot\widetilde{f}'_i=0$ for a general $\widetilde{K}\in\widetilde{\mathcal{K}}$.

**Remark 5.34.** The morphism $\nu_{\widetilde{\mathcal{K}}} : Y \to W_F^9 \subset \mathbb{P}^9$ blows-down the 42 exceptional divisors of $bl''' : Y \to Y''$ and the nine divisors $\mathcal{R}_{ik}$ to curves of $W_F^9$. This follows by the fact that $\widetilde{K} \cdot \Gamma_i$, $\widetilde{K} \cdot \Gamma_i'$, $\widetilde{K} \cdot \Lambda_{ijk,h}$, $\widetilde{K} \cdot \Lambda_{ijk,h}'$, $\widetilde{K} \cdot \mathcal{R}_{ik} \neq 0$ and $\widetilde{K}^2 \cdot \Gamma_i = \widetilde{K}^2 \cdot \Gamma_i' = \widetilde{K}^2 \cdot \Lambda_{ijk,h} = \widetilde{K}^2 \cdot \Lambda_{ijk,h}' = \widetilde{K}^2 \cdot \mathcal{R}_{ik} = 0$, for all $i, j, k \in \{1, 2, 3\}$ with $i < j$ and $h \in \{i, j\}$. Indeed by Remarks 5.28, 5.29 we have

$$\widetilde{K}^2 \cdot \Gamma_i = \widetilde{K} \cdot (-3\mathcal{E} \cdot \Gamma_i - 2 \sum_{\substack{1 \le x < y \le 3 \\ i \in \{x,y\}}} \mathcal{F}_{xy} \cdot \Gamma_i - 4\Gamma_i^2) = 9\mathcal{E}^2 \cdot \Gamma_i + 6 \sum_{\substack{1 \le x < y \le 3 \\ i \in \{x,y\}}} \mathcal{E} \cdot \mathcal{F}_{xy} \cdot \Gamma_i +$$

$$+12\mathcal{E} \cdot \Gamma_i^2 + 6 \sum_{\substack{1 \le x < y \le 3 \\ i \in \{x,y\}}} \mathcal{F}_{xy} \cdot \mathcal{E} \cdot \Gamma_i + 4 \sum_{\substack{1 \le x < y \le 3 \\ i \in \{x,y\}}} \mathcal{F}_{xy}^2 \cdot \Gamma_i + 8 \sum_{\substack{1 \le x < y \le 3 \\ i \in \{x,y\}}} \mathcal{F}_{xy} \cdot \Gamma_i^2 + 12\mathcal{E} \cdot \Gamma_i^2 +$$

$$+8 \sum_{\substack{1 \le x < y \le 3 \\ i \in \{x,y\}}} \mathcal{F}_{xy} \cdot \Gamma_i^2 + 16\Gamma_i^3 = 0 + 6 \cdot 2 - 12 + 6 + 6 + 0 - 8 - 8 - 12 - 8 - 8 + 16 \cdot 2 = 0;$$

$$\widetilde{K}^2 \cdot \Lambda_{ijk,h} = \widetilde{K} \cdot (-2\mathcal{E}_{ijk} \cdot \Lambda_{ijk,h} - 2\mathcal{F}_{ij} \cdot \Lambda_{ijk,h} - \mathcal{R}_{hk} \cdot \Lambda_{ijk,h} - 3\Lambda_{ijk,h}^2) = 4\mathcal{E}_{ijk}^2 \cdot \Lambda_{ijk,h} +$$

$$+4\mathcal{E}_{ijk} \cdot \mathcal{F}_{ij} \cdot \Lambda_{ijk,h} + 2\mathcal{E}_{ijk} \cdot \mathcal{R}_{hk} \cdot \Lambda_{ijk,h} + 6\mathcal{E}_{ijk} \cdot \Lambda_{ijk,h}^2 + 4\mathcal{F}_{ij} \cdot \mathcal{E}_{ijk} \cdot \Lambda_{ijk,h} + 4\mathcal{F}_{ij}^2 \cdot \Lambda_{ijk,h} + 6\mathcal{F}_{ij} \cdot \Lambda_{ijk,h}^2 +$$

$$+2\mathcal{R}_{hk} \cdot \mathcal{E}_{ijk} \cdot \Lambda_{ijk,h} + \mathcal{R}_{hk}^2 \cdot \Lambda_{ijk,h} + 3\mathcal{R}_{hk} \cdot \Lambda_{ijk,h}^2 + 6\mathcal{E}_{ijk} \cdot \Lambda_{ijk,h}^2 + 6\mathcal{F}_{ij} \cdot \Lambda_{ijk,h}^2 + 3\mathcal{R}_{hk} \cdot \Lambda_{ijk,h}^2 + 9\Lambda_{ijk,h}^3 =$$

$$= 0 + 4 + 2 - 6 + 4 + 0 - 6 + 2 + 0 - 3 - 6 - 6 - 3 + 18 = 0;$$

$$\widetilde{K}^2 \cdot \mathcal{R}_{ik} = \widetilde{K} \cdot \Bigg( 7\mathcal{H} \cdot \mathcal{R}_{ik} - 2 \sum_{\substack{1 \le x < y \le 3 \\ i \in \{x,y\}}} \mathcal{E}_{xyk} \cdot \mathcal{R}_{ik} - 2 \sum_{\substack{1 \le x < y \le 3 \\ k \in \{x,y\}}} \mathcal{E}_{xyi}' \cdot \mathcal{R}_{ik} - \mathcal{R}_{ik}^2 - 3 \sum_{\substack{1 \le x < y \le 3 \\ i \in \{x,y\}}} \Lambda_{xyk,i} \cdot \mathcal{R}_{ik} +$$

$$-3 \sum_{\substack{1 \le x < y \le 3 \\ k \in \{x,y\}}} \Lambda_{xyi,k}' \cdot \mathcal{R}_{ik} \Bigg) = 49\mathcal{H}^2 \cdot \mathcal{R}_{ik} - 7\mathcal{H} \cdot \mathcal{R}_{ik}^2 + 4 \sum_{\substack{1 \le x < y \le 3 \\ i \in \{x,y\}}} \mathcal{E}_{xyk}^2 \cdot \mathcal{R}_{ik} + 2 \sum_{\substack{1 \le x < y \le 3 \\ i \in \{x,y\}}} \mathcal{E}_{xyk} \cdot \mathcal{R}_{ik}^2 +$$

$$+6 \sum_{\substack{1 \le x < y \le 3 \\ i \in \{x,y\}}} \mathcal{E}_{xyk} \cdot \mathcal{R}_{ik} \cdot \Lambda_{xyk,i} + 4 \sum_{\substack{1 \le x < y \le 3 \\ k \in \{x,y\}}} \mathcal{E}_{xyi}'^2 \cdot \mathcal{R}_{ik} + 2 \sum_{\substack{1 \le x < y \le 3 \\ k \in \{x,y\}}} \mathcal{E}_{xyi}' \cdot \mathcal{R}_{ik}^2 + 6 \sum_{\substack{1 \le x < y \le 3 \\ k \in \{x,y\}}} \mathcal{E}_{xyi}' \cdot \mathcal{R}_{ik} \cdot \Lambda_{xyi,k}' +$$

$$-7\mathcal{H} \cdot \mathcal{R}_{ik}^2 + 2 \sum_{\substack{1 \le x < y \le 3 \\ i \in \{x,y\}}} \mathcal{E}_{xyk} \cdot \mathcal{R}_{ik}^2 + 2 \sum_{\substack{1 \le x < y \le 3 \\ k \in \{x,y\}}} \mathcal{E}_{xyi}' \cdot \mathcal{R}_{ik}^2 + 3 \sum_{\substack{1 \le x < y \le 3 \\ i \in \{x,y\}}} \mathcal{R}_{ik}^2 \cdot \Lambda_{xyk,i} + \mathcal{R}_{ik}^3 +$$

$$+6 \sum_{\substack{1 \le x < y \le 3 \\ i \in \{x,y\}}} \mathcal{E}_{xyk} \cdot \Lambda_{xyk,i} \cdot \mathcal{R}_{ik} + 9 \sum_{\substack{1 \le x < y \le 3 \\ i \in \{x,y\}}} \Lambda_{xyk,i}^2 \cdot \mathcal{R}_{ik} + 3 \sum_{\substack{1 \le x < y \le 3 \\ i \in \{x,y\}}} \Lambda_{xyk,i} \cdot \mathcal{R}_{ik}^2 +$$

$$+6 \sum_{\substack{1 \le x < y \le 3 \\ k \in \{x,y\}}} \mathcal{E}_{xyi}' \cdot \Lambda_{xyi,k}' \cdot \mathcal{R}_{ik} + 9 \sum_{\substack{1 \le x < y \le 3 \\ k \in \{x,y\}}} \Lambda_{xyi,k}'^2 \cdot \mathcal{R}_{ik} + 3 \sum_{\substack{1 \le x < y \le 3 \\ k \in \{x,y\}}} \Lambda_{xyi,k}' \cdot \mathcal{R}_{ik}^2 =$$

$$= 0 + 7 - 8 - 4 + 12 - 8 - 4 + 12 + 7 - 4 - 4 + 0 + 6 + 12 - 18 + 0 + 12 - 18 + 0 = 0;$$

similarly one can compute $\widetilde{K}^2 \cdot \Gamma_i'$ and $\widetilde{K}^2 \cdot \Lambda_{ijk,h}'$.

**Remark 5.35.** Let $\widetilde{K}$ be a general element of $\widetilde{\mathcal{K}}$. By Remarks 5.29, 5.31 we have

$$\widetilde{K}^2\cdot\mathcal{F}_{12}=\widetilde{K}\cdot\Big(7\mathcal{H}\cdot\mathcal{F}_{12}-3\mathcal{E}\cdot\mathcal{F}_{12}-2\mathcal{E}_{121}\cdot\mathcal{F}_{12}-2\mathcal{E}_{122}\cdot\mathcal{F}_{12}-2\mathcal{E}_{123}\cdot\mathcal{F}_{12}-2\mathcal{F}_{12}^2-3\Lambda_{121,1}\cdot\mathcal{F}_{12}+$$

$$-3\Lambda_{121,2}\cdot\mathcal{F}_{12}-3\Lambda_{122,1}\cdot\mathcal{F}_{12}-3\Lambda_{122,2}\cdot\mathcal{F}_{12}-3\Lambda_{123,1}\cdot\mathcal{F}_{12}-3\Lambda_{123,2}\cdot\mathcal{F}_{12}-4\Gamma_1\cdot\mathcal{F}_{12}-4\Gamma_2\cdot\mathcal{F}_{12}\Big)=$$

$$=49\mathcal{H}^2\cdot\mathcal{F}_{12}-14\mathcal{H}\cdot\mathcal{F}_{12}^2+9\mathcal{E}^2\cdot\mathcal{F}_{12}+6\mathcal{E}\cdot\mathcal{F}_{12}^2+12\mathcal{E}\cdot\mathcal{F}_{12}\cdot\Gamma_1+12\mathcal{E}^2\cdot\mathcal{F}_{12}\cdot\Gamma_2+$$

$$+4\mathcal{E}_{121}^2\cdot\mathcal{F}_{12}+4\mathcal{E}_{122}^2\cdot\mathcal{F}_{12}+4\mathcal{E}_{123}^2\cdot\mathcal{F}_{12}+4\mathcal{E}_{121}\cdot\mathcal{F}_{12}^2+4\mathcal{E}_{122}\cdot\mathcal{F}_{12}^2+4\mathcal{E}_{123}\cdot\mathcal{F}_{12}^2+$$

$$+6\mathcal{E}_{121}\cdot\mathcal{F}_{12}\cdot\Lambda_{121,1}+6\mathcal{E}_{121}\cdot\mathcal{F}_{12}\cdot\Lambda_{121,2}+6\mathcal{E}_{122}\cdot\mathcal{F}_{12}\cdot\Lambda_{122,1}+6\mathcal{E}_{122}\cdot\mathcal{F}_{12}\cdot\Lambda_{122,2}+6\mathcal{E}_{123}\cdot\mathcal{F}_{12}\cdot\Lambda_{123,1}+$$

$$+6\mathcal{E}_{123}\cdot\mathcal{F}_{12}\cdot\Lambda_{123,2}-14\mathcal{F}_{12}^2\cdot\mathcal{H}+6\mathcal{F}_{12}^2\cdot\mathcal{E}+4\mathcal{F}_{12}^2\cdot\mathcal{E}_{121}+4\mathcal{F}_{12}^2\cdot\mathcal{E}_{122}+4\mathcal{F}_{12}^2\cdot\mathcal{E}_{123}+$$

$$+4\mathcal{F}_{12}^3+9\Lambda_{121,1}^2\cdot\mathcal{F}_{12}+9\Lambda_{121,2}^2\cdot\mathcal{F}_{12}+9\Lambda_{122,1}^2\cdot\mathcal{F}_{12}+9\Lambda_{122,2}^2\cdot\mathcal{F}_{12}+9\Lambda_{123,1}^2\cdot\mathcal{F}_{12}+9\Lambda_{123,2}^2\cdot\mathcal{F}_{12}+$$

$$+6\Lambda_{121,1}\cdot\mathcal{F}_{12}^2+6\Lambda_{121,2}\cdot\mathcal{F}_{12}^2+6\Lambda_{122,1}\cdot\mathcal{F}_{12}^2+6\Lambda_{122,2}\cdot\mathcal{F}_{12}^2+6\Lambda_{123,1}\cdot\mathcal{F}_{12}^2+6\Lambda_{123,2}\cdot\mathcal{F}_{12}^2+$$

$$+6\Lambda_{121,1}\cdot\mathcal{F}_{12}\cdot\mathcal{E}_{121}+6\Lambda_{121,2}\cdot\mathcal{F}_{12}\cdot\mathcal{E}_{121}+6\Lambda_{122,1}\cdot\mathcal{F}_{12}\cdot\mathcal{E}_{122}+6\Lambda_{122,2}\cdot\mathcal{F}_{12}\cdot\mathcal{E}_{122}+6\Lambda_{123,1}\cdot\mathcal{F}_{12}\cdot\mathcal{E}_{123}+$$

$$+6\Lambda_{123,2}\cdot\mathcal{F}_{12}\cdot\mathcal{E}_{123}+16\Gamma_1^2\cdot\mathcal{F}_{12}+16\Gamma_2^2\cdot\mathcal{F}_{12}+8\Gamma_1\cdot\mathcal{F}_{12}^2+8\Gamma_2\cdot\mathcal{F}_{12}^2+12\Gamma_1\cdot\mathcal{F}_{12}\cdot\mathcal{E}+12\Gamma_2\cdot\mathcal{F}_{12}\cdot\mathcal{E}=$$

$$=0+14-18-6+12+12-8-8-8-4-4-4+6+6+6+6+6+6+14-6-4-4-4+$$

$$+24-9-9-9-9-9-9+0+0+0+0+0+0+6+6+6+6+6+6-16-16+0+0+12+12=8.$$

Similarly we obtain that $\widetilde{K}^2\cdot\mathcal{F}_{ij}=\widetilde{K}^2\cdot\mathcal{F}'_{ij}=8>0$ for $0\le i<j\le 3$. Thus the curves $\widetilde{K}\cap\mathcal{F}_{ij}$ and $\widetilde{K}\cap\mathcal{F}'_{ij}$ are not contracted by the rational map defined by $\widetilde{\mathcal{K}}|_{\widetilde{K}}$.

**Remark 5.36.** Let us fix a general element $\widetilde{K}\in\widetilde{\mathcal{K}}$ and let us take $S:=\nu_{\widetilde{\mathcal{K}}}(\widetilde{K})$ and $K'':=bl'''(\widetilde{K})\in\mathcal{K}''$. Since $bl''':Y\to Y''$ has no effect on $K''$, then $\widetilde{K}\cap\Gamma_i$, $\widetilde{K}\cap\Gamma'_i$, $\widetilde{K}\cap\Lambda_{ijk,h}$, $\widetilde{K}\cap\Lambda'_{ijk,h}$ are still $(-1)$-curves on $\widetilde{K}$, for all $i,j,k\in\{1,2,3\}$ with $i<j$ and $h\in\{i,j\}$ (see Remark 5.27). By Remarks 5.29, 5.31, we also have that $(\widetilde{K}\cap\mathcal{R}_{ik})|^2_{\widetilde{K}}=\mathcal{R}_{ik}^2\cdot\widetilde{K}=\mathcal{R}_{ik}^2\cdot(7\mathcal{H}-2\sum_{\substack{1\le a<b\le 3,\, i\in\{a,b\}\\ 1\le x<y\le 3,\, k\in\{x,y\}}}(\mathcal{E}_{abk}+\mathcal{E}'_{xyi})-\mathcal{R}_{ik})=-5$. Furthermore we have that $\mathcal{R}_{ik}\cdot_{\widetilde{K}}\Lambda_{abk,i}=1$ and $\mathcal{R}_{ik}\cdot_{\widetilde{K}}\Lambda'_{xyi,k}=1$ for $1\le a<b\le 3$ and $1\le x<y\le 3$ with $i\in\{a,b\}$ and $k\in\{x,y\}$ (use Remark 5.29). Thus we can see the map $\nu_{\widetilde{\mathcal{K}}}|_{\widetilde{K}}:\widetilde{K}\to S$ as the blow-up of $S$ at the six points $\nu_{\widetilde{\mathcal{K}}}(\widetilde{K}\cap\Gamma_i)$ and $\nu_{\widetilde{\mathcal{K}}}(\widetilde{K}\cap\Gamma'_i)$, at the nine points $\nu_{\widetilde{\mathcal{K}}}(\widetilde{K}\cap\mathcal{R}_{ik})$ and at the four points $\nu_{\widetilde{\mathcal{K}}}(\widetilde{K}\cap\Lambda_{abk,i})$, $\nu_{\widetilde{\mathcal{K}}}(\widetilde{K}\cap\Lambda'_{xyi,k})$ which are infinitely near to each $\nu_{\widetilde{\mathcal{K}}}(\widetilde{K}\cap\mathcal{R}_{ik})$ (see Remarks 5.23, 5.32, 5.33, 5.34, 5.35). Then $S$ is a smooth surface.

**Remark 5.37.** The surface $T\cup T'$ is the only sextic surface of $\mathbb{P}^3$ which is singular along the edges of the two trihedra. Let us consider the strict transforms $\widetilde{T}$ and $\widetilde{T}'$ on $Y$ of the trihedra:

$$\widetilde{T}\sim 3\mathcal{H}-3\mathcal{E}-\sum_{\substack{i,j,k=1\\ i<j}}^{3}(2\mathcal{E}_{ijk}+\mathcal{E}'_{ijk})-\sum_{1\le i<j\le 3}2\mathcal{F}_{ij}-\sum_{i,j=1}^{3}\mathcal{R}_{ij}+$$

$$-\sum_{i=1}^{3} 4\Gamma_i - \sum_{\substack{i,j,k\in\{1,2,3\}\\ i<j,\ h=i,j}} (3\Lambda_{ijk,h} + \Lambda'_{ijk,h}),$$

$$\widetilde{T}' \sim 3\mathcal{H} - 3\mathcal{E}' - \sum_{\substack{i,j,k=1\\ i<j}}^{3} (\mathcal{E}_{ijk} + 2\mathcal{E}'_{ijk}) - \sum_{i=1}^{3} 2\mathcal{F}'_{ij} - \sum_{i,j=1}^{3} \mathcal{R}_{ij} +$$

$$-\sum_{i=1}^{3} 4\Gamma'_i - \sum_{\substack{i,j,k\in\{1,2,3\}\\ i<j,\ h=i,j}} (\Lambda_{ijk,h} + 3\Lambda'_{ijk,h}).$$

Let $\widetilde{K}$ be a general element of $\widetilde{\mathcal{K}}$. Then we have that

$$0 \sim (\widetilde{T} + \widetilde{T}')|_{\widetilde{K}} \sim \Big(6\mathcal{H} - \sum_{1\le i<j\le 3} (2\mathcal{F}_{ij} + 2\mathcal{F}'_{ij}) - \sum_{i,j=1}^{3} 2\mathcal{R}_{ij} +$$

$$-\sum_{i=1}^{3} (4\Gamma_i + 4\Gamma'_i) - \sum_{\substack{i,j,k\in\{1,2,3\}\\ i<j,\ h=i,j}} (4\Lambda_{ijk,h} + 4\Lambda'_{ijk,h})\Big)|_{\widetilde{K}}.$$

**Theorem 5.38.** Let $S$ be a general hyperplane section of the threefold $W_F^9 \subset \mathbb{P}^9$. Then $S$ is an Enriques surface.

*Proof.* We recall that $S$ is the image of a general element $\widetilde{K} \in \widetilde{\mathcal{K}}$ via the birational morphism $\nu_{\widetilde{\mathcal{K}}} : Y \to W_F^9 \subset \mathbb{P}^9$. Furthermore $S$ is smooth (see Remark 5.36). By Proposition 5.26 we have that $p_g(\widetilde{K}) - q(\widetilde{K}) = p_a(\widetilde{K}) = 0$. Let us consider the following exact sequence

$$0 \to \mathcal{O}_Y(-\widetilde{K}) \to \mathcal{O}_Y \to \mathcal{O}_{\widetilde{K}} \to 0.$$

Since $Y$ is a smooth rational threefold and $\widetilde{K}$ is a big and nef divisor on $Y$, by Serre Duality and by the Kawamata-Viehweg vanishing theorem we have $h^i(Y, \mathcal{O}_Y(-\widetilde{K})) = 0$ for $i = 1, 2$, and so $q(\widetilde{K}) = h^1(\widetilde{K}, \mathcal{O}_{\widetilde{K}}) = h^1(Y, \mathcal{O}_Y) = 0$. Thus we also obtain $p_g(\widetilde{K}) = 0$. It remains to prove that $2K_S \sim 0$. Since

$$K_Y = bl'''^*(K_{Y''}) + \sum_{i=1}^{3}(\Gamma_i + \Gamma'_i) + \sum_{\substack{i,j,k=1\\ i<j,\ h=i,j}} (\Lambda_{ijk,h} + \Lambda'_{ijk,h}) \sim -4\mathcal{H} + 2\mathcal{E} + 2\mathcal{E}' +$$

$$+2\sum_{\substack{i,j,k=1\\ i<j}}^{3} (\mathcal{E}_{ijk}+\mathcal{E}'_{ijk}) + \sum_{lei<j\le 3} (\mathcal{F}_{ij}+\mathcal{F}'_{ij}) + \sum_{i,j=1}^{3} \mathcal{R}_{ij} + \sum_{i=1}^{3} 3(\Gamma_i+\Gamma'_i) + \sum_{\substack{i,j,k=1\\ i<j,\ h=i,j}}^{3} 3(\Lambda_{ijk,h}+\Lambda'_{ijk,h})$$

(see [27, p.187]), then, by the adjunction formula, we obtain that

$$2K_{\widetilde{K}} = 2(K_Y + \widetilde{K})|_{\widetilde{K}} \sim \Big(6\mathcal{H} - \sum_{1\le j\le 3} 2(\mathcal{F}_{ij} + \mathcal{F}'_{ij}) - \sum_{i=1}^{3} 2(\Gamma_i + \Gamma'_i)\Big)|_{\widetilde{K}}.$$

Furthermore, by Remark 5.37, we have

$$2K_{\widetilde{K}} \sim \Big(\widetilde{T}+\widetilde{T}'+\sum_{i,j=1}^{3}2\mathcal{R}_{ij}+\sum_{i=1}^{3}2(\Gamma_i+\Gamma_i')+\sum_{\substack{i,j,k=1\\ i<j,\ h=i,j}}^{3}4(\Lambda_{ijk,h}+\Lambda'_{ijk,h})\Big)|_{\widetilde{K}} \sim$$

$$\sim \Big(\sum_{i,j=1}^{3}2\mathcal{R}_{ij}+\sum_{i=1}^{3}2(\Gamma_i+\Gamma_i')+\sum_{\substack{i,j,k=1\\ i<j,\ h=i,j}}^{3}4(\Lambda_{ijk,h}+\Lambda'_{ijk,h})\Big)|_{\widetilde{K}} =$$

$$= \Big(\sum_{i,k=1}^{3}2\Big(\mathcal{R}_{ik}+\sum_{\substack{a,b,x,y\in\{1,2,3\}\\ a<b,\,x<y\\ i\in\{a,b\},\ k\in\{x,y\}}}(\Lambda_{abk,i}+\Lambda'_{xyi,k})\Big)+\sum_{i=1}^{3}2(\Gamma_i+\Gamma_i')+\sum_{\substack{i,j,k=1\\ i<j,\ h=i,j}}^{3}2(\Lambda_{ijk,h}+\Lambda'_{ijk,h})\Big)|_{\widetilde{K}}.$$

Finally, by Remark 5.36, we obtain $2K_S \sim (\nu_{\widetilde{S}})_*(2K_{\widetilde{K}}) \sim 0$. □

One can prove that $W_F^9 \subset \mathbb{P}^9$ is not a cone over a general hyperplane section, as in the proof of Theorem 5.15. So $W_F^9 \subset \mathbb{P}^9$ satisfies the Assumption (*) of § 3.3. Furthermore, if $p$ is the genus of a curve section of $W_F^9$, we have that $16 = 2p-2$ by the adjunction formula. Then $W_F^9$ is an Enriques-Fano threefold of genus $p=9$, since $W_F^9 \subset \mathbb{P}^9$ is (projectively) normal by Theorem 3.8 and Proposition 3.11.

□

### 5.3.2 Singularities of $W_F^9$

We recall that the eight divisors $\mathcal{E}$, $\mathcal{E}'$, $\widetilde{f}_1$, $\widetilde{f}_2$, $\widetilde{f}_3$, $\widetilde{f}_1'$, $\widetilde{f}_2'$, $\widetilde{f}_3'$ are contracted by $\nu_{\widetilde{\mathcal{K}}} : Y \to W_F^9 \subset \mathbb{P}^9$ to points of $W_F^9$ (see Remarks 5.32, 5.33). Let us define

$$P_1 := \nu_{\widetilde{\mathcal{K}}}(\mathcal{E}'),\ P_2 := \nu_{\widetilde{\mathcal{K}}}(\widetilde{f}_1),\ P_3 := \nu_{\widetilde{\mathcal{K}}}(\widetilde{f}_2),\ P_4 := \nu_{\widetilde{\mathcal{K}}}(\widetilde{f}_3),$$

$$P_1' := \nu_{\widetilde{\mathcal{K}}}(\mathcal{E}),\ P_2' := \nu_{\widetilde{\mathcal{K}}}(\widetilde{f}_1'),\ P_3' := \nu_{\widetilde{\mathcal{K}}}(\widetilde{f}_2'),\ P_4' := \nu_{\widetilde{\mathcal{K}}}(\widetilde{f}_3').$$

**Lemma 5.39.** The 18 divisors $\mathcal{E}_{ijk}$ and $\mathcal{E}'_{ijk}$ are mapped by $\nu_{\widetilde{\mathcal{K}}} : Y \to W_F^9 \subset \mathbb{P}^9$ to the six points $P_2$, $P_3$, $P_4$, $P_2'$, $P_3'$ and $P_4'$ of $W_F^9$ in the following way:

$$P_{i+1} = \nu_{\widetilde{\mathcal{K}}}(\widetilde{f}_i) = \nu_{\widetilde{\mathcal{K}}}(\mathcal{E}'_{rsi}), \quad P'_{i+1} = \nu_{\widetilde{\mathcal{K}}}(\widetilde{f}_i') = \nu_{\widetilde{\mathcal{K}}}(\mathcal{E}_{rsi}),$$

for all $i,r,s \in \{1,2,3\}$ and $r<s$.

*Proof.* By Remark 5.32 we have that $\nu_{\widetilde{\mathcal{K}}}(\mathcal{E}_{ijk})$ and $\nu_{\widetilde{\mathcal{K}}}(\mathcal{E}'_{ijk})$ are points of $W_F^9$ for all $i,j,k \in \{1,2,3\}$ and $i<j$. Since $\widetilde{f}_i \cap \mathcal{E}'_{rsi} \neq \emptyset$ for all $i,r,s \in \{1,2,3\}$ and $r<s$, then the three divisors $\mathcal{E}'_{rsi}$ are mapped to the same point $\nu_{\widetilde{\mathcal{K}}}(\widetilde{f}_i) = P_{i+1}$. Similarly the three divisors $\mathcal{E}_{rsi}$ are mapped to the same point $\nu_{\widetilde{\mathcal{K}}}(\widetilde{f}_i') = P'_{i+1}$. □

**Proposition 5.40.** The points $P_1,\dots,P_4,P_1',\dots,P_4'$ are eight quadruple points of $W_F^9$ whose tangent cone is a cone over a Veronese surface.

*Proof.* The analysis of the points $P'_1$ and $P_1$ follows by Remark 5.30 as in the proof of Proposition 5.16. Let us fix now $1 \leq i \leq 3$. Let us find the tangent cone to $W^9_F$ at $P_{i+1}$. Similarly one can study the tangent cone to $W^9_F$ at $P'_{i+1}$. The hyperplane sections of $W^9_F \subset \mathbb{P}^9$ passing through $P_{i+1}$ correspond to the elements of $\widetilde{\mathcal{K}}$ containing $\widetilde{f}_i \cup \mathcal{E}'_{12i} \cup \mathcal{E}'_{13i} \cup \mathcal{E}'_{23i}$ (see Lemma 5.39). Let $\widetilde{\mathcal{K}}_i := \widetilde{\mathcal{K}} - \widetilde{f}_i - \mathcal{E}'_{12i} - \mathcal{E}'_{13i} - \mathcal{E}'_{23i}$ be the sublinear system of $\widetilde{\mathcal{K}}$ defined by these elements. Let us study $\widetilde{\mathcal{K}}_i|_{\widetilde{f}_i} = |\mathcal{O}_{\widetilde{f}_i}(-\widetilde{f}_i - \mathcal{E}'_{12i} - \mathcal{E}'_{13i} - \mathcal{E}'_{23i})|$. Let us consider the case $i = 1$. Since

$$\widetilde{f}_1 \sim_Y \mathcal{H} - \mathcal{E}_v - \sum_{j=1}^{3} \mathcal{E}_{13j} - \sum_{j=1}^{3} \mathcal{E}_{12j} - \sum_{1 \leq r < s \leq 3} \mathcal{E}'_{rs1} - \mathcal{F}_{13} - \mathcal{F}_{12} - \sum_{j=1}^{3} \mathcal{R}_{1j} - 2\Gamma_1 - \Gamma_2 - \Gamma_3 +$$

$$- \sum_{j=1}^{3} (2\Lambda_{13j,1} + \Lambda_{13j,3}) - \sum_{j=1}^{3} (2\Lambda_{12j,1} + \Lambda_{12j,2}) - \sum_{1 \leq r < s \leq 3} (\Lambda'_{rs1,r} + \Lambda'_{rs1,s}),$$

we have that

$$\widetilde{f}_1|_{\widetilde{f}_1} \sim_{\widetilde{f}_1} \left( \mathcal{H} - \sum_{1 \leq r < s \leq 3} \mathcal{E}'_{rs1} - \mathcal{F}_{13} - \mathcal{F}_{12} - \sum_{j=1}^{3} \mathcal{R}_{1j} - 2\Gamma_1 - \sum_{j=1}^{3} 2\Lambda_{13j,1} - \sum_{j=1}^{3} 2\Lambda_{12j,1} \right) |_{\widetilde{f}_1}.$$

Let $\mathcal{L}_1$ be the pullback on $\widetilde{f}_1$ of the linear equivalence class of the lines of the face $f_1 \cong \mathbb{P}^2$. By abuse of notation, let us denote by $\widetilde{\gamma}_1$, $\widetilde{\lambda}_{131,1}$, $\widetilde{\lambda}_{132,1}$, $\widetilde{\lambda}_{133,1}$, $\widetilde{\lambda}_{121,1}$, $\widetilde{\lambda}_{122,1}$, $\widetilde{\lambda}_{123,1}$ the $(-1)$-curves on $\widetilde{f}_1$ given by $\Gamma_1|_{\widetilde{f}_1}$, $\Lambda_{131,1}|_{\widetilde{f}_1}$, $\Lambda_{132,1}|_{\widetilde{f}_1}$, $\Lambda_{133,1}|_{\widetilde{f}_1}$, $\Lambda_{121,1}|_{\widetilde{f}_1}$, $\Lambda_{122,1}|_{\widetilde{f}_1}$, $\Lambda_{123,1}|_{\widetilde{f}_1}$. Let us also consider the $(-1)$-curves on $\widetilde{f}_1$ defined by $\epsilon_{rs1} := \mathcal{E}'_{rs1}|_{\widetilde{f}_1}$ for $1 \leq r < s \leq 3$. Then we have $\widetilde{f}_1|_{\widetilde{f}_1} \sim \mathcal{L}_1 - \sum_{1 \leq r < s \leq 3} \epsilon'_{rs1} - (\mathcal{L}_1 - \widetilde{\gamma}_1 - \sum_{j=1}^{3} \widetilde{\lambda}_{13j,1}) - (\mathcal{L}_1 - \widetilde{\gamma}_1 - \sum_{j=1}^{3} \widetilde{\lambda}_{12j,1}) - (3\mathcal{L}_1 - \sum_{1 \leq r < s 3} 2\epsilon'_{rs1} - \sum_{j=1}^{3} \widetilde{\lambda}_{13j,1} - \sum_{j=1}^{3} \widetilde{\lambda}_{12j,1}) - 2\widetilde{\gamma}_1 - \sum_{j=1}^{3} 2\widetilde{\lambda}_{13j,1} - \sum_{j=1}^{3} 2\widetilde{\lambda}_{12j,1} = -4\mathcal{L}_1 + \sum_{1 \leq r < s \leq 3} \epsilon'_{rs1}$. Similarly $\widetilde{f}_i|_{\widetilde{f}_i} \sim -4\mathcal{L}_i + \sum_{1 \leq r < s \leq 3} \epsilon'_{rsi}$, for $i = 2, 3$. Thus we obtain $\widetilde{\mathcal{K}}_i|_{\widetilde{f}_i} = |\mathcal{O}_{\widetilde{f}_i}(4\mathcal{L}_i - \sum_{1 \leq r < s \leq 3} 2\epsilon'_{rsi})|$, which is isomorphic to the linear system of the quartic plane curves on $f_i$ with double points at the three points $q'_{rsi} = l'_{rs} \cap f_i$ for $1 \leq r < s \leq 3$. By applying a quadratic transformation, we obtain that $\widetilde{\mathcal{K}}_i|_{\widetilde{f}_i} \cong |\mathcal{O}_{\mathbb{P}^2}(2)|$, whose image is a Veronese surface $V_i$. Furthermore we have that $\widetilde{\mathcal{K}}_i|_{\mathcal{E}'_{rsi}} = |\mathcal{O}_{\mathcal{E}'_{rsi}}(-\widetilde{f}_i - \mathcal{E}'_{rsi})| = |\mathcal{O}_{\mathcal{E}'_{rsi}}(2\mathcal{L}'_{rsi} - 2\widetilde{\alpha}'_{rsi})| \cong \mathbb{P}^2$ for $1 \leq r < s \leq 3$ (see Remark 5.30). Since $\widetilde{\mathcal{K}}_i|_{\mathcal{E}'_{rsi}}$ is isomorphic to the linear system of the conics of $E'_{rsi}$ with node at the point $E'_{rsi} \cap \widetilde{l}'_{ij}$, then its image is a conic $C'_{rsi}$. Since $V_i \cup C'_{12i} \cup C'_{13i} \cup C'_{23i} = \mathbb{P}(TC_{P_{i+1}} W^9_F)$, then it must be $C'_{12i}, C'_{13i}, C'_{23i} \subset V_i = \mathbb{P}(TC_{P_{i+1}} W^9_F)$. Therefore $\widetilde{f}_i$ is contracted by $\nu_{\widetilde{\mathcal{K}}}$ to the point $P_{i+1}$, which is a quadruple point whose tangent cone tangent is a cone over a Veronese surface, and the divisors $\mathcal{E}'_{12i}$, $\mathcal{E}'_{13i}$, $\mathcal{E}'_{23i}$ are contracted in three conics contained in the Veronese surface given by the exceptional divisor of the minimal resolution of $P_{i+1}$. $\square$

We recall that $\nu_{\mathcal{K}} : \mathbb{P}^3 \dashrightarrow W^9_F \subset \mathbb{P}^9$ is an isomorphism outside $T \cup T'$ (see Remark 5.23). Then $P_1$, $P_2$, $P_3$, $P_4$, $P'_1$, $P'_2$, $P'_3$ and $P'_4$ are the only singular points of $W^9_F$ (see Remarks 5.33, 5.34, 5.35). Furthermore $\nu_{\widetilde{\mathcal{K}}} : Y \to W^9_F$ is a desingularization of $W^9_F$ but it is not the minimal one: indeed the proof of Proposition 5.40 says us that

$\nu_{\widetilde{\mathcal{K}}} : Y \to W_F^9$ is the blow-up of the minimal desingularization of $W_F^9$ along curves (conics) contained in the minimal resolutions of $P_2$, $P_3$, $P_4$, $P_2'$, $P_3'$ and $P_4'$. Finally, by recalling Definition 4.4, we have the following result.

**Theorem 5.41.** Each singular point of $W_F^9$ is associated with *at least* $m = 4$ of the other singular points.

*Proof.* We know that the 42 exceptional divisors of $bl''' : Y \to Y''$ are mapped by $\nu_{\widetilde{\mathcal{K}}} : Y \to W_F^9 \subset \mathbb{P}^9$ to curves of $W_F^9$ (see Remark 5.34). In particular they are mapped to lines of $W_F^9$ (use similar arguments to the ones in the proof of Theorem 5.17). Since $\Gamma_i \cap \mathcal{E} \neq \emptyset$ and $\Gamma_i \cap \widetilde{f}_i \neq \emptyset$ for $1 \leq i \leq 3$, we have that $P_1'$ is associated with $P_{i+1}$. Similarly $P_1$ is associated with $P_{i+1}'$. One can verify that the other 36 exceptional divisors are mapped to nine lines in the following way:

$$\langle P_{i+1}, P_{j+1}' \rangle = \langle \nu_{\widetilde{\mathcal{K}}}(\mathcal{E}_{rsi}'), \nu_{\widetilde{\mathcal{K}}}(\mathcal{E}_{khj}) \rangle = \nu_{\widetilde{\mathcal{K}}}(\Lambda_{rsi,j}') = \nu_{\widetilde{\mathcal{K}}}(\Lambda_{khj,i})$$

for $i, j, r, s, k, h \in \{1, 2, 3\}$ and $r < s$ and $k < h$. So $P_2$, $P_3$, $P_4$, $P_2'$, $P_3'$, $P_4'$ are associated with each other as in Figure 24. It remains to show that $P_1 = \nu_{\widetilde{\mathcal{K}}}(\mathcal{E}')$ is associated with $P_1' = \nu_{\widetilde{\mathcal{K}}}(\mathcal{E})$. Let us consider the line $l_{vv'} := \langle v, v' \rangle \subset \mathbb{P}^3$ joining the two vertices of the trihedra $T$ and $T'$. Let $\widetilde{l}_{vv'}$ be its strict transform on $Y$. We obtain that $\nu_{\widetilde{\mathcal{K}}}(\widetilde{l}_{vv'}) = \langle P_1, P_1' \rangle \subset W_F^9$, since $\widetilde{l}_{vv'} \cap \mathcal{E} \neq \emptyset$, $\widetilde{l}_{vv'} \cap \mathcal{E}' \neq \emptyset$ and $\deg(\nu_{\widetilde{\mathcal{K}}}(\widetilde{l}_{vv'})) = \widetilde{K} \cdot (\mathcal{H} - \mathcal{E} - \mathcal{E}' - \sum_{i=1}^3 \Gamma_i - \sum_{i=1}^3 \Gamma_i')^2 = 1$. □

**Remark 5.42.** Thanks to a computational anaylisis with Macaulay2, we can say that each singular point of $W_F^9$ is associated with *exactly* $m = 4$ of the other singular points, as in Figure 24 of Appendix A. This follows by Remark 6.10 since the embedding of the BS-EF 3-fold $W_{BS}^9$ in $\mathbb{P}^9$ is the F-EF 3-fold $W_F^9$ (see Theorem 6.11).

## 5.4 F-EF 3-fold of genus 7

### 5.4.1 Construction of $W_F^7$

Let us take a tetrahedron $T = \bigcup_{i=0}^3 f_i \subset \mathbb{P}^3$ and let us denote by $v_i$ the vertex opposite to the face $f_i$, for $0 \leq i \leq 3$. Let $l_{ij}$ be the edge $f_i \cap f_j$, for $0 \leq i < j \leq 3$. Furthermore let us fix a general plane $\pi$ of $\mathbb{P}^3$. The plane $\pi$ intersects each edge $l_{ij}$ of $T$ at one point, which is denoted by $p_{ij} := l_{ij} \cap \pi$. In the plane $\pi$ there is a 3-dimensional linear system of cubic curves passing through the six points $p_{ij}$ (see [17, §9.2.2]). Let us fix a general element $\delta$ of this linear system (Figure 7): it is an elliptic smooth cubic plane curve. Let us consider the linear system $\mathcal{X}$ of the sextic surfaces in $\mathbb{P}^3$ double along the six edges of the tetrahedron $T$ and containing the cubic plane curve $\delta$.

**Proposition 5.43.** The linear system $\mathcal{X}$ defined as above has $\dim \mathcal{X} = 7$.

*Proof.* The linear system $\mathcal{X}$ is a sublinear system of the linear system $\mathcal{S}$ of the sextic surfaces double along the six edges of $T$. In particular we have that $\mathcal{S} = |\mathcal{I}_{\gamma^2|\mathbb{P}^3}(6)|$ and $\mathcal{X} = |\mathcal{I}_{\gamma^2 \cup \delta|\mathbb{P}^3}(6)|$, where $\gamma$ is the sextic reducible curve given by the union of the edges of $T$. We also have that $\mathcal{S}$ cuts on $\delta$ a complete linear system $|\mathcal{O}_\delta(D)|$. Indeed we recall that $\mathcal{S}$ contains a sublinear system whose fixed part is given by the tetrahedron

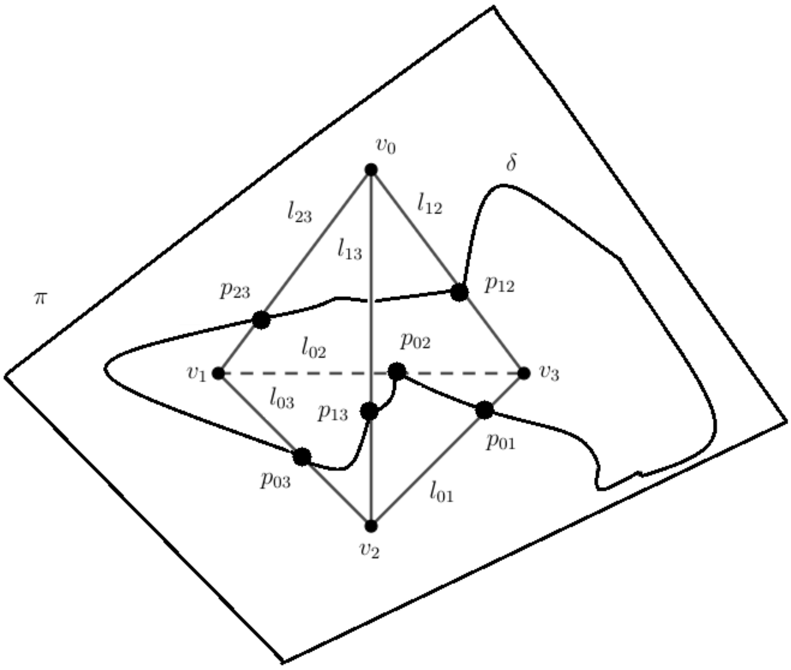


*Figure 7:* *Cubic plane curve $\delta$ in a general plane $\pi$ intersecting each edge of the tetrahedron $T$ at one point.*

and whose movable part, given by the quadric surfaces of $\mathbb{P}^3$, cuts a complete linear system on $\delta$. Hence we have the following exact sequence

$$0 \to H^0(\mathcal{I}_{\gamma^2 \cup \delta|\mathbb{P}^3}(6)) \to H^0(\mathcal{I}_{\gamma^2|\mathbb{P}^3}(6)) \to H^0(\mathcal{O}_\delta(D)) \to 0.$$

Let $\Sigma$ be a general element of $\mathcal{S}$. The cubic plane curve $\delta$ intersects $\Sigma$, outside the base locus of $\mathcal{S}$, in $3 \cdot 6 - 2 \cdot 6 = 6$ points. Hence $\deg D = 6$. We recall that $\dim H^0(\mathbb{P}^3, \mathcal{I}_{\gamma^2|\mathbb{P}^3}(6)) = \dim \mathcal{S} + 1 = 14$. Since $\deg D = 6 > 2p_g(\delta) - 2 = 0$, then $\dim H^1(\delta, \mathcal{O}_\delta(D)) = 0$ (see [29, Example 1.3.4]) and we have $\dim H^0(\delta, \mathcal{O}_\delta(D)) = 6$ by Riemann-Roch. So the above exact seguence implies that

$$\dim \mathcal{X} = \dim H^0(\mathbb{P}^3, \mathcal{I}_{\gamma^2 \cup \delta|\mathbb{P}^3}(6)) - 1 = 14 - 6 - 1 = 7.$$

$\square$

**Remark 5.44.** Let $\nu_{\mathcal{S}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{13}$ be the rational map defined by the linear system $\mathcal{S}$ of the sextic surfaces of $\mathbb{P}^3$ singular along the edges of $T$, whose image is the F-EF 3-fold $W_F^{13}$. Let $W_F^7$ be the image of $\mathbb{P}^3$ via the rational map defined by the linear system $\mathcal{X}$. Then $W_F^7$ is the projection of $W_F^{13}$ from the linear subspace of $\mathbb{P}^{13}$ spanned by the sextic elliptic curve $\nu_{\mathcal{S}}(\delta)$.

**Lemma 5.45.** The rational map $\nu_{\mathcal{X}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^7$ defined by $\mathcal{X}$ is a birational map onto the image.

*Proof.* Let $X$ be a general element of $\mathcal{X}$. The linear system $\mathcal{X}$ contains a sublinear system $\overline{\mathcal{X}} \subset \mathcal{X}$ whose fixed part is given by $T \cup \pi$ and such that $\overline{\mathcal{X}}|_X$ coincides with the linear system on $X$ cut out by the planes of $\mathbb{P}^3$. Then we obtain the birationality of the maps defined by $\mathcal{X}|_X$ and by $\mathcal{X}$. $\square$

**Remark 5.46.** The proof of Lemma 5.45 tells us that the linear system $\mathcal{X}$ is very ample outside the tetrahedron $T$ and the plane $\pi$. So $\nu_{\mathcal{X}} : \mathbb{P}^3 \dashrightarrow \nu_{\mathcal{X}}(\mathbb{P}^3) \subset \mathbb{P}^7$ is an isomorphism outside $T \cup \pi$.

The Proposition 5.43 proves the existence of sextic surfaces of $\mathbb{P}^3$ double along the edges of the tetrahedron $T$ (shortly, *Enriques sextics*) and containing a given cubic plane curve $\delta$. However, a priori, these surfaces could have further singularities and their desingularizations could be not Enriques surfaces. Let us study the surfaces of $\mathcal{X}$.

Up to a change of coordinates, we can consider in $\mathbb{P}^3_{[s_0:s_1:s_2:s_3]}$ the tetrahedron $T = \{s_0s_1s_2s_3 = 0\}$ with faces $f_i = \{s_i = 0\}$ for $0 \leq i \leq 3$. Let us fix the cubic plane curve $\widehat{\delta} := \{\sum_{i=0}^{3} s_i = 0, s_1^2s_2 + s_1s_2^2 + s_1^2s_3 + s_1s_2s_3 + s_2^2s_3 + s_1s_3^2 + s_2s_3^2 = 0\}$, which intersects the edges of $T$ at one point each. Thanks to Macaulay2, we can construct the linear system $\widehat{\mathcal{X}}$ of the sextic surfaces of $\mathbb{P}^3$ which are singular along the edges of $T$ and which contain the curve $\widehat{\delta}$ (see Code B.7 of Appendix B). Let us take $\widehat{X} := \{s_0^2s_1^2s_2^2 - s_0^3s_1s_2s_3 - s_0s_1^3s_2s_3 - 2s_0s_1s_2^3s_3 + s_0^2s_1^2s_3^2 + 2s_0^2s_2^2s_3^2 + s_0s_1s_2^2s_3^2 + 2s_1^2s_2^2s_3^2 - 2s_0s_1s_2s_3^3 = 0\} \in \widehat{\mathcal{X}}$. By the computational analysis, we see that $\widehat{X}$ has singular points only along the edges of $T$. In particular the tangent cone to $\widehat{X}$ at a vertex of $T$ is given by the union of the three faces of $T$ containing that vertex; the tangent cone to $\widehat{X}$ at a point $p \in l_{ij}$, with $p \neq v_k$ and $p \neq v_h$, is the union of two planes containing $l_{ij}$, where $i, j, k, h$ are four distinct indices in $\{0, 1, 2, 3\}$. Then $\widehat{X}$ has ordinary singularities along the edges of $T$ (see Definition 3.4) and no further singularities. The same happens for a general surface of $\widehat{\mathcal{X}}$. Let $\mathcal{D}$ be the family of the cubic plane curves of $\mathbb{P}^3$ intersecting the edges of $T$ at one point each. We have that $\mathcal{D}$ is an irreducible variety of dimension 6. Then what is true for the special cubic plane curve $\widehat{\delta} \in \mathcal{D}$ is also true for the general cubic plane curve $\delta \in \mathcal{D}$. Therefore there exist Enriques sextics in $\mathbb{P}^3$, with ordinary singularities along the edges of $T$ and no further singularities, that contain $\delta$. Let $X$ be such a general surface and let us take its minimal desingularization $n : X^\nu \to X$. It follows that $X^\nu$ is an Enriques surface (see [16, p.275]). Furthermore let $E$ be the strict transform of $\delta$ on $X^\nu$. Then $E$ is an elliptic curve such that $E \cdot H = 3$, where $H$ is the pullback of the general hyperplane section of $X$. If $\delta$ moved in a linear system on $X$, then $E$ would move in an elliptic pencil on $X^\nu$. Thus $E$ should be 2-divisible (see [2, Lemma 17.1]) and $E \cdot H = 3$ would be a contradiction. So $\delta$ does not move in any linear system on $X$ and, with a compute of parameters, one can see that the general Enriques sextic in $\mathbb{P}^3$ contains some cubic plane curve of $\mathcal{D}$. These arguments prove that the image of a general $X \in \mathcal{X}$ via the rational map $\nu_{\mathcal{X}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^7$ is an Enriques surface. Finally one can prove that $W_F^7 = \nu_{\mathcal{X}}(\mathbb{P}^3) \subset \mathbb{P}^7$ is not a cone over a general hyperplane section, as in the proof of Theorem 5.15. So $W_F^7 \subset \mathbb{P}^7$ satisfies Assumption (*) of § 3.3. Furthermore $\deg W_F^7 = 12$ (see Code B.7 of Appendix B) and, if $p$ is the genus of a curve section of $W_F^7$, we have that $12 = 2p - 2$ by the adjunction formula. Thus $W_F^7 \subset \mathbb{P}^7$ is (projectively) normal (see Theorem 3.8 and Proposition 3.11) and we obtain the following theorem.

**Theorem 5.47.** [23, §4] The image of $\mathbb{P}^3$ via the rational map defined by $\mathcal{X}$ is an Enriques-Fano threefold $W_F^7 \subset \mathbb{P}^7$ of genus $p = 7$.

By Remark 5.44, the above theorem also follows by [10, Lemma 4.4, Lemma 4.6].

### 5.4.2 Singularities of $W_F^7$

In order to describe the geometry and the singularities of $W_F^7$, we will use the techniques of the proof of Theorems 5.4, 5.24.

**Remark 5.48.** Let $X$ be a general element of $\mathcal{X}$ and let us take four distinct indices $i,j,k,h \in \{0,1,2,3\}$. As we said in § 5.4.1, we have that $TC_{v_i}X = f_j \cup f_k \cup f_h$ and, if $p \in l_{ij}$ with $p \neq v_k$ and $p \neq v_h$, we have that $TC_pX$ is the union of two variable planes $\pi_{p,X}, \pi'_{p,X} \in |\mathcal{I}_{l_{ij}|\mathbb{P}^3}(1)|$ depending on the choice of $p$ and of $X$ and which can also coincide. In particular if $p = p_{ij}$, then one of the two planes of $TC_{p_{ij}}X$ is tangent to $\delta$ at $p_{ij}$ and we will denote this plane by $\pi_{ij}$.

Let us blow-up $\mathbb{P}^3$ at the vertices of $T$ and at the six points $p_{ij}$, for $0 \leq i < j \leq 3$. We obtain a smooth threefold $Y'$ and a birational morphism $bl' : Y' \to \mathbb{P}^3$ with exceptional divisors $E_i := (bl')^{-1}(v_i)$, $E_{ij} := (bl')^{-1}(p_{ij})$. Let $\mathcal{X}'$ be the strict transform of $\mathcal{X}$ and let us denote by $H$ the pullback on $Y'$ of the hyperplane class on $\mathbb{P}^3$. Then an element of $\mathcal{X}'$ is linearly equivalent to $6H - 3\sum_{i=0}^3 E_i - 2\sum_{0\leq i<j\leq 3} E_{ij}$. Let $\widetilde{f_i}$ be the strict transform of the face $f_i$ and let $\widetilde{\pi}_{ij}$ the strict transform of the plane $\pi_{ij}$ defined in Remark 5.48, for $0 \leq i < j \leq 3$. We denote by $\gamma_{ki} := E_k \cap \widetilde{f_j}$ the line cut out by $\widetilde{f_i}$ on $E_k$ and by $\lambda_{ij} := E_{ij} \cap \widetilde{\pi}_{ij}$ the line cut out by $\widetilde{\pi}_{ij}$ on $E_{ij}$, for distinct indices $i,j,k \in \{0,1,2,3\}$ with $i < j$. We have that $\gamma_{ki}$ and $\lambda_{ij}$ are $(-1)$-curves respectively on $\widetilde{f_i}$ and $\widetilde{\pi}_{ij}$. Let $X'$ be the strict transform of a general $X \in \mathcal{X}$. By Remark 5.48 we have that $X' \cap E_k = \bigcup_{\substack{i=0\\ i\neq k}}^3 \gamma_{ki}$ for all $0 \leq k \leq 3$.

**Remark 5.49.** We observe that $X' \cap E_{ij} = \lambda_{ij} \cup \beta_{ij,X}$, where $\beta_{ij,X}$ moves in the pencil of the lines of $E_{ij}$ through the point $E_{ij} \cap \widetilde{l}_{ij}$ and it depends on the choice of $X$, for all $0 \leq i < j \leq 3$ (see Remark 5.48).

Let us take the strict transforms $\widetilde{l}_{ij}$ of the six edges of $T$ and the strict transform $\widetilde{\delta}$ of the cubic plane curve $\delta$. The base locus of $\mathcal{X}'$ is given by the union of the six curves $\widetilde{l}_{ij}$ (along which a general $X' \in \mathcal{X}$' has double points), of the curve $\widetilde{\delta}$, of the twelve curves $\gamma_{ij}$ and the six curves $\lambda_{ij}$ (see Remark 5.48). Let us blow-up the strict transforms of the edges of $T$ and of the cubic plane curve $\delta$. We obtain a smooth threefold $Y''$ and a birational morphism $bl'' : Y'' \to Y'$ with exceptional divisors $(bl'')^{-1}(\widetilde{\delta}) =: F_\delta \cong \mathbb{P}(\mathcal{N}_{\widetilde{\delta}|Y'})$ and $(bl'')^{-1}(\widetilde{l}_{ij}) =: F_{ij} \cong \mathbb{P}(\mathcal{N}_{\widetilde{l}_{ij}|Y'}) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) \cong \mathbb{F}_0$, for $0 \leq i < j \leq 3$.

**Remark 5.50.** The divisor $F_{\widetilde{\delta}}$ is a smooth elliptic ruled surface, since it is a $\mathbb{P}^1$-bundle over the elliptic curve $\widetilde{\delta}$. We also have that $\deg(\mathcal{N}_{\widetilde{\delta}|Y'}) = 0$. Indeed, since $\delta$ is the complete intersection of the plane $\pi$ and of a cubic surface passing through the points $p_{ij}$, then we have $\mathcal{N}_{\widetilde{\delta}|Y'} \cong \mathcal{O}_{\widetilde{\delta}}(H - \sum_{0\leq i<j\leq 3} E_{ij}) \oplus \mathcal{O}_{\widetilde{\delta}}(3H - \sum_{0\leq i<j\leq 3} E_{ij})$ and $\deg(\mathcal{N}_{\widetilde{\delta}|Y'}) = (3-6) + (9-6) = 0$.

Since $bl'' : Y'' \to Y'$ has no effect on $\widetilde{f_i}$, we will use the same symbols to indicate its strict transforms on $Y''$; furthermore let us denote by $\widetilde{E}_i$ and $\widetilde{E}_{ij}$ respectively the strict transforms of $E_i$ and $E_{ij}$, for $0 \leq i < j \leq 3$.

**Remark 5.51.** Let us take the curves $\alpha_{kij} := \widetilde{E}_k \cap F_{ij}$, $\alpha_{ij} := \widetilde{E}_{ij} \cap F_{ij}$, $\alpha'_{ij} := \widetilde{E}_{ij} \cap F_\delta$, for distinct indices $i, j, k \in \{0, 1, 2, 3\}$ with $i < j$ (see Figure 8). We have that $\alpha_{kij}$ is a $(-1)$-curve on $\widetilde{E}_k$ and a fibre on $F_{ij}$; $\alpha_{ij}$ is a $(-1)$-curve on $\widetilde{E}_{ij}$ and a fibre on $F_{ij}$; $\alpha'_{ij}$ is a $(-1)$-curve on $\widetilde{E}_{ij}$ and a fibre on $F_\delta$.

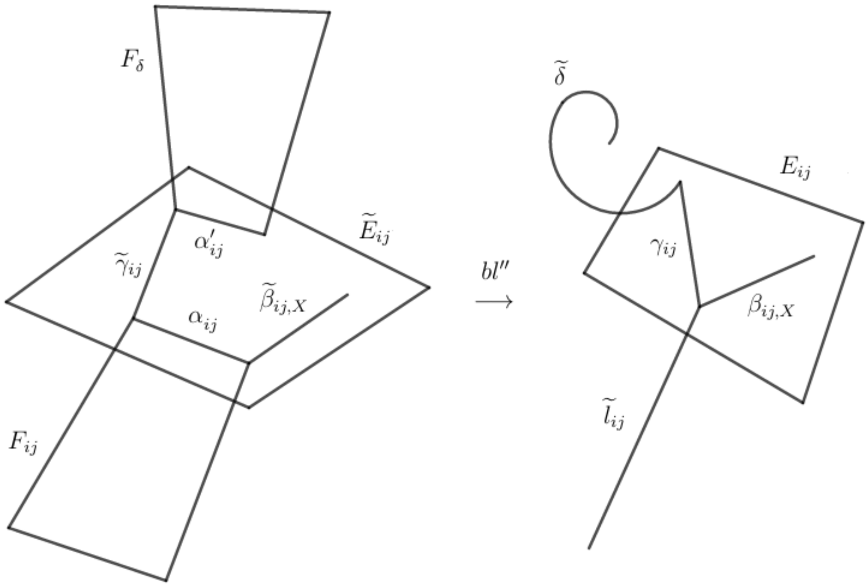


*Figure 8:* Description of $bl'''|_{\widetilde{E}_{ij}} : \widetilde{E}_{ij} \to E_{ij}$.

Let $\mathcal{X}''$ be the strict transform of $\mathcal{X}'$ and let $X''$ be an element of $\mathcal{X}''$. Then

$$X'' \sim 6H - 3\sum_{i=0}^{3} \widetilde{E}_i - 2 \sum_{0 \le i < j \le 3} E_{ij} - 2 \sum_{0 \le i < j \le 3} F_{ij} - F_\delta,$$

where, by abuse of notation, $H$ denotes the pullback $bl''^* H$. By Remark 5.48, we have that the base locus of $\mathcal{X}''$ is given by the disjoint union of the strict transforms $\widetilde{\gamma}_{ki}$ and $\widetilde{\lambda}_{ij}$ of the 18 lines $\gamma_{ki}$ and $\lambda_{ij}$, for distinct indices $i, j, k \in \{0, 1, 2, 3\}$ and $i < j$.

**Remark 5.52.** We have that $\widetilde{\gamma}_{ki}^2|_{\widetilde{E}_k} = -1$, $\widetilde{\gamma}_{ki}^2|_{\widetilde{f}_i} = -1$, $\widetilde{\lambda}_{ij}^2|_{\widetilde{E}_{ij}} = -1$. Furthermore, if $X''$ is the strict transform of a general $X' \in \mathcal{X}'$, we also have that the twelve $\widetilde{\gamma}_{ki}$ are $(-1)$-curves on $X''$ for $i, k \in \{0, 1, 2, 3\}$ and $i \neq k$ (see Remark 5.7). Finally we want to show that the 6 curves $\widetilde{\lambda}_{ij}$ are $(-1)$-curves on $X''$ too. We observe that $X'' \cap \widetilde{E}_{ij} = \widetilde{\lambda}_{ij} \cup \widetilde{\beta}_{ij,X''}$, where $\widetilde{\beta}_{ij,X''}$ is the strict transform of the curve of Remark 5.49, which moves in a pencil and depends on $X''$. Since $\widetilde{\lambda}_{ij}$ and $\widetilde{\beta}_{ij,X''}$ are disjoint, we have $(\widetilde{\lambda}_{ij})^2|_{X''} + (\widetilde{\beta}_{ij,X''})^2|_{X''} = (X'' \cap \widetilde{E}_{ij})^2|_{X''} = \widetilde{E}_{ij}^2 \cdot X'' = \widetilde{E}_{ij} \cdot (\widetilde{E}_{ij} \cdot X'') = \widetilde{E}_{ij} \cdot \widetilde{\lambda}_{ij} + \widetilde{E}_{ij} \cdot \widetilde{\beta}_{ij,X''}$. Hence $(\widetilde{\lambda}_{ij})^2|_{X''} = \widetilde{\lambda}_{ij} \cdot \widetilde{E}_{ij} = \widetilde{\pi}_{ij} \cdot \widetilde{E}_{ij}^2 = -1$.

Finally let us consider $bl''' : Y \to Y''$ the blow-up of $Y''$ along the above 18 curves, with exceptional divisors $\Gamma_{ki} := bl'''^{-1}(\widetilde{\gamma}_{ki})$, $\Lambda_{ij} := bl'''^{-1}(\widetilde{\lambda}_{ij})$, for distinct indices $i, j, k \in \{0, 1, 2, 3\}$ with $i < j$. We denote by $\mathcal{E}_i$ and $\mathcal{E}_{ij}$ respectively the strict transform of $\widetilde{E}_i$ and $\widetilde{E}_{ij}$; by $\mathcal{F}_{ij}$ the strict transform of $F_{ij}$; by $\mathcal{F}_\delta$ the strict transform of $F_\delta$; by $\mathcal{H}$ the pullback of $H$.

**Remark 5.53.** We have that

$$\Gamma_{ki} = \mathbb{P}(\mathcal{N}_{\widetilde{\gamma}_{ki}|Y''}) \cong \mathbb{P}(\mathcal{O}_{\widetilde{\gamma}_{ki}}(\widetilde{E}_k) \oplus \mathcal{O}_{\widetilde{\gamma}_{ki}}(\widetilde{f}_i)) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{F}_0,$$

$$\Lambda_{ij} = \mathbb{P}(\mathcal{N}_{\widetilde{\lambda}_{ij}|Y''}) \cong \mathbb{P}(\mathcal{O}_{\widetilde{\lambda}_{ij}}(\widetilde{E}_{ij}) \oplus \mathcal{O}_{\widetilde{\lambda}_{ij}}(\widetilde{\pi}_{ij})) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{F}_0.$$

Furthermore we have $\Gamma_{ki}^3 = -\deg(\mathcal{N}_{\widetilde{\gamma}_{ki}|Y''}) = 2$, and $\Lambda_{ij}^3 = -\deg(\mathcal{N}_{\widetilde{\lambda}_{ij}|Y''}) = 2$ (see [27, Chap 4, §6] and [32, Lemma 2.2.14]).

**Remark 5.54.** Let us take distinct indices $i, j, k \in \{0, 1, 2, 3\}$ with $i < j$. The divisor $\mathcal{F}_{ij}$ intersects $\Gamma_{ki}$, $\Gamma_{kj}$, $\Lambda_{ij}$ each along a $\mathbb{P}^1$ which is a $(-1)$-curve on $\mathcal{F}_{ij}$ and a fibre on $\Gamma_{ki}$, $\Gamma_{kj}$, $\Lambda_{ij}$. Similarly we have $\Lambda_{ij}^2 \cdot \mathcal{F}_\delta = -1$ and $\Lambda_{ij} \cdot \mathcal{F}_\delta^2 = 0$. Let us consider the strict transforms $\widetilde{\alpha}_{kij}$, $\widetilde{\alpha}_{ij}$, $\widetilde{\alpha}'_{ij}$ of the curves defined in Remark 5.51. Then we have

$$\widetilde{\alpha}_{kij}^2|_{\mathcal{E}_k} = \mathcal{F}_{ij}^2 \cdot \mathcal{E}_k = -1, \quad \widetilde{\alpha}_{ij}^2|_{\mathcal{E}_{ij}} = {\mathcal{F}_{ij}}^2 \cdot \mathcal{E}_{ij} = -1, \quad \widetilde{\alpha}_{ij}'^2|_{\mathcal{E}'_{ij}} = \mathcal{F}_\delta^2 \cdot \mathcal{E}_{ij} = -1,$$

$$\widetilde{\alpha}_{kij}^2|_{\mathcal{F}_{ij}} = \mathcal{E}_k^2 \cdot \mathcal{F}_{ij} = -2, \quad \widetilde{\alpha}_{ij}^2|_{\mathcal{F}_{ij}} = \mathcal{E}_{ij}^2 \cdot \mathcal{F}_{ij} = -1, \quad \widetilde{\alpha}_{ij}^2|_{\mathcal{F}_\delta^2} = \mathcal{E}_{ij}^2 \cdot \mathcal{F}_\delta = -1.$$

Finally we recall that a general line of $\mathbb{P}^3$ does not intersect the edges of $T$ and the curve $\delta$; instead a general plane of $\mathbb{P}^3$ intersects each edge of $T$ at one point and the curve $\delta$ at 3 points. Hence we have $\mathcal{H}^2 \cdot \mathcal{F}_{ij} = \mathcal{H}^2 \cdot \mathcal{F}_\delta = 0$, $\mathcal{F}_{ij}^2 \cdot \mathcal{H} = -1$ and $\mathcal{F}_\delta^2 \cdot \mathcal{H} = -3$.

**Remark 5.55.** We recall that by construction we have $bl'''^*(E_k) = \mathcal{E}_k + \sum_{\substack{t=0 \\ t \neq k}}^3 \Gamma_{kt}$ and $bl'''^*(E_{ij}) = \mathcal{E}_{ij} + \Lambda_{ij}$, and, by abuse of notation, we denote $\mathcal{E}_k \cap \Gamma_{ki}$ and $\mathcal{E}_{ij} \cap \Lambda_{ij}$ by $\widetilde{\gamma}_{ki}$ and $\widetilde{\lambda}_{ij}$, for distinct $i, j, k \in \{1, 2, 3\}$ with $i < j$. Let us denote by $\mathcal{L}_{ki}$ and $\mathcal{L}_{ij}$ respectively the strict transform on $Y$ of a general line of $E_k$ and $E_{ij}$. By using similar arguments to the ones in Remark 5.30 we obtain that $\mathcal{E}_k^3 = 4$ and $\mathcal{E}_{ij}^3 = 2$, since we have

$$\mathcal{E}_k|_{\mathcal{E}_k} \sim -(\mathcal{L}_k + \sum_{\substack{i=0 \\ i \neq k}}^3 \widetilde{\gamma}_{ki}) \sim -(4\mathcal{L}_k - 2 \sum_{0 \leq i < j \leq 3} \widetilde{\alpha}_{ij}),$$

$$\mathcal{E}_{ij}|_{\mathcal{E}_{ij}} \sim -(\mathcal{L}_{ij} + \widetilde{\lambda}_{ij}) \sim -(2\mathcal{L}_{ij} - \widetilde{\alpha}_{ij} - \widetilde{\alpha}'_{ij}).$$

**Remark 5.56.** By using similar arguments to the ones in Remark 5.11 we have $\mathcal{F}_{ij}^3 = -\deg(\mathcal{N}_{\widetilde{l}_{ij}|Y'}) = 4$, for $0 \leq i < j \leq 3$, and $\mathcal{F}_\delta^3 = -\deg(\mathcal{N}_{\widetilde{\delta}|Y'}) = 0$ (see Remark 5.50).

Let $\widetilde{X}$ be the strict transform on $Y$ of an element of $\mathcal{X}''$: then

$$\widetilde{X} \sim 6\mathcal{H} - \sum_{k=0}^3 3\mathcal{E}_k - 2 \sum_{0 \leq i < j \leq 3} \mathcal{E}_{ij} - 2 \sum_{0 \leq i < j \leq 3} \mathcal{F}_{ij} - \mathcal{F}_\delta - \sum_{\substack{i,k=0 \\ i \neq k}}^3 4\Gamma_{ki} - \sum_{0 \leq i < j \leq 3} 3\Lambda_{ij}.$$

Let us take the linear system $\widetilde{\mathcal{X}} := |\mathcal{O}_Y(\widetilde{X})|$ on $Y$. It is base point free and it defines a birational morphism $\nu_{\widetilde{\mathcal{X}}} : Y \to W_F^7 \subset \mathbb{P}^7$. Furthermore we have the following diagram:

$$\begin{array}{ccccccc} Y & & & & & \xrightarrow{\nu_{\widetilde{\mathcal{X}}}} & \\ \downarrow{\scriptstyle bl'''} & & & & & & \\ Y'' & \xrightarrow{bl''} & Y' & \xrightarrow{bl'} & \mathbb{P}^3 & \overset{\nu_{\mathcal{X}}}{\dashrightarrow} & W_F^7 \subset \mathbb{P}^7. \end{array}$$

**Remark 5.57.** The divisors $\mathcal{E}_i$ and the strict transforms $\widetilde{f}_i$ on $Y$ of the faces of $T$ are contracted by $\nu_{\widetilde{\mathcal{X}}} : Y \to W_F^7 \subset \mathbb{P}^7$ to points of $W_F^7$, for $0 \leq i \leq 3$, since $\widetilde{X} \cdot \mathcal{E}_i = \widetilde{X} \cdot \widetilde{f}_i = 0$ for a general $\widetilde{X} \in \widetilde{\mathcal{X}}$.

**Remark 5.58.** The 18 exceptional divisors of $bl''' : Y \to Y''$ and the six divisors $\mathcal{E}_{ij}$, for $0 \leq i, j \leq 3$, are contracted by the morphism $\nu_{\widetilde{\mathcal{X}}} : Y \to W_F^7 \subset \mathbb{P}^7$ to curves of $W_F^7$. This follows by the fact that $\widetilde{X} \cdot \Gamma_{ki}$, $\widetilde{X} \cdot \Lambda_{ij}$, $\widetilde{X} \cdot \mathcal{E}_{ij} \neq 0$ and $\widetilde{X}^2 \cdot \Gamma_{ki} = \widetilde{X}^2 \cdot \Lambda_{ij} = \widetilde{X}^2 \cdot \mathcal{E}_{ij} = 0$ for distinct $i, j, k \in \{0, 1, 2, 3\}$ and $i < j$ (use Remarks 5.53, 5.54 and calculations similar to the ones in Remarks 5.13, 5.34).

**Remark 5.59.** Let us fix $0 \leq i < j \leq 3$ and let $\widetilde{X}$ be a general element of $\widetilde{\mathcal{X}}$. Since $\widetilde{X}^2 \cdot \mathcal{F}_{ij} = 3 > 0$ and $\widetilde{X}^2 \cdot \mathcal{F}_\delta = 6 > 0$ (use Remarks 5.54, 5.56 and calculations similar to the ones in Remarks 5.14, 5.35), then the curves $\widetilde{X} \cap \mathcal{F}_{ij}$ and $\widetilde{X} \cap \mathcal{F}_\delta$ are not contracted by the rational map defined by $\widetilde{\mathcal{X}}|_{\widetilde{X}}$.

We still define $P_{i+1} := \nu_{\widetilde{\mathcal{X}}}(\mathcal{E}_i)$ and $P'_{i+1} := \nu_{\widetilde{\mathcal{X}}}(\widetilde{f}_i)$, as in § 5.2. They are quadruple points of $W_F^7$ whose tangent cone is a cone over a Veronese surface. The proof is similar to the one of Proposition 5.16. We recall that $\nu_{\mathcal{X}} : \mathbb{P}^3 \dashrightarrow W_F^7 \subset \mathbb{P}^7$ is an isomorphism outside $T \cup \pi$ (see Remark 5.46). Then $P_1$, $P_2$, $P_3$, $P_4$, $P'_1$, $P'_2$, $P'_3$ and $P'_4$ are the only singular points of $W_F^7$ (see Remarks 5.57, 5.58, 5.59).

**Lemma 5.60.** The six divisors $\mathcal{E}_{ij}$, with $0 \leq i < j \leq 3$, are mapped by $\nu_{\widetilde{\mathcal{X}}} : Y \to W_F^7 \subset \mathbb{P}^7$ to lines of $W_F^7$. In particular we have $\nu_{\widetilde{\mathcal{X}}}(\mathcal{E}_{ij}) = \left\langle P'_{i+1}, P'_{j+1} \right\rangle$.

*Proof.* We know that the above 6 divisors are mapped by $\nu_{\widetilde{\mathcal{X}}} : Y \to W_F^7 \subset \mathbb{P}^7$ to curves (see Remark 5.58). Let us show that these curves are lines. Let $\widetilde{X}$ be a general element of $\widetilde{\mathcal{X}}$ and let us consider the divisor $\mathcal{E}_{ij}$ for a fixed pair of indices $0 \leq i < j \leq 3$. We observe that $\widetilde{\mathcal{X}}|_{\mathcal{E}_{ij}} \cong |\mathcal{O}_{\mathcal{E}_{ij}}(\widetilde{\beta}_{ij,\widetilde{X}})| \cong \mathbb{P}^1$ (see Remark 5.49), so $\nu_{\widetilde{\mathcal{X}}}(\mathcal{E}_{ij}) \subset W_F^7$ is a line. Since $\mathcal{E}_{ij} \cap \widetilde{f}_i \neq \emptyset$ and $\mathcal{E}_{ij} \cap \widetilde{f}_j \neq \emptyset$, then $\nu_{\widetilde{\mathcal{X}}}(\mathcal{E}_{ij})$ is the line joining the points $P'_{i+1}$ and $P'_{j+1}$. □

By recalling Definition 4.4 we have the following result.

**Theorem 5.61.** Each of the eight points $P_1$, $P_2$, $P_3$, $P_4$, $P'_1$, $P'_2$, $P'_3$, $P'_4$ is associated with $m = 6$ of the others, as in Figure 22 of Appendix A.

*Proof.* The 12 divisors $\Gamma_{ki}$, for $i, k \in \{0, 1, 2, 3\}$ and $i \neq k$, are mapped by $\nu_{\widetilde{\mathcal{X}}} : Y \to W_F^7 \subset \mathbb{P}^7$ to lines of $W_F^7$ joining the points $P_1$, $P_2$, $P_3$, $P_4$, $P'_1$, $P'_2$, $P'_3$, $P'_4$ as in Figure 26 of Appendix A (see Remark 5.58 and use arguments of the proof of Theorem 5.17). We also recall that the six lines $\nu_{\widetilde{\mathcal{X}}}(\mathcal{E}_{ij})$ joins the points $P'_1$, $P'_2$, $P'_3$, $P'_4$ two by two (see Lemma 5.60). It remains to show that $\langle P_1, P_2 \rangle$, $\langle P_1, P_3 \rangle$, $\langle P_1, P_4 \rangle$, $\langle P_2, P_3 \rangle$, $\langle P_2, P_4 \rangle$, $\langle P_3, P_4 \rangle \subset W_F^7$ and that $\langle P_1, P'_1 \rangle$, $\langle P_2, P'_2 \rangle$, $\langle P_3, P'_3 \rangle$, $\langle P_4, P'_4 \rangle$ , $\not\subset W_F^7$. This follows by a computational analysis with Macaulay2 (see Code B.7 of Appendix B). □

## 5.5 F-EF 3-fold of genus 6

### 5.5.1 Construction of $W_F^6$

Let us consider five general points $q_1$, $q_2$, $q_3$, $q_4$, $q_5$ in $\mathbb{P}^3$. We have the following result.

**Theorem 5.62.** There are three twisted cubics $C_1$, $C_2$ and $C_3$, three quadric surfaces $Q_6$, $Q_7$ and $Q_8$ of $\mathbb{P}^3$ and three lines $r_1$, $r_2$, and $r_3$ such that $Q_6$ and $Q_7$ are smooth and

$$C_1 \cap C_2 \cap C_3 = \{q_1, \dots q_5\},\ Q_6 \cap Q_7 = C_1 \cup r_1,\ Q_6 \cap Q_8 = C_2 \cup r_2,\ Q_7 \cap Q_8 = C_3 \cup r_3,$$

where $r_i$ intersects $C_i$ at two points $a_i'$ and $a_i''$, for $1 \leq i \leq 3$. Furthermore, by taking three distinct indices $i, j, k \in \{1, 2, 3\}$ with $i < j$, we have the following three possibilities:

(i) $Q_8$ is smooth and the three lines $r_1$, $r_2$ and $r_3$ intersect pairwise at three distinct points $b_{ij} := r_i \cap r_j$ such that $b_{ij} = r_i \cap C_k = r_j \cap C_k$ (see Figure 9);

(ii) $Q_8$ is smooth and the three lines $r_1$, $r_2$, $r_3$ intersect at a same point $b$; moreover, up to renaming the points of $r_k \cap C_k$ and of $C_1 \cap C_2 \cap C_3$, we have that $r_k \cap C_i = r_k \cap C_j = a_k'' = q_k$ (see left side of Figure 10);

(iii) $Q_8$ is a cone and the three lines $r_1$, $r_2$, $r_3$ intersect at the vertex $v$ of $Q_8$; moreover, up to renaming the points of $r_k \cap C_k$ and of $C_1 \cap C_2 \cap C_3$, we have that $v = q_1 = a_k'' = r_k \cap C_i = r_k \cap C_j$ (see right side of Figure 10).

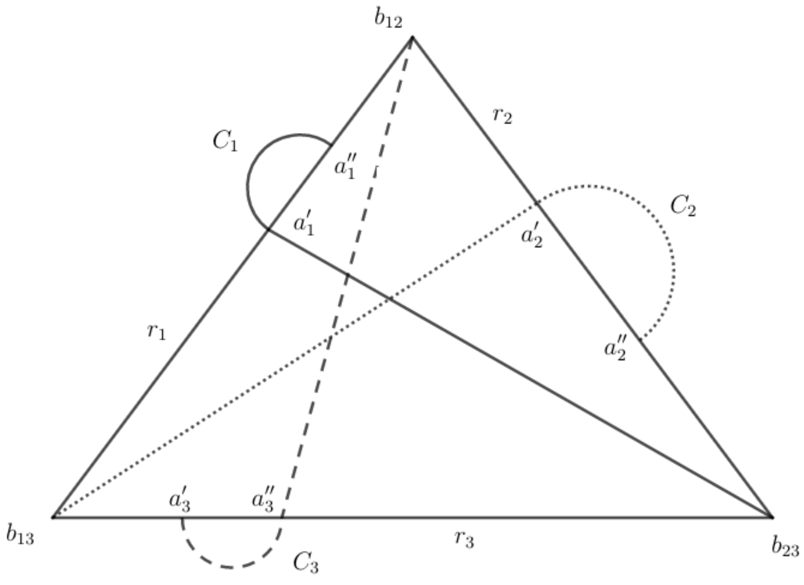


*Figure 9:* *Description of the intersection points between the twisted cubics $C_1$, $C_2$, $C_3$ and their chords $r_1$, $r_2$, $r_3$, in the case (i) of Theorem 5.62. We recall that $C_1 \cap C_2 \cap C_3 = \{q_1, \dots, q_5\}$, even if it is not represented in the figure.*

*Proof.* The proof is divided into 6 steps, given by the Lemmas 5.63, 5.64, 5.65, 5.66, 5.67, 5.68 below.

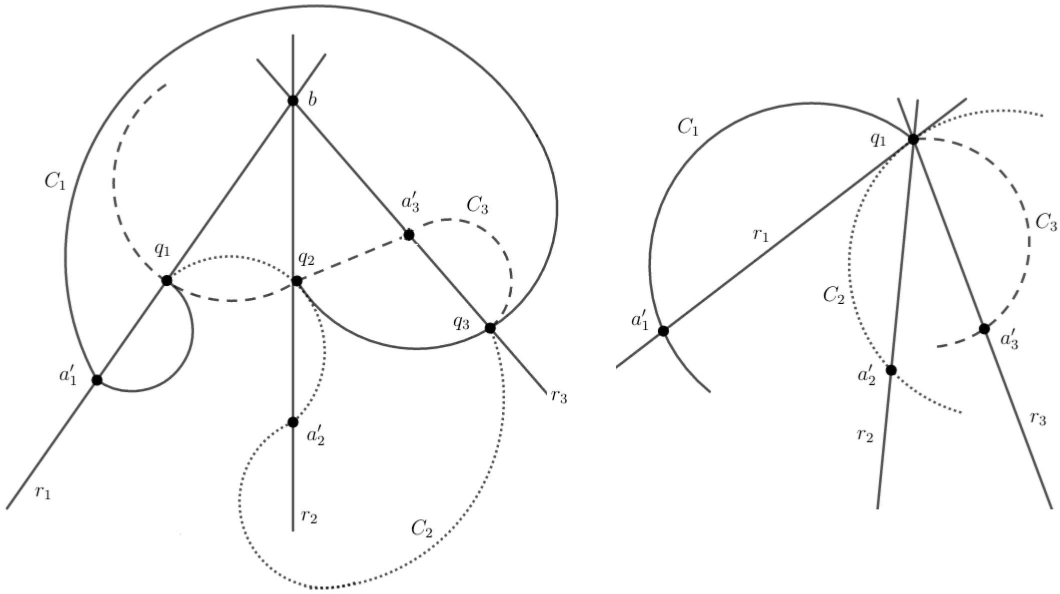


*Figure 10: Description of the intersection points between the twisted cubics $C_1$, $C_2$, $C_3$ and their chords $r_1$, $r_2$, $r_3$, in the case (ii) of Theorem 5.62 (on the left) and in the case (iii) (on the right). We recall that $C_1 \cap C_2 \cap C_3 = \{q_1, \dots, q_5\}$, even if some of these points are not represented in the figure.*

**Lemma 5.63.** There are three twisted cubics $C_1$, $C_2$ and $C_3$ passing through the five general points $q_1, \dots, q_5$ and there are three quadric surfaces $Q_6$, $Q_7$ and $Q_8$ and three lines $r_1$, $r_2$, and $r_3$ such that $Q_6$ and $Q_7$ are smooth and $Q_6 \cap Q_7 = C_1 \cup r_1$, $Q_6 \cap Q_8 = C_2 \cup r_2$, $Q_7 \cap Q_8 = C_3 \cup r_3$.

*Proof.* Let us consider the two-dimensional family $\mathcal{C}$ of the twisted cubics passing through $q_1, \dots, q_5$. Let us take a general twisted cubic $C_1 \in \mathcal{C}$ and two smooth quadric surfaces $Q_6$, $Q_7$ containing $C_1$, i.e. two general elements $Q_6, Q_7 \in |\mathcal{I}_{C_1|\mathbb{P}^3}(2)| \cong \mathbb{P}^2$. It is known that there exists a line $r_1$ such that $Q_6 \cap Q_7 = C_1 \cup r_1$ (see [28, Example 1.11]). Since $Q_6$ is a smooth quadric surface in $\mathbb{P}^3$, then a quadric section of $Q_6 \subset \mathbb{P}^3$ is linearly equivalent to $2f_1 + 2f_2$, where $f_1$ and $f_2$ represent the two rulings of $Q_6$ and satisfy the relations $f_1^2 = 0 = f_2^2$ and $f_1 \cdot f_2 = 1$. In particular, since $Q_6 \cap Q_7 \sim 2f_1 + 2f_2$, we can suppose that $r_1 \sim f_1$ and $C_1 \sim f_1 + 2f_2$. We also have that $|\mathcal{O}_{Q_6}(f_1 + 2f_2)| \cong \mathbb{P}^5$. By the generality of $q_1, \dots, q_5$ we may assume that $C_1$ is the unique twisted cubic in $|\mathcal{O}_{Q_6}(f_1 + 2f_2)|$ through $q_1, \dots, q_5$. Similarly let us take the unique twisted cubic $C_2$ through $q_1, \dots, q_5$ in $|\mathcal{O}_{Q_6}(2f_1 + f_2)|$. So each smooth quadric surface passing through $q_1, \dots, q_5$ contains exactly two twisted cubics passing through them: let $C_3$ be the other twisted cubic in $Q_7$ through $q_1, \dots, q_5$. Let us define $\Lambda_{C_i} := |\mathcal{I}_{C_i|\mathbb{P}^3}(2)| \cong \mathbb{P}^2$ for $i = 1, 2, 3$. Since $\Lambda_{C_i} \subset |\mathcal{I}_{\{q_1,\dots,q_5\}|\mathbb{P}^3}(2)| \cong \mathbb{P}^4$ for $i = 1, 2, 3$, then $\dim \Lambda_{C_2} \cap \Lambda_{C_3} \geq 0$. So there exists a quadric surface $Q_8 \in \Lambda_{C_2} \cap \Lambda_{C_3}$ such that $C_2 \subset Q_6 \cap Q_8$ and $C_3 \subset Q_7 \cap Q_8$, and there are two lines $r_2$ and $r_3$ such $Q_6 \cap Q_8 = C_2 \cup r_2$ and $Q_7 \cap Q_8 = C_3 \cup r_3$. □

Let us fix now three twisted cubics, three lines and three quadric surfaces as in Lemma 5.63. It must be $r_1 \not\subset Q_8$, $r_2 \not\subset Q_7$ and $r_3 \not\subset Q_6$. Let us take $i, j, k \in \{1, 2, 3\}$ with $k \neq i$ and $i < j$. By construction we have $r_i \cdot C_i = 2$, so we let $\{a'_i, a''_i\} := r_i \cap C_i$. Furthemore $r_i \cdot C_k = 1$, so we define $a_{ik} := r_i \cap C_k$.

**Lemma 5.64.** The line $r_i$ intersects the line $r_j$ for all $1 \leq i < j \leq 3$.

*Proof.* By construction we have that $r_1 \cap r_2 \neq \emptyset$ and $r_1 \cap r_3 \neq \emptyset$. Furthermore it must be $r_2 \cap r_3 \neq \emptyset$. Indeed if $Q_8$ is a cone, then $r_2$ and $r_3$ intersect at the vertex; if $Q_8$ is smooth, then $C_2$ and $C_3$ are not linearly equivalent and $r_2$ and $r_3$ belong to different rulings. $\square$

By Lemma 5.64 we have two possibilities: the three lines $r_1$, $r_2$ and $r_3$ intersect pairwise at 3 distinct points $\{r_1 \cap r_2, r_1 \cap r_2, r_2 \cap r_3\}$ or they intersect at a same point $r_1 \cap r_2 \cap r_3$.

**Lemma 5.65.** Let $Q \subset \mathbb{P}^3$ be a quadric cone with vertex $v$. If $C$ is a twisted cubic contained in $Q$, then $v \in C$.

*Proof.* Let us suppose that $v \notin C$. Let $H$ be a general plane of $\mathbb{P}^3$ such that $v \notin H$ and let us take the projection map $\pi_v : \mathbb{P}^3 \dashrightarrow H \cong \mathbb{P}^2$ from the point $v$ to the plane $H$. Since $v \notin C$, then $\pi_v(C)$ is a cubic plane curve. Furthermore $\pi_v(C)$ has to be contained in $\pi_v(Q)$, which is a conic. So we have a contradiction. $\square$

**Lemma 5.66.** Let us suppose that the three lines $r_1$, $r_2$ and $r_3$ intersect pairwise at 3 distinct points and let us denote them by $b_{ij} := r_i \cap r_j$ for $1 \leq i < j \leq 3$. Then the quadric surface $Q_8 \subset \mathbb{P}^3$ is smooth and we have $b_{ij} = r_i \cap C_k = r_j \cap C_k$ for all $1 \leq k \leq 3$ such that $k \neq i$ and $k \neq j$.

*Proof.* Let us suppose that $Q_8$ is a cone with vertex $v$. Then $r_2$, $r_3$, $C_2$ and $C_3$ must pass through $v$ (see Lemma 5.65). In particular there exist a point in $r_2 \cap C_2$, a point in $r_3 \cap C_3$ and a point in $\{q_1, \dots, q_5\}$, for example $a''_2$, $a''_3$ and $q_5$, such that $v = a''_2 = a''_3 = b_{23} = a_{21} = a_{31} = a_{23} = a_{32} = q_5$. Moreover, we have that $r_1 \cap Q_8 = (r_1 \cap Q_6) \cap Q_8 = r_1 \cap (C_2 \cup r_2) = \{a_{12}, b_{12}\}$, since $r_1 \subset Q_6$. Similarly $r_1 \cap Q_8 = \{a_{13}, b_{13}\}$, since $r_1 \subset Q_7$ (see Lemma 5.63). Since $b_{12} \neq b_{13}$ by hypothesis, it must be $b_{12} = a_{13}$ and $b_{13} = a_{12}$. This implies $b_{12} = r_2 \cap C_3 = v$ and $b_{13} = r_3 \cap C_2 = v$, which is a contradiction. Hence $Q_8$ is a smooth quadric surface of $\mathbb{P}^3$. Finally we observe that $r_1 \cap Q_8 = \{b_{13}, a_{13}\} = \{b_{12}, a_{12}\}$, $r_2 \cap Q_7 = \{b_{12}, a_{21}\} = \{b_{23}, a_{23}\}$ and $r_3 \cap Q_6 = \{b_{13}, a_{31}\} = \{b_{23}, a_{32}\}$. Since $b_{12}$, $b_{13}$, $b_{23}$ are three distinct points by hypothesis, then it must be $b_{13} = a_{12} = a_{32}$, $b_{12} = a_{13} = a_{23}$ and $b_{23} = a_{21} = a_{31}$. $\square$

**Lemma 5.67.** Let us suppose that the three lines $r_1$, $r_2$ and $r_3$ intersect at the same point $b$. If $Q_8$ is smooth, then we obtain the assertion (ii) of Theorem 5.62.

*Proof.* Let $i, j, k$ be three distinct indices in $\{1, 2, 3\}$. Since $r_k \subset Q_{k+i+3}$, we have that $r_k \cap Q_{i+j+3} = (r_k \cap Q_{k+i+3}) \cap Q_{i+j+3} = r_k \cap (C_i \cup r_i) = \{a_{ki}, b\}$ (see Lemma 5.63). Similarly we have that $r_k \cap Q_{i+j+3} = \{a_{kj}, b\}$, since $r_k \subset Q_{k+j+3}$. So it must be $a_{ki} = a_{kj}$. This implies that there are three points in $\{q_1, \dots, q_5\}$, namely $q_1$, $q_2$, $q_3$, and there exist a point in $r_k \cap C_k$, for example $a''_k$, such that $a_{ki} = a_{kj} = a''_k = q_k$. $\square$

**Lemma 5.68.** Let us suppose that the three lines $r_1$, $r_2$ and $r_3$ intersect at the same point $b$. If $Q_8$ is a cone, then we obtain the assertion (iii) of Theorem 5.62.

*Proof.* If $Q_8$ is a cone with vertex $v$, then $v = r_2 \cap r_3 = r_1 \cap r_2 \cap r_3 = b$. Since $C_2, C_3 \subset Q_8$, then $v \in C_2 \cap C_3 = C_1 \cap C_2 \cap C_3 = \{q_1, \dots, q_5\}$ (see Lemma 5.65). Thus we have $v = a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32}$. Furthermore there exist a point in each $r_1 \cap C_1$, $r_2 \cap C_2$, $r_3 \cap C_3$, for example $a_1''$, $a_2''$, $a_3''$, such that $v = a_1'' = a_2'' = a_3''$. □

□

Let us see that by choosing sufficiently general objects, we can exclude the cases (ii) and (iii) of Theorem 5.62. Let us take the two-dimensional family $\mathcal{C}_{q_1,\dots,q_5}$ of the twisted cubics of $\mathbb{P}^3$ passing through the fixed points $q_1, \dots, q_5$. For all $C \in \mathcal{C}_{q_1,\dots,q_5}$ we define $\Lambda_C := |\mathcal{I}_{C|\mathbb{P}^3}(2)| \cong \mathbb{P}^2$, which is a plane in $|\mathcal{I}_{\{q_1,\dots,q_5\}|\mathbb{P}^3}(2)| \cong \mathbb{P}^4$. We recall that, if we fix a general $C \in \mathcal{C}_{q_1,\dots,q_5}$ and if $Q \in \Lambda_C$ is general, then $Q$ is smooth and contains exactly two twisted cubics $C, C' \in C_{q_1,\dots,q_5}$ (see proof of Lemma 5.63). We can consider the map $\varphi_C : \Lambda_C \to \mathcal{C}_{q_1,\dots,q_5}$ which sends a general $Q \in \Lambda_C$ to the other twisted cubic $C'$ in $Q$ passing through $q_1, \dots, q_5$. This map is well defined and it has fibres of dimension 0: indeed by Bezout's Theorem we have that two quadric surfaces of $\mathbb{P}^3$ intersecting along $C \cup C'$ have to coincide, since $C \cup C'$ is a curve of degree 6. Hence $\varphi_C$ is a birational map. In other words, the correspondence $C' \leftrightarrow Q$ is $1:1$ between an open set of $\mathcal{C}_{q_1,\dots,q_5}$ and an open set of $\Lambda_C$. Let us *fix* now a *general* $C_1 \in \mathcal{C}_{q_1,\dots,q_5}$ and a *general* smooth quadric surface $Q_6 \in \Lambda_{C_1}$. Then $C_2 := \varphi_{C_1}(Q_6) \in \mathcal{C}_{q_1,\dots,q_5}$ is fixed too, since it is uniquely determined by $Q_6$. Let us take another *general* $Q_7 \in \Lambda_{C_1}$, which is another smooth quadric surface of $\mathbb{P}^3$ containing $C_1$. We may assume that $Q_7$ is sufficiently general in order to have that $Q_7$ intersects $Q_6$ along the union of $C_1$ and a line $r_1$ not passing through $q_1, \dots, q_5$. Let us define $C_3 := \varphi_{C_1}(Q_7) \in \mathcal{C}_{q_1,\dots,q_5}$. Then $\dim_{\mathbb{P}^4} \Lambda_{C_2} \cap \Lambda_{C_3} \geq 0$ and, by Bezout's theorem, if $Q_8 \in \Lambda_{C_2} \cap \Lambda_{C_3}$ then $Q_8$ is *unique*. In particular $Q_8$ is uniquely determined by $C_3$ which is uniquely determined by $Q_7$. Let $r_2$, $r_3$ be the lines such that $Q_6 \cap Q_8 = C_2 \cup r_2$ and $Q_7 \cap Q_8 = C_3 \cup r_3$. Since $\{q_1, \dots, q_5\} \cap r_1 = \emptyset$ by construction, we may suppose to fix three twisted cubics $C_1$, $C_2$, $C_3$, three lines $r_1$, $r_2$, $r_3$ and three smooth quadric surfaces $Q_6$, $Q_7$, $Q_8$ in $\mathbb{P}^3$ satisfying the property (i) of Theorem 5.62.

By the generality of $Q_6, Q_7 \in \Lambda_{C_1} \cong \mathbb{P}^2$, we may also assume that $r_i$ is a *chord* of $C_i$ for $i = 1, 2$, i.e. $a_1' \neq a_1''$ and $a_2' \neq a_2''$. Let us explain this. We recall that $C_1$ and $C_2$ are the only twisted cubics in $Q_6$ through $q_1, \dots, q_5$. In particular, if $f_1$ and $f_2$ represent the two ruling of $Q_6$, then we have that $C_1 \sim_{Q_6} f_1 + 2f_2$ and $C_2 \sim_{Q_6} 2f_1 + f_2$. For any choice of $R_1 \in |\mathcal{O}_{Q_6}(f_1)|$ and $R_2 \in |\mathcal{O}_{Q_6}(f_2)|$ we have that $h^0(\mathcal{I}_{R_1 \cup C_1|\mathbb{P}^3}(2)) = h^0(\mathcal{I}_{R_2 \cup C_2|\mathbb{P}^3}(2)) = 2$. Furthermore, by the Hurwitz formula applied to $|\mathcal{O}_{C_i}(f_i)|$ for $i = 1, 2$, there exist only two lines $R_{i,1}, R_{i,2} \in |\mathcal{O}_{Q_6}(f_i)|$ which are tangent to $C_i$. Let us consider the four lines $L_{1,1} = |\mathcal{I}_{R_{1,1} \cup C_1}(2)|$, $L_{1,2} = |\mathcal{I}_{R_{1,2} \cup C_1}(2)|$, $L_{2,1} = |\mathcal{I}_{R_{2,1} \cup C_2}(2)|$, $L_{2,2} = |\mathcal{I}_{R_{2,2} \cup C_2}(2)|$. By the generality of $Q_7 \in \Lambda_{C_1} \cong \mathbb{P}^2$ and by using the fact the $Q_8$ is uniquely determined by $Q_7$, we may assume that $Q_7 \notin L_{1,1} \cup L_{1,2}$ and $Q_8 \notin L_{2,1} \cup L_{2,2}$. So $Q_7 \cap Q_6 = C_1 \cup r_1$ with $r_1$ transverse to $C_1$ and $Q_8 \cap Q_6 = C_2 \cup r_2$ with $r_2$ transverse to $C_2$. Instead we cannot exclude the case where $r_3$ is *tangent* to $C_3$, i.e. $r_3 \cap C_3 = \{a_3', a_3''\}$ with $a_3' = a_3''$. It would be interesting to study the above case, but we will analyze the case in which $r_3$ is a *chord* of $C_3$, since it is the situation mentioned by Fano in order to describe his Enriques-Fano threefold of genus 6 (see [23, §3]).

So let us fix now three twisted cubics $C_1$, $C_2$, $C_3$, three chords $r_1$, $r_2$, $r_3$ and three smooth quadric surfaces $Q_6$, $Q_7$, $Q_8$ satisfying (i) of Theorem 5.62. Let $\mathcal{P}$ be the linear system of the septic surfaces of $\mathbb{P}^3$ double along the three twisted cubics $C_1$, $C_2$ and $C_3$ passing through $q_1$, $q_2$, $q_3$, $q_4$, $q_5$.

**Remark 5.69.** The surface $Q_{i+j+3}$ is the unique quadric surface of $\mathbb{P}^3$ containing $C_i \cup C_j \cup r_i \cup r_j$ for all $1 \leq i < j \leq 3$. Indeed we have $h^0(\mathcal{I}_{C_i \cup C_j \cup r_i \cup r_j|\mathbb{P}^3}(2)) = 1$, by the smoothness of the three quadric surfaces and by following exact sequence

$$0 \to \mathcal{I}_{Q_{i+j+3}|\mathbb{P}^3}(2) \to \mathcal{I}_{C_i \cup C_j \cup r_i \cup r_j|\mathbb{P}^3}(2) \to \mathcal{I}_{C_i \cup C_j \cup r_i \cup r_j|Q_{i+j+3}}(2) \to 0.$$

**Remark 5.70.** An element $P \in \mathcal{P}$ contains the lines $r_1$, $r_2$, $r_3$. Assume the contrary. Let us fix three distinct indices $i, j, k \in \{1, 2, 3\}$. By Bezout's Theorem, $P \cap r_i$ is given by 7 points. Furthermore $r_i$ is a line through four double points of $P$, i.e. $r_i \cap C_j$, $r_i \cap C_k$, $a'_i$ and $a''_i$ (see Figure 9). So we obtain that $P \cap r_i$ contains at least 8 points, counted with multiplicity, which is a contradiction. It must be $r_i \subset P$.

Let $g_{i+j+3} := g_{i+j+3}(s_0, s_1, s_2, s_3)$ be the quadratic homogeneous polynomial defining the smooth quadric surface $Q_{i+j+3} \subset \mathbb{P}^3_{[s_0,\ldots,s_3]}$ for $1 \leq i < j \leq 3$.

**Lemma 5.71.** The linear system $\mathcal{P}$ has equation $g_6 g_7 f_8 + g_6 g_8 f_7 + g_7 g_8 f_6 = 0$, where $f_6 \in H^0(\mathbb{P}^3, \mathcal{I}_{C_1 \cup C_2|\mathbb{P}^3}(3))$, $f_7 \in H^0(\mathbb{P}^3, \mathcal{I}_{C_1 \cup C_3|\mathbb{P}^3}(3))$ and $f_8 \in H^0(\mathbb{P}^3, \mathcal{I}_{C_2 \cup C_3|\mathbb{P}^3}(3))$.

*Proof.* Let $F := F(s_0, s_1, s_2, s_3)$ be the homogeneous polynomial of degree 7 defining a general element $P$ of $\mathcal{P}$ in $\mathbb{P}^3_{[s_0,\ldots,s_3]}$. We recall that the intersection of an irreducible septic surface of $\mathbb{P}^3$ with a quadric surface is a curve of degree 14. In particular, $P$ intersects each quadric surface $Q_{i+j+3}$ along the curve of degree 14 given by the two double twisted cubics $C_i$ and $C_j$ plus the two lines $r_i$ and $r_j$, for $1 \leq i < j \leq 3$. This implies that it must be $P \cap Q_{i+j+3} = \{g_{i+k+3} g_{j+k+3} f_{i+j+3} = 0,\ g_{i+j+3} = 0\} = 2C_i + 2C_j + r_i + r_j$ for some $f_{i+j+3} \in H^0(\mathbb{P}^3, \mathcal{I}_{C_i \cup C_j|\mathbb{P}^3}(3))$, where $1 \leq k \leq 3$ with $k \neq i$ and $k \neq j$. Then it must be $F = g_6 h_5 + g_7 g_8 f_6$, where $h_5$ is a homogeneous polynomial of degree 5 such that $h_5 = g_7 h_3 + g_8 f_7$, where $h_3$ is a homogeneous polynomial of degree 3 such that $h_3 = g_8 h_1 + f_8$, where $h_1$ is a homogeneous polynomial of degree 1. Thus we have $F = g_6 g_7 g_8 h_1 + g_6 g_7 f_8 + g_6 g_8 f_7 + g_7 g_8 f_6$. Since $g_{i+j+3} h_1 \in H^0(\mathbb{P}^3, \mathcal{I}_{C_i \cup C_j|\mathbb{P}^3}(3))$ for $1 \leq i < j \leq 3$, we obtain that $F$ has the expression of the statement. $\square$

**Lemma 5.72.** Let us take $1 \leq i < j \leq 3$. Then $\dim H^0(\mathbb{P}^3, \mathcal{I}_{C_i \cup C_j|\mathbb{P}^3}(3)) = 5$ and a general element of $|\mathcal{I}_{C_i \cup C_j|\mathbb{P}^3}(3)| \cong \mathbb{P}^4$ corresponds to a smooth irreducible surface.

*Proof.* Let us consider the following exact sequence

$$0 \to \mathcal{I}_{Q_{i+j+3}|\mathbb{P}^3}(3) \to \mathcal{I}_{C_i \cup C_j|\mathbb{P}^3}(3) \to \mathcal{I}_{C_i \cup C_j|Q_{i+j+3}}(3) \to 0.$$

Since $h^1(\mathcal{I}_{Q_{i+j+3|\mathbb{P}^3}}(3)) = h^1(\mathcal{O}_{\mathbb{P}^3}(1)) = 0$, $h^0(\mathcal{I}_{Q_{i+j+3}|\mathbb{P}^3}(3)) = h^0(\mathcal{O}_{\mathbb{P}^3}(1)) = 4$ and $h^0(\mathcal{I}_{C_i \cup C_j|Q_{i+j+3}}(3)) = h^0(\mathcal{O}_{Q_{i+j+3}}) = 1$, then we obtain $h^0(\mathcal{I}_{C_i \cup C_j|\mathbb{P}^3}(3)) = 5$. Let $S_3$ be now a general element of $|\mathcal{I}_{C_i \cup C_j|\mathbb{P}^3}(3)|$. We may assume that $Q_{i+j+3} \not\subset S_3$ and so that $S_3$ is irreducible, since $C_i \cup C_j$ is not degenerate and the only quadric surface containing this curve is $Q_{i+j+3}$ (see Remark 5.69). We want to show that $S_3$ is smooth. First let us see that $C_i \cup C_j$ is the base scheme of $|\mathcal{I}_{C_i \cup C_j|\mathbb{P}^3}(3)|$: we have to show that $h^0(\mathcal{I}_{C_i \cup C_j \cup \{x\}|\mathbb{P}^3}(3)) = h^0(\mathcal{I}_{C_i \cup C_j|\mathbb{P}^3}(3)) - 1 = 4$ for a point $x \in \mathbb{P}^3 \setminus (C_i \cup C_j)$. This is exactly what happens: indeed $x \notin Q_{i+j+3}$ (otherwise $x \in S'_3 \cap Q_{i+j+3} = C_i \cup C_j$ for $S'_3 \in |\mathcal{I}_{C_i \cup C_j \cup \{x\}|\mathbb{P}^3}(3)|$, which is a contradiction) and so we have the following exact sequence

$$0 \to \mathcal{I}_{Q_{i+j+3} \cup \{x\}|\mathbb{P}^3}(3) \to \mathcal{I}_{C_i \cup C_j \cup \{x\}|\mathbb{P}^3}(3) \to \mathcal{I}_{C_i \cup C_j|Q_{i+j+3}}(3) \to 0$$

from which $h^0(\mathcal{I}_{C_i \cup C_j \cup \{x\}|\mathbb{P}^3}(3)) = 4$, since $h^1(\mathcal{I}_{Q_{i+j+3} \cup \{x\}|\mathbb{P}^3}(3)) = h^1(\mathcal{I}_{\{x\}|\mathbb{P}^3}(1)) = 0$ and $h^0(\mathcal{I}_{Q_{i+j+3} \cup \{x\}|\mathbb{P}^3}(3)) = h^0(\mathcal{I}_{\{x\}|\mathbb{P}^3}(1)) = 3$. Let $p \in (C_i \cup C_j) \setminus \{q_1, \ldots, q_5\}$. If $S_3$ were singular at $p$, then $S_3 \cap Q_{i+j+3} = C_i \cup C_j$ would be singular at $p$, which is a contradiction. Let $p \in \{q_1, \ldots, q_5\}$. If $H$ is a plane such that $q_h \notin H$ for all $1 \leq h \leq 5$, then $Q_{i+j+3} \cup H \in |\mathcal{I}_{C_i \cup C_j|\mathbb{P}^3}(3)|$. Since $Q_{i+j+3} \cup H$ is smooth at $p$, then the general element of $|\mathcal{I}_{C_i \cup C_j|\mathbb{P}^3}(3)|$ is a cubic surface smooth at $p$. Thus $S_3$ is smooth. □

**Remark 5.73.** There exists a septic surface in $\mathcal{P}$ containing $Q_6$ but not $Q_7$. By Lemmas 5.71, 5.72 it is sufficient to take a septic surface defined by the equation $g_6 g_7 f_8 + g_6 g_8 f_7 + g_7 g_8 g_6 h = 0$ with $f_8 \in H^0(\mathbb{P}^3, \mathcal{I}_{C_2 \cup C_3|\mathbb{P}^3}(3))$ and $h \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ and where $f_7$ is a general (irreducible) element of $H^0(\mathbb{P}^3, \mathcal{I}_{C_1 \cup C_3|\mathbb{P}^3}(3))$. One can also construct a septic surface in $\mathcal{P}$ containing $Q_6$ and $Q_7$ but not $Q_8$. By Lemmas 5.71, 5.72 it is sufficient to take a septic surface with equation $g_6 g_7 f_8 + g_6 g_8 g_7 h' + g_7 g_8 g_6 h = 0$ where $h, h' \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ and where $f_8$ is a general (irreducible) element of $H^0(\mathbb{P}^3, \mathcal{I}_{C_2 \cup C_3|\mathbb{P}^3}(3))$.

A priori we have that $\dim \mathcal{P} \leq 14$, since the equation of $\mathcal{P}$ depends by 15 parameters which can be linearly dependent (see Lemmas 5.71, 5.72). However we have the following result.

**Proposition 5.74.** The linear system $\mathcal{P}$ defined as above has $\dim \mathcal{P} = 6$.

*Proof.* Let us consider the sublinear system of the septic surfaces of $\mathcal{P}$ containing $Q_{i+j+3}$ for $1 \leq i < j \leq 3$. The movable part of this linear system is isomorphic to the linear system $\mathcal{T}$ of the quintic surfaces of $\mathbb{P}^3$ containing the two twisted cubics $C_i, C_j \subset Q_{i+j+3}$, containing the line $r_k$, and with double points along the twisted cubic $C_k$, where $1 \leq k \leq 3$ and $k \neq i$ and $k \neq j$. We want to show that $\operatorname{codim}(\{P \in \mathcal{P} | P \supset Q_{i+j+3}\}, \mathcal{P}) = 1$. In order to do it, let $V \subset H^0(\mathcal{O}_{\mathbb{P}^3}(7))$ and $K \subset H^0(\mathcal{O}_{\mathbb{P}^3}(5))$ be the subspaces such that $\mathcal{P} = \mathbb{P}(V)$ and $\mathcal{T} = \mathbb{P}(K)$. From

$$0 \to \mathcal{O}_{\mathbb{P}^3}(5) \to \mathcal{O}_{\mathbb{P}^3}(7) \to \mathcal{O}_{Q_{i+j+3}}(7) \to 0,$$

we obtain

$$\begin{array}{ccccccccc} 0 \to & H^0(\mathcal{O}_{\mathbb{P}^3}(5)) & \to & H^0(\mathcal{O}_{\mathbb{P}^3}(7)) & \to & H^0(\mathcal{O}_{Q_{i+j+3}}(7)) & \to 0 \\ & \cup & & \cup & & \cup & \\ 0 \to & K & \to & V & \to & V|_{Q_{i+j+3}} & \to 0. \end{array}$$

We have to show that $\operatorname{codim}(K, V) = 1$, which is equivalent to find $\dim V|_{Q_{i+j+3}} = 1$. This follows by the fact that $\dim \mathcal{P}|_{Q_{i+j+3}} = 0$, since $\mathcal{P}|_{Q_{i+j+3}}$ has only fixed part

$2C_i+2C_j+r_i+r_j$. Then we have that codim $(\{P \in \mathcal{P} | P \supset Q_{i+j+3}\}, \mathcal{P}) = 1$ and, since containing the three quadrics $Q_6$, $Q_7$ and $Q_8$ imposes independent conditions (see Remark 5.73), we also obtain codim$(\{P \in \mathcal{P} | P \supset Q_6, Q_7, Q_8\}, \mathcal{P}) = 3$. Furthermore each element of $\{P \in \mathcal{P} | P \supset Q_6, Q_7, Q_8\}$ is of the form $Q_6 \cup Q_7 \cup Q_7 \cup \pi$, where $\pi$ is a general plane of $\mathbb{P}^3$. Thus we have $\dim\{P \in \mathcal{P} | P \supset Q_6, Q_7, Q_8\} = \dim |\mathcal{O}_{\mathbb{P}^3}(1)| = 3$ and finally $\dim \mathcal{P} = 3 + 3 = 6$. □

**Remark 5.75.** Let us fix $1 \leq i < j \leq 3$ and let us consider the quadric surface $Q_{i+j+3} \subset \mathbb{P}^3$. Since $Q_{i+j+3}$ is smooth, then the tangent space to $Q_{i+j+3}$ at the point $p \in Q_{i+j+3}$ is a plane of $\mathbb{P}^3$, which is spanned by the two lines of $Q_{i+j+3}$ intersecting at $p$. Let us take the point $p = q_h$ for some $1 \leq h \leq 5$. Since the twisted cubics $C_i$ and $C_j$ are contained in $Q_{i+j+3}$ and they pass through $q_h$, then the tangent plane to $Q_{i+j+3}$ at $q_h$ has to contain the tangent lines to $C_i$ and $C_j$ at $q_h$. We recall that $C_i \cdot C_j = (f_1+2f_2)\cdot(2f_1+f_2) = 5$, where $f_1$ and $f_2$ represent the two rulings of $Q_{i+j+3}$. Since $q_1, \ldots, q_5$ are distinct by construction, the intersection of $C_i$ and $C_j$ at each $q_h$ is transverse. Then we have $T_{q_h}C_i \neq T_{q_h}C_j$ and $t_{hij} := T_{q_h}Q_{i+j+3} = \langle T_{q_h}C_i, T_{q_h}C_j \rangle$. In particular we have that $T_{q_h}Q_6 = \langle T_{q_h}C_1, T_{q_h}C_2 \rangle$ and $T_{q_h}Q_7 = \langle T_{q_h}C_1, T_{q_h}C_3 \rangle$. By the generality of $q_1, \ldots, q_5$ and by the generality of $Q_6$ and $Q_7$, we may assume $T_{q_h}Q_6 \cap T_{q_h}Q_7 = T_{q_h}C_1$. Thus $T_{q_h}C_1$, $T_{q_h}C_2$ and $T_{q_h}C_3$ are linearly independent.

**Proposition 5.76.** Let $P$ be a general element of $\mathcal{P}$ and let us take $1 \leq h \leq 5$ ad three distinct indices $i, j, k \in \{1, 2, 3\}$ with $i < j$. Then we have that

(i) $TC_{q_h}P = \bigcup_{1 \leq i < j \leq 3} T_{q_h}Q_{i+j+3} = \bigcup_{1 \leq i < j \leq 3} t_{hij}$ and $P$ has triple points at the five points $q_1, \ldots, q_5$;

(ii) if $p \in C_k$ with $p \notin \{q_1, \ldots, q_5\}$, $p \notin r_k \cap C_k$ and $p \neq b_{ij}$, then $TC_pP$ is the union of two variable planes $\pi_{p,P}$ and $\pi'_{p,P}$ containing $T_pC_k$ and depending on the choice of the point $p$ and of the surface $P$;

(iii) if $p \in r_k \cap C_k$, then $TC_pP = \pi_p \cup \pi_{p,P}$, where the plane $\pi_p := T_pQ_{i+k+3} = T_pQ_{j+k+3}$ contains $T_pC_k$ and $r_k$, and where $\pi_{p,P}$ is a plane containing $T_pC_k$ and depending on the choice of $p \in \{a'_k, a''_k\}$ and of $P$;

(iv) $TC_{b_{ij}}P = \pi_{ij,i} \cup \pi_{ij,j}$, where the plane $\pi_{ij,i} := T_pQ_{i+k+3}$ contains $r_i$ and $T_{b_{ij}}C_k$, and where the plane $\pi_{ij,j} := T_pQ_{j+k+3}$ contains $r_j$ and $T_{b_{ij}}C_k$;

(v) if $p \in r_k$ with $p \notin r_k \cap C_k$ and $p \neq r_k \cap r_i$, then $TC_pP$ is a variable plane depending on the choice of $p$ and $P$.

*Proof.* We may assume that $P$ has equation $g_6g_7f_8 + g_6g_8f_7 + g_7g_8f_6 = 0$ for a smooth irreducible $f_6 \in H^0(\mathbb{P}^3, \mathcal{I}_{C_1 \cup C_2 | \mathbb{P}^3}(3))$, a smooth irreucible $f_7 \in H^0(\mathbb{P}^3, \mathcal{I}_{C_1 \cup C_3 | \mathbb{P}^3}(3))$ and a smooth irreducible $f_8 \in H^0(\mathbb{P}^3, \mathcal{I}_{C_2 \cup C_3 | \mathbb{P}^3}(3))$ (see Lemmas 5.71, 5.72). Let $p$ be a point of $P$ and let us consider an open affine set $U_p \cong \mathbb{A}^3 \subset \mathbb{P}^3$ containing $p$. By abuse of notation, let us denote by $F := g_6g_7f_8 + g_6g_8f_7 + g_7g_8f_6$ the polynomial of degree 7 defining $P \cap U_p$. In order to compute the tangent cone to $P$ at the point $p$, we have to take the minimal degree homogeneous part of the Taylor series of $F$ at $p$. In the following, if $h$ is a polynomial, then $h_d(p)$ will denote the homogeneous part of degree

$d$ of the Taylor series of $h$ at $p$. By using this notation, if $p$ is a point of the quadric $Q_{i+j+3}$, we have that $T_pQ_{i+j+3} = \{g_{i+j+3,1}(p) = 0\}$, for $1 \leq i < j \leq 3$. Let us study $TC_pP$ case by case.

(i) Let us take $p \in \{q_1, \ldots, q_5\}$. Then $TC_pP$ has equation $g_{6,1}(p)g_{7,1}(p)f_{8,1}(p) + g_{6,1}(p)g_{8,1}(p)f_{7,1}(p)+g_{7,1}(p)g_{8,1}(p)f_{6,1}(p) = 0$, where $\{f_{8,1}(p) = 0\} = \langle T_pC_2, T_pC_3\rangle$, $\{f_{7,1}(p) = 0\} = \langle T_pC_1, T_pC_3\rangle$, $\{f_{6,1}(p) = 0\} = \langle T_pC_1, T_pC_2\rangle$ (see Remark 5.75). So we obtain $TC_pP = \bigcup_{1\leq i<j\leq 3} T_pQ_{i+j+3}$.

(ii) Let us take $p \in C_k$ such that $p \notin \{q_1, \ldots, q_5\}$, $p \notin r_k \cap C_k$ and $p \neq b_{ij}$ for distinct indices $i, j, k \in \{1, 2, 3\}$ with $i < j$. Let us suppose $k = 1$. Then $TC_pP$ has equation $c_1g_{6,1}(p)g_{7,1}(p) + c_2g_{6,1}(p)f_{7,1}(p) + c_3g_{7,1}(p)f_{6,1}(p) = 0$, where $c_1$, $c_2$, $c_3$ are constants (depending on the choice of $p$ and $P$). Since $\{g_{6,1}(p)g_{7,1}(p) = 0\}$, $\{g_{6,1}(p)f_{7,1}(p) = 0\}$ and $\{g_{7,1}(p)f_{6,1}(p) = 0\}$ are three reducible quadric surfaces given by two planes containing the line $T_pC_1$, then $TC_pP$ is singular along $T_pC_1$ and so it is the union of two planes containing $T_pC_1$. Similarly for $k = 2, 3$.

(iii) Let us take $p \in r_k \cap C_k$ for $1 \leq k \leq 3$. Let us suppose $k = 1$. Then $TC_pP$ has equation $c_1g_{6,1}(p)g_{7,1}(p) + c_2g_{6,1}(p)f_{7,1}(p) + c_3g_{7,1}(p)f_{6,1}(p) = 0$, where $c_1$, $c_2$, $c_3$ are constants (depending on the choice of $p$ and $P$) and where $\{g_{6,1}(p) = 0\} = T_pQ_6 \supset T_pC_1 \cup r_1$, $\{g_{7,1}(p) = 0\} = T_pQ_7 \supset T_pC_1 \cup r_1$, $\{f_{68,1}(p) = 0\} \supset T_pC_1$ and $\{f_{78,1}(p) = 0\} \supset T_pC_1$. In this case we also have $T_pQ_6 = T_pQ_7$, otherwise it would be $1 \leq \dim T_p(Q_6 \cap Q_7) \leq \dim(T_pQ_6 \cap T_pQ_7) = 1$ and $Q_6 \cap Q_7 = r_1 \cup C_1$ would be smooth at $p \in r_1 \cap C_1$, which is a contradiction. Thus $T_pP$ is the union of the plane $T_pQ_6(= TpQ_7)$, which contains $T_pC_1$ and $r_1$, and a plane containing $T_pC_1$. Similarly for $k = 2, 3$.

(iv) Let us take $p \in r_i\cap r_j\cap C_k$ for three distinct indices $i, j, k \in \{1, 2, 3\}$ and $i < j$. Let us suppose $i = 1$, $j = 2$, $k = 3$. Then $TC_pP$ has equation $g_{7,1}(p)g_{8,1}(p) = 0$, where $\{g_{7,1}(p) = 0\} = T_pQ_7 \supset T_pC_3 \cup r_1$ and $\{g_{8,1}(p) = 0\} \supset T_pC_3 \cup r_2$. Thus $TC_pP$ is the union of $T_pQ_7 \cup T_pQ_8$. Similarly by taking $(i, j, k) \in \{(1, 3, 2), (2, 3, 1)\}$.

(v) Let us take $p \in r_k$ with $p \notin r_k \cap C_k$ and $p \neq r_k \cap r_i$ for $1 \leq i, k \leq 3$ and $i \neq k$. Let us suppose $k = 1$. Then $TC_pP$ has equation $c_1g_{6,1}(p)+c_2g_{7,1}(p) = 0$ where $c_1$ and $c_2$ are constants depending on the choice of $p$ and $P$. Similarly for $k = 2, 3$.

□

**Lemma 5.77.** The rational map $\nu_{\mathcal{P}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^6$ defined by $\mathcal{P}$ is birational onto the image.

*Proof.* It is sufficient to prove that the map defined by $\mathcal{P}$ on a general $P \in \mathcal{P}$ is birational onto the image. This actually happens because $\mathcal{P}|_P$ contains a sublinear system that defines a birational map. Indeed $\mathcal{P}$ contains a sublinear system $\overline{\mathcal{P}} \subset \mathcal{P}$ whose fixed part is given by $Q_6 \cup Q_7 \cup Q_8$ and such that $\overline{\mathcal{P}}|_P$ coincides with the linear system on $P$ cut out by the planes of $\mathbb{P}^3$. □

**Remark 5.78.** The proof of Lemma 5.77 tells us that the linear system $\mathcal{P}$ is very ample outside the three quadric surfaces $Q_6$, $Q_7$, $Q_8$. So $\nu_{\mathcal{P}} : \mathbb{P}^3 \dashrightarrow \nu_{\mathcal{P}}(\mathbb{P}^3) \subset \mathbb{P}^6$ is an isomorphism outside $Q_6 \cup Q_7 \cup Q_8$.

**Theorem 5.79.** [23, §3] The image of $\mathbb{P}^3$ via the rational map defined by $\mathcal{P}$ is an Enriques-Fano threefold $W_F^6$ of genus $p = 6$.

*Proof.* We will prove the theorem by using the same techniques of the proof of Theorems 5.4, 5.24. In particular the proof is divided into several steps, given by the Remarks 5.80, 5.81, the Proposition 5.82, the Remarks 5.83,..., 5.92 and the Theorem 5.93 below.

First we blow-up $\mathbb{P}^3$ at the five points $q_1$, $q_2$, $q_3$, $q_4$, $q_5$, at the six points $a'_1$, $a'_2$, $a'_3$, $a''_1$, $a''_2$, $a''_3$ and at the three points $b_{12}$, $b_{13}$, $b_{23}$. We obtain a smooth threefold $Y'$ and a birational morphism $bl' : Y' \to \mathbb{P}^3$ with exceptional divisors $E_h := (bl')^{-1}(q_h)$, $E_{ij} = (bl')^{-1}(b_{ij})$, $E'_i := (bl')^{-1}(a'_i)$, $E''_i := (bl')^{-1}(a''_i)$, for $1 \leq h \leq 5$ and $1 \leq i < j \leq 3$. Let $\mathcal{P}'$ be the strict transform of $\mathcal{P}$ and let us denote by $H$ the pullback on $Y'$ of the hyperplane class on $\mathbb{P}^3$. Then an element of $\mathcal{P}'$ is linearly equivalent to $7H - 3\sum_{h=0}^{5} E_i - 2\sum_{i=1}^{3}(E'_i + E''_i) - 2\sum_{1\leq i<j\leq 3} E_{ij}$. Let $\widetilde{t}_{hij}$, $\widetilde{\pi}_{a'_i}$, $\widetilde{\pi}_{a''_i}$, $\widetilde{\pi}_{ij,i}$, $\widetilde{\pi}_{ij,j}$ be the strict transforms of the planes defined in Remark 5.75 and Proposition 5.76, for $1 \leq h \leq 5$ and $1 \leq i < j \leq 3$. Let us consider the following 27 lines on $Y'$:

$$\gamma_{hij} := E_h \cap \widetilde{t}_{hij},\ \lambda_{ij,i} := E_{ij} \cap \widetilde{\pi}_{ij,i},\ \lambda_{ij,j} := E_{ij} \cap \widetilde{\pi}_{ij,j},\ \lambda'_i = E'_i \cap \pi_{a'_i},\ \lambda''_i = E''_i \cap \pi_{a''_i}.$$

They are respectively $(-1)$-curves on $\widetilde{t}_{hij}$, $\widetilde{\pi}_{ij,i}$, $\widetilde{\pi}_{ij,j}$, $\widetilde{\pi}_{a'_i}$ and $\widetilde{\pi}_{a''_i}$. Let $P'$ be the strict transform of a general $P \in \mathcal{P}$. By Proposition 5.76 (i) and (iv) we have that $P' \cap E_h = \bigcup_{\leq i<j\leq 3} \gamma_{hij}$ and $P' \cap E_{ij} = \lambda_{ij,i} \cup \lambda_{ij,j}$, for all $1 \leq h \leq 5$ and $1 \leq i < j \leq 3$.

**Remark 5.80.** Let us fix $1 \leq i \leq 3$. We have that $P' \cap E'_i = \lambda'_i \cup \beta'_{i,P}$ and $P' \cap E''_i = \lambda''_i \cup \beta''_{i,P}$, where the curve $\beta'_{i,P}$ moves in the pencil of the lines of $E'_i$ through the point $E'_i \cap \widetilde{C}_i$ and the curve $\beta''_{i,P}$ moves in the pencil of the lines of $E''_i$ through the point $E''_i \cap \widetilde{C}_i$, and both lines depend on the choice of $P$ (see Proposition 5.76 (iii)).

Let us take the strict transforms $\widetilde{C}_i$ and $\widetilde{r}_i$ of the three twisted cubics and of their chords, for $1 \leq i \leq 3$. The base locus of $\mathcal{P}'$ is given by the union of the three curves $\widetilde{C}_i$ (along which a general $P' \in \mathcal{P}'$ has double points), of the three curves $\widetilde{r}_i$, and of the 27 curves $\gamma_{hij}$, $\lambda_{ij,i}$, $\lambda_{ij,j}$, $\lambda'_i$, $\lambda''_i$ defined above (see Proposition 5.76). Let us blow-up $Y'$ along the strict transforms of the three twisted cubics and of their chords. We obtain a smooth threefold $Y''$ and a birational morphism $bl'' : Y'' \to Y'$ with exceptional divisors $(bl'')^{-1}(\widetilde{C}_i) := F_i \cong \mathbb{P}(\mathcal{N}_{\widetilde{C}_i|Y'}) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)) \cong \mathbb{F}_0$ and $(bl'')^{-1}(\widetilde{r}_i) := R_i \cong \mathbb{P}(\mathcal{N}_{\widetilde{r}_i|Y'}) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)) \cong \mathbb{F}_0$ for $1 \leq i \leq 3$, since $\mathcal{N}_{C_i|\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5)$ (see [22, Proposition 6]). Let us denote by $\widetilde{E}_h$, $\widetilde{E}'_i$, $\widetilde{E}''_i$ and $\widetilde{E}_{ij}$ respectively the strict transforms of $E_h$, $E'_i$, $E''_i$, $E_{ij}$, for $1 \leq h \leq 5$ and $0 \leq i < j \leq 3$.

**Remark 5.81.** Let us consider the curves $\alpha_{hi} := \widetilde{E}_h \cap F_i$, $\alpha'_i := \widetilde{E}'_i \cap F_i$, $\alpha''_i := \widetilde{E}''_i \cap F_i$, $\alpha_{ijk} := \widetilde{E}_{ij} \cap F_k$, $\rho'_i := \widetilde{E}'_i \cap R_i$, $\rho''_i := \widetilde{E}''_i \cap R_i$, $\rho_{ij,i} := \widetilde{E}_{ij} \cap R_i$, $\rho_{ij,j} := \widetilde{E}_{ij} \cap R_j$ for $1 \leq h \leq 5$ and distinct indices $i, j, k \in \{1, 2, 3\}$ with $i < j$. Each of these curves is a fibre on the exceptional divisor of $bl'' : Y'' \to Y'$ which contains it, and is a $(-1)$-curve on the strict transform of the exceptional divisor of $bl : Y' \to \mathbb{P}^3$ containing it.

Let $\mathcal{P}''$ be the strict transform of $\mathcal{P}'$. If $P''$ is an element of $\mathcal{P}''$, then

$$P'' \sim 7H - 3\sum_{h=1}^{5} \widetilde{E}_h - 2\sum_{i=1}^{3}(\widetilde{E}'_i + \widetilde{E}''_i) - 2\sum_{1\le i<j\le 3} \widetilde{E}_{ij} - 2\sum_{i=1}^{3} F_i - \sum_{i=1}^{3} R_i,$$

where, by abuse of notation, $H$ denotes the pullback $bl''^{*}H$.

**Proposition 5.82.** A general element $P'' \in \mathcal{P}''$ is a smooth surface with zero arithmetic genus $p_a(P'') = 0$.

*Proof.* The smoothness of $P''$ is shown in [27, p.620-621], since $P''$ is the blow-up of a surface $P \in \mathcal{P}$ with ordinary singularities along its singular curves (see Definition 3.4 and Proposition 5.76). We have to compute $p_a(P'') = \chi(\mathcal{O}_{Y''}(K_{Y''} + P''))$ (see proof of Proposition 5.26). Since

$$K_{Y''} \sim -4H + 2\sum_{h=1}^{5} \widetilde{E}_h + 2\sum_{i=1}^{3}(\widetilde{E}'_i + \widetilde{E}''_i) + 2\sum_{1\le i<j\le 3} \widetilde{E}_{ij} + \sum_{i=1}^{3} F_i + \sum_{i=1}^{3} R_i$$

(see [27, p.187]), then $K_{Y''} + P'' \sim 3H - \sum_{h=1}^{5} \widetilde{E}_h - \sum_{i=1}^{3} F_i$, by the adjunction formula. By denoting the fibre class of $F_i$ by $f_i$ for $i = 1, 2, 3$, we have the following two exact sequences:

$$0 \to \mathcal{O}_{Y''}(3H - \sum_{h=1}^{5} \widetilde{E}_h) \to \mathcal{O}_{Y''}(3H) \to \oplus_{h=1}^{5}\mathcal{O}_{\widetilde{E}_h} \to 0,$$

$$0 \to \mathcal{O}_{Y''}(K_{Y''} + P'') \to \mathcal{O}_{Y''}(3H - \sum_{h=1}^{5} \widetilde{E}_h) \to \oplus_{i=1}^{3}\mathcal{O}_{F_i}(4f_i) \to 0.$$

So we obtain $\chi(\mathcal{O}_{Y''}(K_{Y''} + P'')) = \binom{6}{3} - 5 - 3\cdot 5 = 0$. □

By Proposition 5.76, we have that the base locus of $\mathcal{P}''$ is given by the disjoint union of the strict transforms $\widetilde{\gamma}_{hij}$, $\widetilde{\lambda}_{ij,i}$, $\widetilde{\lambda}_{ij,j}$, $\widetilde{\lambda}'_i$ and $\widetilde{\lambda}''_i$ of the 27 lines $\gamma_{hij}$, $\lambda_{ij,x}$, $\lambda'_i$, $\lambda''_i$ for $1 \le h \le 5$ and $1 \le i < j \le 3$.

**Remark 5.83.** We observe that $\widetilde{\gamma}^2_{hij}|_{\widetilde{E}_h} = \widetilde{\lambda}^2_{ij,i}|_{\widetilde{E}_{ij}} = \widetilde{\lambda}^2_{ij,j}|_{\widetilde{E}_{ij}} = \widetilde{\lambda}'^2_i|_{\widetilde{E}'_i} = \widetilde{\lambda}''^2_i|_{\widetilde{E}''_i} = -1$. Furthermore, by using similar arguments to the ones in Remark 5.7 and Remark 5.52, we have that the 27 curves $\widetilde{\gamma}_{hij}$, $\widetilde{\lambda}_{ij,i}$, $\widetilde{\lambda}_{ij,j}$, $\widetilde{\lambda}'_i$ and $\widetilde{\lambda}''_i$ are $(-1)$-curves on the strict transform $P''$ of a general $P' \in \mathcal{P}'$. Moreover $P''$ contains other $(-1)$-curves that depend on $P''$ itself: they are the strict transforms $\widetilde{\beta}'_{i,P}$ and $\widetilde{\beta}''_{i,P}$ of the curves defined in Remark 5.80.

Finally let us consider $bl''' : Y \to Y''$ the blow-up of $Y''$ along the above 27 curves, with exceptional divisors $\Gamma_{hij} := bl'''^{-1}(\widetilde{\gamma}_{hij})$, $\Lambda_{ij,i} := bl'''^{-1}(\widetilde{\lambda}_{ij,i})$, $\Lambda_{ij,j} := bl'''^{-1}(\widetilde{\lambda}_{ij,j})$, $\Lambda'_i := bl'''^{-1}(\widetilde{\lambda}'_i)$, $\Lambda''_i := bl'''^{-1}(\widetilde{\lambda}''_i)$ for $1 \le h \le 5$ and $1 \le i < j \le 3$. We denote by $\mathcal{E}_h$, $\mathcal{E}'_i$, $\mathcal{E}''_i$ and $\mathcal{E}_{ij}$ respectively the strict transform of $\widetilde{E}_h$, $\widetilde{E}'_i$, $\widetilde{E}''_i$ and $\widetilde{E}_{ij}$; by $\mathcal{F}_i$ the strict transform of $F_i$; by $\mathcal{R}_i$ the strict transform of $R_i$; by $\mathcal{H}$ the pullback of $H$.

**Remark 5.84.** We have that

$$\Gamma_{hij} = \mathbb{P}(\mathcal{N}_{\widetilde{\gamma}_{hij}|Y''}) \cong \mathbb{P}(\mathcal{O}_{\widetilde{\gamma}_{hij}}(\widetilde{E}_h) \oplus \mathcal{O}_{\widetilde{\gamma}_{hij}}(\widetilde{t}_{hij})) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{F}_0,$$

$$\Lambda_{ij,i} = \mathbb{P}(\mathcal{N}_{\widetilde{\lambda}_{ij,i}|Y''}) \cong \mathbb{P}(\mathcal{O}_{\widetilde{\lambda}_{ij,i}}(\widetilde{E}_{ij}) \oplus \mathcal{O}_{\widetilde{\lambda}_{ij,i}}(\widetilde{\pi}_{ij,i})) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{F}_0,$$

$$\Lambda_{ij,j} = \mathbb{P}(\mathcal{N}_{\widetilde{\lambda}_{ij,j}|Y''}) \cong \mathbb{P}(\mathcal{O}_{\widetilde{\lambda}_{ij,j}}(\widetilde{E}_{ij}) \oplus \mathcal{O}_{\widetilde{\lambda}_{ij,j}}(\widetilde{\pi}_{ij,j})) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{F}_0,$$

$$\Lambda'_i = \mathbb{P}(\mathcal{N}_{\widetilde{\lambda}'_i|Y''}) \cong \mathbb{P}(\mathcal{O}_{\widetilde{\lambda}'_i}(\widetilde{E}'_i) \oplus \mathcal{O}_{\widetilde{\lambda}'_i}(\widetilde{\pi}_{a'_i})) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{F}_0,$$

$$\Lambda''_i = \mathbb{P}(\mathcal{N}_{\widetilde{\lambda}''_i|Y''}) \cong \mathbb{P}(\mathcal{O}_{\widetilde{\lambda}''_i}(\widetilde{E}''_i) \oplus \mathcal{O}_{\widetilde{\lambda}''_i}(\widetilde{\pi}_{a''_i})) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{F}_0,$$

Furthermore we have $\Gamma^3_{hij} = -\deg(\mathcal{N}_{\widetilde{\gamma}_{hij}|Y''}) = 2$, $\Lambda^3_{ij,i} = -\deg(\mathcal{N}_{\widetilde{\lambda}_{ij,i}|Y''}) = 2$, $\Lambda^3_{ij,j} = -\deg(\mathcal{N}_{\widetilde{\lambda}_{ij,j}|Y''}) = 2$, $\Lambda'^3_i = -\deg(\mathcal{N}_{\widetilde{\lambda}'_i|Y''}) = 2$, $\Lambda''^3_i = -\deg(\mathcal{N}_{\widetilde{\lambda}''_i|Y''}) = 2$ (see [27, Chap 4, §6] and [32, Lemma 2.2.14]).

**Remark 5.85.** Let us take $1 \leq h \leq 5$ and distinct indices $i, j, k \in \{1, 2, 3\}$ with $i < j$. The divisor $\mathcal{F}_k$ intersects $\Gamma_{hst}$, $\Lambda'_k$, $\Lambda''_k$, $\Lambda_{ij,i}$, $\Lambda_{ij,j}$ each along a $\mathbb{P}^1$ which is a $(-1)$-curve on $\mathcal{F}_i$ and a fibre on $\Gamma_{hst}$, $\Lambda'_k$, $\Lambda''_k$, $\Lambda_{ij,i}$, $\Lambda_{ij,j}$, where $1 \leq s < t \leq 3$ and $k \in \{s, t\}$. Similarly we have $\Lambda'^2_i \cdot \mathcal{R}_i = \Lambda''^2_i \cdot \mathcal{R}_i = \Lambda^2_{ij,i} \cdot \mathcal{R}_i = \Lambda^2_{ij,j} \cdot \mathcal{R}_j = -1$ and $\Lambda'_i \cdot \mathcal{R}^2_i = \Lambda''_i \cdot \mathcal{R}^2_i = \Lambda_{ij,i} \cdot \mathcal{R}^2_i = \Lambda^2_{ij,j} \cdot \mathcal{R}_j = 0$. Let us consider the strict transforms $\widetilde{\alpha}_{hi}$, $\widetilde{\alpha}_{ijk}$, $\widetilde{\alpha}'_i$, $\widetilde{\alpha}''_i$, $\widetilde{\rho}_{ij,i}$, $\widetilde{\rho}_{ij,j}$, $\widetilde{\rho}'_i$, $\widetilde{\rho}''_i$ of the curves defined in Remark 5.81. Then we have

$$\widetilde{\alpha}^2_{hi}|_{\mathcal{E}_h} = \mathcal{F}^2_i \cdot \mathcal{E}_h = -1,\ \widetilde{\alpha}^2_{ijk}|_{\mathcal{E}_{ij}} = {\mathcal{F}_k}^2 \cdot \mathcal{E}_{ij} = -1,\ \widetilde{\alpha}'^2_i|_{\mathcal{E}'_i} = \mathcal{F}^2_i \cdot \mathcal{E}'_i = -1,\ \widetilde{\alpha}''^2_i|_{\mathcal{E}''_i} = \mathcal{F}^2_i \cdot \mathcal{E}''_i = -1,$$

$$\widetilde{\alpha}^2_{hi}|_{\mathcal{F}_i} = \mathcal{E}^2_h \cdot \mathcal{F}_i = -2,\ \widetilde{\alpha}^2_{ijk}|_{\mathcal{F}_k} = \mathcal{E}^2_{ij} \cdot \mathcal{F}_k = -2,\ \widetilde{\alpha}'^2_i|_{\mathcal{F}^2_i} = \mathcal{E}'^2_i \cdot \mathcal{F}_i = -1,\ \widetilde{\alpha}''^2_i|_{\mathcal{F}^2_i} = \mathcal{E}''^2_i \cdot \mathcal{F}_i = -1,$$

$$\widetilde{\rho}^2_{ij,i}|_{\mathcal{E}_{ij}} = {\mathcal{R}_i}^2 \cdot \mathcal{E}_{ij} = -1,\ \widetilde{\rho}^2_{ij,j}|_{\mathcal{E}_{ij}} = {\mathcal{R}_j}^2 \cdot \mathcal{E}_{ij} = -1,\ \widetilde{\rho}'^2_i|_{\mathcal{E}'_i} = \mathcal{R}^2_i \cdot \mathcal{E}'_i = -1,\ \widetilde{\rho}''^2_i|_{\mathcal{E}''_i} = \mathcal{R}^2_i \cdot \mathcal{E}''_i = -1,$$

$$\widetilde{\rho}^2_{ij,i}|_{\mathcal{R}_i} = \mathcal{E}^2_{ij} \cdot \mathcal{R}_i = -1,\ \widetilde{\rho}^2_{ij,j}|_{\mathcal{R}_j} = \mathcal{E}^2_{ij} \cdot \mathcal{R}_j = -1,\ \widetilde{\rho}'^2_i|_{\mathcal{R}_i} = \mathcal{E}'^2_i \cdot \mathcal{R}_i = -1,\ \widetilde{\rho}''^2_i|_{\mathcal{R}_i} = \mathcal{E}''^2_i \cdot \mathcal{R}_i = -1.$$

Finally we recall that a general line of $\mathbb{P}^3$ does not intersect the three twisted cubics $C_1$, $C_2$, $C_3$ and their chords; instead a general plane of $\mathbb{P}^3$ intersects each twisted cubic at three points and each chord at one point. Hence we have $\mathcal{H}^2 \cdot \mathcal{F}_i = \mathcal{H}^2 \cdot \mathcal{R}_i = 0$, $\mathcal{F}^2_i \cdot \mathcal{H} = -3$ and $\mathcal{R}^2_i \cdot \mathcal{H} = -1$.

**Remark 5.86.** We recall that by construction we have $bl''''^*(\widetilde{E}_h) = \mathcal{E}_h + \sum_{1 \leq i < j \leq 3} \Gamma_{hij}$, $bl''''^*(\widetilde{E}_{ij}) = \mathcal{E}_{ij} + \Lambda_{ij,i} + \Lambda_{ij,j}$, $bl''''^*(\widetilde{E}'_i) = \mathcal{E}'_i + \Lambda'_i$, $bl''''^*(\widetilde{E}''_i) = \mathcal{E}''_i + \Lambda''_i$. By abuse of notation, we denote $\mathcal{E}_h \cap \Gamma_{hij}$, $\mathcal{E}_{ij} \cap \Lambda_{ij,x}$, $\mathcal{E}'_i \cap \Lambda'_i$ and $\mathcal{E}''_i \cap \Lambda''_i$ respectively by $\widetilde{\gamma}_{hij}$, $\widetilde{\lambda}_{ij,i}$, $\widetilde{\lambda}_{ij,j}$, $\widetilde{\lambda}'_i$, $\widetilde{\lambda}''_i$ for $1 \leq h \leq 5$ and $1 \leq i < j \leq 3$. Let $\mathcal{L}_h$, $\mathcal{L}_{ij}$, $\mathcal{L}'_i$, $\mathcal{L}''_i$ be respectively the strict transform on $Y$ of a general line of $E_h$, $E_{ij}$, $E'_i$ and $E''_i$. By using similar arguments to the ones in Remark 5.30 we obtain that $\mathcal{E}^3_h = 4$, $\mathcal{E}^3_{ij} = 3$ and $\mathcal{E}'^3_i = \mathcal{E}''^3_i = 2$, since we have $\mathcal{E}_h|_{\mathcal{E}_h} \sim -(\mathcal{L}_h + \sum_{1 \leq i < j \leq 3} \widetilde{\gamma}_{hij}) \sim -(4\mathcal{L}_h - 2\sum_{i=1}^{3} \widetilde{\alpha}_{hi})$, $\mathcal{E}_{ij}|_{\mathcal{E}_{ij}} \sim -(\mathcal{L}_{ij} + \widetilde{\lambda}_{ij,i} + \widetilde{\lambda}_{ij,j}) \sim -(3\mathcal{L}_{ij} - 2\widetilde{\alpha}_{ijk} - \widetilde{\rho}_{ij,i} - \widetilde{\rho}_{ij,j})$, $\mathcal{E}'_i|_{\mathcal{E}'_i} \sim -(\mathcal{L}'_i + \widetilde{\lambda}'_i) \sim -(2\mathcal{L}'_i - \widetilde{\alpha}'_i - \widetilde{\rho}'_i)$ and $\mathcal{E}''_i|_{\mathcal{E}''_i} \sim -(\mathcal{L}''_i + \widetilde{\lambda}''_i) \sim -(2\mathcal{L}''_i - \widetilde{\alpha}''_i - \widetilde{\rho}''_i)$.

**Remark 5.87.** By using similar arguments to the ones in Remark 5.11 we have $\mathcal{F}^3_i = -\deg(\mathcal{N}_{\widetilde{C}_i|Y'}) = 6$ and $\mathcal{R}^3_i = -\deg(\mathcal{N}_{\widetilde{r}_i|Y'}) = 6$ for $1 \leq i \leq 3$.

Let $\widetilde{P}$ be the strict transform on $Y$ of an element of $\mathcal{P}''$: then

$$\mathcal{P} \sim 7\mathcal{H} - 3\sum_{h=1}^{5}\mathcal{E}_h - 2\sum_{i=1}^{3}(\mathcal{E}'_i + \mathcal{E}''_i) - 2\sum_{1\le i<j\le 3}\mathcal{E}_{ij} - 2\sum_{i=1}^{3}\mathcal{F}_i - \sum_{i=1}^{3}\mathcal{R}_i +$$

$$-4\sum_{\substack{h=1\\ 1\le i<j\le 3}}^{5}\Gamma_{hij} - 3\sum_{i=1}^{3}(\Lambda'_i + \Lambda''_i) - 3\sum_{1\le i<j\le 3}(\Lambda_{ij,i} + \Lambda_{ij,j}).$$

Let us take the linear system $\widetilde{\mathcal{P}} := |\mathcal{O}_Y(\widetilde{P})|$ on $Y$. It is base point free and it defines a morphism $\nu_{\widetilde{\mathcal{P}}} : Y \to \mathbb{P}^6$ birational onto the image $W_F^6 := \nu_{\widetilde{\mathcal{P}}}(Y)$, which is a threefold of degree $\deg W_F^6 = 10$. It follows by Lemma 5.77 and by the fact that $\widetilde{P}^3 = 10$ (use Remarks 5.84, 5.85, 5.86, 5.87 and calculations similar to the ones in § 5.2, 5.3). Then we have the following diagram:

$$\begin{array}{ccccccc}
Y & & & & & \overset{\nu_{\widetilde{\mathcal{P}}}}{\searrow} & \\
\downarrow{\scriptstyle bl'''} & & & & & & \\
Y'' & \xrightarrow{bl''} & Y' & \xrightarrow{bl'} & \mathbb{P}^3 & \overset{\nu_{\mathcal{P}}}{\dashrightarrow} & W_F^6 \subset \mathbb{P}^6.
\end{array}$$

It remains to show that the general hyperplane section of the threefold $W_F^6$ is an Enriques surface.

**Remark 5.88.** Let $\widetilde{Q}_6$, $\widetilde{Q}_7$ and $\widetilde{Q}_8$ be the strict transforms on $Y$ of the quadric surfaces $Q_6$, $Q_7$, $Q_8$. By construction we have $\widetilde{P} \cdot \mathcal{E}_h = \widetilde{P} \cdot \mathcal{E}_{ij} = \widetilde{P} \cdot \widetilde{Q}_{i+j+3} = 0$ for a general $\widetilde{P} \in \widetilde{\mathcal{P}}$ and for all $1 \le h \le 5$ and $1 \le i < j \le 3$.

**Remark 5.89.** The 27 exceptional divisors of $bl''' : Y \to Y''$, the six divisors $\mathcal{E}_i$ and $\mathcal{E}'_i$, and the three divisors $\mathcal{R}_i$ are contracted by the morphism $\nu_{\widetilde{\mathcal{P}}} : Y \to W_F^6 \subset \mathbb{P}^6$ to curves of $W_F^6$. This follows by the fact that $\widetilde{P} \cdot \Gamma_{hij}$, $\widetilde{P} \cdot \Lambda_{ij,i}$, $\widetilde{P} \cdot \Lambda_{ij,j}$, $\widetilde{P} \cdot \Lambda'_i$, $\widetilde{P} \cdot \Lambda''_i$, $\widetilde{P} \cdot \mathcal{E}'_i$, $\widetilde{P} \cdot \mathcal{E}''_i$, $\widetilde{P} \cdot \mathcal{R}_i \neq 0$ and $\widetilde{P}^2 \cdot \Gamma_{hij} = \widetilde{P}^2 \cdot \Lambda_{ij,i} = \widetilde{P}^2 \cdot \Lambda_{ij,j} = \widetilde{P}^2 \cdot \Lambda'_i = \widetilde{P}^2 \cdot \Lambda''_i = \widetilde{P}^2 \cdot \mathcal{E}'_i = \widetilde{P}^2 \cdot \mathcal{E}''_i = \widetilde{P}^2 \cdot \mathcal{R}_i = 0$ for $1 \le h \le 5$ and $1 \le i < j \le 3$ (use Remarks 5.84, 5.85 and calculations similar to the ones in Remarks 5.13, 5.34).

**Remark 5.90.** Let us fix $0 \le i \le 3$ and let $\widetilde{P}$ be a general element of $\widetilde{\mathcal{P}}$. Since $\widetilde{P}^2 \cdot \mathcal{F}_i = 10 > 0$ (use Remarks 5.85, 5.87 and calculations similar to the ones in Remarks 5.14, 5.35), then the curve $\widetilde{P} \cap \mathcal{F}_i$ is not contracted by the rational map defined by $\widetilde{\mathcal{P}}|_{\widetilde{P}}$.

**Remark 5.91.** Let us fix $1 \le h \le 5$ and $1 \le i < j \le 3$. Let us consider a general element $\widetilde{P} \in \widetilde{\mathcal{P}}$ and let us take $S := \nu_{\widetilde{\mathcal{P}}}(\widetilde{P})$ and $P'' := bl'''(\widetilde{P}) \in \mathcal{P}''$. Since $bl''' : Y \to Y''$ has no effect on $P''$, then $\widetilde{P} \cap \Gamma_{hij}$, $\widetilde{P} \cap \Lambda_{ij,i}$, $\widetilde{P} \cap \Lambda_{ij,j}$, $\widetilde{P} \cap \Lambda'_i$, $\widetilde{P} \cap \Lambda''_i$, $\widetilde{P} \cap \mathcal{E}'_i \cong \widetilde{\beta}'_{i,P}$ and $\widetilde{P} \cap \mathcal{E}''_i \cong \widetilde{\beta}''_{i,P}$ are still $(-1)$-curves on $\widetilde{P}$ (see Remark 5.83). We also have that $(\widetilde{P} \cap \mathcal{R}_i)|^2_{\widetilde{P}} = \mathcal{R}_i^2 \cdot \widetilde{P} = -5$ (use Remarks 5.85, 5.87). Furthermore $\widetilde{P} \cap \mathcal{R}_i$ intersects the four curves $\widetilde{P} \cap \Lambda'_i$, $\widetilde{P} \cap \Lambda''_i$, $\widetilde{P} \cap \Lambda_{ij,i}$, $\widetilde{P} \cap \Lambda_{st,i}$ at one point each, for $1 \le s < t \le 3$ and $i \in \{s, t\}$ (use Remark 5.85). Thus we can see the map $\nu_{\widetilde{\mathcal{P}}}|_{\widetilde{P}} : \widetilde{P} \to S$ as the blow-up of $S$ at the 21 points $\nu_{\widetilde{\mathcal{P}}}(\widetilde{P} \cap \Gamma_{hij})$, $\nu_{\widetilde{\mathcal{P}}}(\widetilde{P} \cap \mathcal{E}'_i)$, $\nu_{\widetilde{\mathcal{P}}}(\widetilde{P} \cap \mathcal{E}''_i)$, at the three points $\nu_{\widetilde{\mathcal{P}}}(\widetilde{P} \cap \mathcal{R}_i)$

and at the four points $\nu_{\widetilde{\mathcal{P}}}(\widetilde{P}\cap\Lambda'_i)$, $\nu_{\widetilde{\mathcal{P}}}(\widetilde{P}\cap\Lambda''_i)$, $\nu_{\widetilde{\mathcal{P}}}(\widetilde{P}\cap\Lambda_{ij,i})$, $\nu_{\widetilde{\mathcal{P}}}(\widetilde{P}\cap\Lambda_{st,i})$ which are infinitely near to each $\nu_{\widetilde{\mathcal{P}}}(\widetilde{P}\cap\mathcal{R}_i)$ (see Remarks 5.78, 5.88, 5.89, 5.90). Then $S$ is a smooth surface.

**Remark 5.92.** The surface $Q_6\cup Q_7\cup Q_8$ is the only sextic surface of $\mathbb{P}^3$ which is singular along the three twisted cubic $C_1$, $C_2$, $C_3$. Let us consider the strict transforms $\widetilde{Q}_6$, $\widetilde{Q}_7$ and $\widetilde{Q}_8$ on $Y$ of these quadric surfaces. Then we have

$$\widetilde{Q}_6+\widetilde{Q}_7+\widetilde{Q}_8\sim 6\mathcal{H}-\sum_{h=1}^{5}3\mathcal{E}_h-\sum_{i=1}^{3}2(\mathcal{E}'_i+\mathcal{E}''_i)-\sum_{1\le i<j\le 3}3\mathcal{E}_{ij}-\sum_{i=1}^{3}2\mathcal{F}_i-\sum_{i=1}^{3}2\mathcal{R}_i+$$

$$-\sum_{\substack{h=1\\1\le i<j\le 3}}^{5}4\Gamma_{hij}-\sum_{i=1}^{3}4(\Lambda'_i+\Lambda''_i)-\sum_{1\le i<j\le 3}4(\Lambda_{ij,i}+\Lambda_{ij,j}).$$

If $\widetilde{P}$ is a general element of $\widetilde{\mathcal{P}}$, then $0\sim(\widetilde{Q}_6+\widetilde{Q}_7+\widetilde{Q}_8)|_{\widetilde{P}}\sim\Big(6\mathcal{H}-\sum_{i=1}^{3}2(\mathcal{E}'_i+\mathcal{E}''_i)-\sum_{i=1}^{3}2\mathcal{F}_i-\sum_{i=1}^{3}2\mathcal{R}_i-4\sum_{\substack{h=1,\dots,5\\1\le i<j\le 3}}\Gamma_{hij}-4\sum_{i=1}^{3}(\Lambda'_i+\Lambda''_i)-4\sum_{1\le i<j\le 3}(\Lambda_{ij,i}+\Lambda_{ij,j})\Big)|_{\widetilde{P}}$.

**Theorem 5.93.** Let $S$ be a general hyperplane section of the threefold $W^6_F\subset\mathbb{P}^6$. Then $S$ is an Enriques surface.

*Proof.* We recall that $S$ is the image of a general element $\widetilde{P}\in\widetilde{\mathcal{P}}$, via the birational morphism $\nu_{\widetilde{\mathcal{P}}}:Y\to W^6_F\subset\mathbb{P}^6$. Furthermore $S$ is smooth (see Remark 5.91). By Proposition 5.82 and by using the arguments of Theorem 5.93, we have that $q(\widetilde{P})=p_g(\widetilde{P})=0$. It remains to prove that $2K_S\sim 0$. Since by [27, p.187] we have that

$$K_Y=bl'''^*(K_{Y''})+\sum_{\substack{h=1\\1\le i<j\le 3}}^{5}\Gamma_{hij}+\sum_{i=1}^{3}(\Lambda'_i+\Lambda''_i)+\sum_{1\le i<j\le 3}(\Lambda_{ij,i}+\Lambda_{ij,j})\sim$$

$$\sim -4\mathcal{H}+\sum_{h=1}^{5}2\mathcal{E}_h+\sum_{i=1}^{3}2(\mathcal{E}'_i+\mathcal{E}''_i)+\sum_{1\le i<j\le 3}2\mathcal{E}_{ij}+\sum_{i=1}^{3}\mathcal{F}_i+\sum_{i=1}^{3}\mathcal{R}_i+$$

$$+3\sum_{\substack{h=1\\1\le i<j\le 3}}^{5}\Gamma_{hij}+3\sum_{i=1}^{3}(\Lambda'_i+\Lambda''_i)+3\sum_{1\le i<j\le 3}(\Lambda_{ij,i}+\Lambda_{ij,j}),$$

then we obtain that $2K_{\widetilde{P}}=2(K_Y+\widetilde{P})|_{\widetilde{P}}\sim(6\mathcal{H}-\sum_{i=1}^{3}2\mathcal{F}_i-2\sum_{\substack{h=1,\dots,5\\1\le i<j\le 3}}\Gamma_{hij})|_{\widetilde{P}}$. Furthermore, by Remark 5.92, we have

$$2K_{\widetilde{P}}\sim\Big(\sum_{i=1}^{3}2(\mathcal{E}'_i+\mathcal{E}''_i)+\sum_{i=1}^{3}2\mathcal{R}_i+\sum_{\substack{h=1\\1\le i<j\le 3}}^{5}2\Gamma_{hij}+\sum_{i=1}^{3}4(\Lambda'_i+\Lambda''_i)+\sum_{1\le i<j\le 3}4(\Lambda_{ij,i}+\Lambda_{ij,j})\Big)|_{\widetilde{P}}.$$

Finally, by Remark5.91, we have $2K_S\sim(\nu_{\widetilde{\mathcal{P}}})_*(2K_{\widetilde{P}})\sim 0$. □

One can prove that $W_F^6 \subset \mathbb{P}^6$ is not a cone over a general hyperplane section, as in the proof of Theorem 5.15. So $W_F^6 \subset \mathbb{P}^6$ satisfies Assumption (*) of § 3.3 and we can obtain an Enriques-Fano threefold in the sense of Definition 3.1 by taking its normalization.

□

It would be interesting to verify with modern techniques if the general hyperplane section of $W_F^6 \subset \mathbb{P}^6$ actually is a Reye congruence, as stated by Fano in [23, §3] (see also [13, Proposition 3]).

### 5.5.2 Singularities of $W_F^6$

We recall that the divisors $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, $\mathcal{E}_4$, $\mathcal{E}_5$, $\widetilde{Q}_6$, $\widetilde{Q}_7$, $\widetilde{Q}_8$ are contracted by $\nu_{\widetilde{\mathcal{P}}} : Y \to W_F^6 \subset \mathbb{P}^6$ to points of $W_F^6$ (see Remark 5.88). Let us define $P_h := \nu_{\widetilde{\mathcal{P}}}(\mathcal{E}_h)$ for $1 \leq h \leq 5$ and $P_6 := \nu_{\widetilde{\mathcal{P}}}(\widetilde{Q}_6)$, $P_7 := \nu_{\widetilde{\mathcal{P}}}(\widetilde{Q}_7)$, $P_8 := \nu_{\widetilde{\mathcal{P}}}(\widetilde{Q}_8)$.

**Remark 5.94.** By Remark 5.88 we have that $\nu_{\widetilde{\mathcal{P}}}(\mathcal{E}_{ij})$ is a point of $W_F^6$ for all $1 \leq i < j \leq 3$. In particular we have $\nu_{\widetilde{\mathcal{P}}}(\mathcal{E}_{ij}) = \nu_{\widetilde{\mathcal{P}}}(\widetilde{Q}_{i+j+3})$, since $\widetilde{Q}_{i+j+3} \cap \mathcal{E}_{ij} \neq \emptyset$. Indeed one can verify that $\widetilde{Q}_{i+j+3} \cap \mathcal{E}_{ij}$ is the strict transform of the line of $E_{ij}$ joining the points $E_{ij} \cap \widetilde{r}_i$ and $E_{ij} \cap \widetilde{r}_j$.

**Proposition 5.95.** The eight points $P_1, \ldots, P_8$, defined as above, are quadruple points of $W_F^6$ whose tangent cone is a cone over a Veronese surface.

*Proof.* The analysis of the points $P_1$, $P_2$, $P_3$, $P_4$ and $P_5$ follows by Remark 5.86, as in the proof of Proposition 5.16. Let us fix now three distinct indices $i, j, k \in \{1, 2, 3\}$ with $i < j$. The hyperplane sections of $W_F^6 \subset \mathbb{P}^6$ passing through $P_{i+j+3}$ correspond to the elements of $\widetilde{\mathcal{P}}$ containing $\widetilde{Q}_{i+j+3} \cup \mathcal{E}_{ij}$ (see Remark 5.94). Let $\widetilde{\mathcal{P}}_{ij} := \widetilde{\mathcal{P}} - \widetilde{Q}_{i+j+3} - \mathcal{E}_{ij}$ be the sublinear system of $\widetilde{\mathcal{P}}$ defined by these elements. Let us study $\widetilde{\mathcal{P}}_{ij}|_{\widetilde{Q}_{i+j+3}} = |\mathcal{O}_{\widetilde{Q}_{i+j+3}}(-\widetilde{Q}_{i+j+3} - \mathcal{E}_{ij})|$. If we consider the case $(i, j) = (1, 2)$, we have

$$\widetilde{Q}_6 \sim_Y 2\mathcal{H} - \sum_{h=1}^{5} \mathcal{E}_h - \sum_{t=1,2} (\mathcal{E}'_t + \mathcal{E}''_t) - \sum_{1 \leq r < s \leq 3} \mathcal{E}_{rs} - \sum_{t=1,2} \mathcal{F}_t - \sum_{t=1,2} \mathcal{R}_t +$$

$$- \sum_{h=1}^{5} (2\Gamma_{h12} + \Gamma_{h13} + \Gamma_{h23}) - \sum_{t=1,2} 2(\Lambda'_t + \Lambda''_t) - \sum_{1 \leq r < s \leq 3} (\Lambda_{rs,r} + \Lambda_{rs,s}),$$

and so

$$\widetilde{Q}_6|_{\widetilde{Q}_6} \sim_{\widetilde{Q}_6} \left( 2\mathcal{H} - \sum_{1 \leq r < s \leq 3} \mathcal{E}_{rs} - \sum_{t=1,2} (\mathcal{F}_t + \mathcal{R}_t) - \sum_{h=1}^{5} 2\Gamma_{h12} - \sum_{t=1,2} 2(\Lambda'_t + \Lambda''_t) \right) |_{\widetilde{Q}_6}.$$

Let $\mathcal{C}_6$ be the pullback on $\widetilde{Q}_6$ of the linear equivalence class of the hyperplane sections of $Q_6$. By abuse of notation, let us denote by $\widetilde{\gamma}_{h12}$, $\widetilde{\lambda}'_t$, $\widetilde{\lambda}''_t$ the $(-1)$-curves on $\widetilde{Q}_6$ given by $\Gamma_{h12}|_{\widetilde{Q}_6}$, $\Lambda'_t|_{\widetilde{Q}_6}$, $\Lambda''_t|_{\widetilde{Q}_6}$ for $1 \leq h \leq 5$ and $t = 1, 2$. Let us also consider the $(-1)$-curves on $\widetilde{Q}_6$ defined by $\epsilon_{rs} := \mathcal{E}'_{rs}|_{\widetilde{Q}_6}$ for $1 \le r < s \le 3$. Then we have $\widetilde{Q}_6|_{\widetilde{Q}_6} \sim_{\widetilde{Q}_6} 2\mathcal{C}_6 - \sum_{1\le r<s\le 3}\epsilon_{rs} - (2\mathcal{C}_6 - \sum_{h=1}^5 \widetilde{\gamma}_{h12} - \widetilde{\lambda}'_1 - \widetilde{\lambda}''_1 - \epsilon_{23} - \epsilon_{12}) - (2\mathcal{C}_6 - \sum_{h=1}^5 \widetilde{\gamma}_{h12} - \widetilde{\lambda}'_2 - \widetilde{\lambda}''_2 - \epsilon_{13} - \epsilon_{12}) - \sum_{h=1}^5 2\widetilde{\gamma}_{h12} - \sum_{t=1,2}(\widetilde{\gamma}'_t + \widetilde{\gamma}''_t) = -2\mathcal{C}_6 + \epsilon_{12} = -2\mathcal{C}_{i+j+3} + \epsilon_{ij}$. Similarly for $(i,j) \in \{(1,3),(2,3)\}$. Thus we have $\widetilde{\mathcal{P}}_{ij}|_{\widetilde{Q}_{i+j+3}} = |\mathcal{O}_{\widetilde{Q}_{i+j+3}}(2\mathcal{C}_{i+j+3} - 2\epsilon_{ij})|$, which is the linear system of the quadric sections of $Q_{i+j+3}$ with node at $r_i \cap r_j$. It is known that $Q_{i+j+3}$ is the image of $\mathbb{P}^2$ via the rational map $\psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ defined by the linear system of the conics passing through two fixed points $x_1$ and $x_2$. The quadric sections of $Q_{i+j+3}$ with node at $r_i \cap r_j$ correspond to the quartic plane curves with node at the points $x_1$ and $x_2$ and at a third fixed point $x_3 := \psi^{-1}(r_i \cap r_j)$. By applying a quadratic transformation, we obtain that $\widetilde{\mathcal{P}}_{ij}|_{\widetilde{Q}_{i+j+3}} \cong |\mathcal{O}_{\mathbb{P}^2}(2)|$, whose image is a Veronese surface $V_{ij}$. Furthermore we have that $\widetilde{\mathcal{P}}_{ij}|_{\mathcal{E}_{ij}} = |\mathcal{O}_{\mathcal{E}_{ij}}(-\widetilde{Q}_{i+j+3} - \mathcal{E}_{ij})| = |\mathcal{O}_{\mathcal{E}_{ij}}(2\mathcal{L}_{ij} - 2\widetilde{\alpha}_{ijk})| \cong \mathbb{P}^2$ (see Remark 5.86). Since $\widetilde{\mathcal{P}}_{ij}|_{\mathcal{E}_{ij}}$ is isomorphic to the linear system of the conics of $E_{ij}$ with node at the point $E_{ij} \cap \widetilde{C}_k$, then its image is a conic $C_{ij}$. Since $V_{ij} \cup C_{ij} = \mathbb{P}(TC_{P_{i+j+3}} W^6_F)$, then it must be $C_{ij} \subset V_{ij} = \mathbb{P}(TC_{P_{i+j+3}} W^6_F)$. Therefore $\widetilde{Q}_{i+j+3}$ is contracted by $\nu_{\widetilde{\mathcal{P}}}$ to the point $P_{i+j+3}$, which is a quadruple point whose tangent cone tangent is a cone over a Veronese surface, and the divisor $\mathcal{E}_{ij}$ is contracted in a conic contained in the Veronese surface given by the exceptional divisor of the minimal resolution of $P_{i+j+3}$. □

We recall that $\nu_{\mathcal{P}} : \mathbb{P}^3 \dashrightarrow W^6_F \subset \mathbb{P}^6$ is an isomorphism outside $Q_6 \cup Q_7 \cup Q_8$ (see Remark 5.78). Then $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$, $P_7$ and $P_8$, are the only singular points of $W^6_F$ (see Remarks 5.88, 5.89, 5.90). Furthermore $\nu_{\widetilde{\mathcal{P}}} : Y \to W^6_F$ is a desingularization of $W^6_F$ but it is not the minimal one: indeed the proof of Proposition 5.95 says us that $\nu_{\widetilde{\mathcal{P}}} : Y \to W^6_F$ is the blow-up of the minimal desingularization of $W^6_F$ along curves (conics) contained in the minimal resolutions of $P_6$, $P_7$ and $P_8$. Finally, by recalling Definition 4.4, we have the following result.

**Theorem 5.96.** The eight points $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$, $P_7$, $P_8$ are all associated with each other, as in Figure 21 of Appendix A.

*Proof.* Let us fix $1 \le h < t \le 5$ and $1 \le i < j \le 3$. Let us consider the line $l_{ht} \subset \mathbb{P}^3$ joining the points $q_h$ and $q_t$. Let $\widetilde{l}_{ht}$ be its strict transform on $Y$. We obtain that $\nu_{\widetilde{\mathcal{P}}}(\widetilde{l}_{ht}) = \langle P_h, P_t \rangle \subset W^6_F$, since $\widetilde{l}_{ht} \cap \mathcal{E}_h \neq \emptyset$, $\widetilde{l}_{ht} \cap \mathcal{E}_t \neq \emptyset$ and $\deg(\nu_{\widetilde{\mathcal{P}}}(\widetilde{l}_{ht})) = \widetilde{P} \cdot (\mathcal{H} - \mathcal{E}_h - \mathcal{E}_t - \sum_{1\le i<j\le 3}\Gamma_{hij} - \sum_{1\le i<j\le 3}\Gamma_{tij})^2 = 1$. So $P_h$ is associated with $P_t$. We recall now that $\Gamma_{hij}$, $\Lambda_{ij,i}$, $\Lambda_{ij,j}$, $\Lambda'_i$, $\Lambda''_i$ and $\mathcal{R}_i$ are mapped by $\nu_{\widetilde{\mathcal{P}}} : Y \to W^6_F \subset \mathbb{P}^6$ to curves of $W^6_F$ (see Remark 5.89). In particular $\Gamma_{hij}$, $\Lambda_{ij,i}$, $\Lambda_{ij,j}$, $\Lambda'_i$, $\Lambda''_i$ are mapped to lines of $W^6_F$ (use similar arguments of proof of Theorem 5.17). We also have that $\nu_{\widetilde{\mathcal{P}}}(\mathcal{R}_i) = \nu_{\widetilde{\mathcal{P}}}(\Lambda'_i) = \nu_{\widetilde{\mathcal{P}}}(\Lambda''_i) = \nu_{\widetilde{\mathcal{P}}}(\Lambda_{ij,i}) = \nu_{\widetilde{\mathcal{P}}}(\Lambda_{st,i})$ for $1 \le s < t \le 3$ and $i \in \{s,t\}$ (see Remark 5.91). Since $\Gamma_{hij} \cap \mathcal{E}_h \neq \emptyset$ and $\Gamma_{hij} \cap \widetilde{Q}_{i+j+3} \neq \emptyset$, then $\langle P_h, P_{i+j+3} \rangle = \nu_{\widetilde{\mathcal{P}}}(\Gamma_{hij})$. So each $P_h$ is associated with each $P_{i+j+3}$. Finally $P_6$, $P_7$ and $P_8$ are mutually associated, since $\nu_{\widetilde{\mathcal{P}}}(\mathcal{R}_1) = \langle P_6, P_7 \rangle$, $\nu_{\widetilde{\mathcal{P}}}(\mathcal{R}_2) = \langle P_6, P_8 \rangle$, $\nu_{\widetilde{\mathcal{P}}}(\mathcal{R}_3) = \langle P_7, P_8 \rangle$ (see Remark 5.94). □

# 6 Computational analysis of the BS-EF 3-folds with very ample hyperplane sections

## 6.1 Abstract

We recall that a fixed BS-EF 3-fold $(W, \mathcal{L})$ is an Enriques-Fano threefold given by the quotient $\pi : X \to X/\sigma =: W$ of a smooth Fano threefold $X$ under an involution $\sigma : X \to X$ with eight fixed points (see [1]). The quotient map $\pi : X \to W$ is defined by the sublinear system of $|-K_X|$ given by the $\sigma$-invariant elements. The images of the eight fixed points of $\sigma$ are eight quadruple points of $W$ whose tangent cone is a cone over a Veronese surface. We will computationally analyze the BS-EF 3-folds with very ample hyperplane sections (see [1, Theorem A]). By calling *configuration* the way in which the eight singular points are associated, we will find the following facts:

(i) the ideal of $W^6_{BS} \subset \mathbb{P}^6$ is generated by cubics; the eight singular points of $W^6_{BS} \subset \mathbb{P}^6$ are similar and they have the same configuration of the ones of the F-EF 3-fold $W^6_F$ (see § 6.2); however, it is not yet known if the two threefolds $W^6_{BS}$ and $W^6_F$ coincide;

(ii) the ideal of $W^7_{BS} \subset \mathbb{P}^7$ is generated by quadrics and cubics; the eight singular points of $W^7_{BS} \subset \mathbb{P}^7$ are similar and they have the same configuration of the ones of the F-EF 3-fold $W^7_F$ (see § 6.3); however, it is not yet known if the two threefolds $W^7_{BS}$ and $W^7_F$ coincide;

(iii) the ideal of $W^8_{BS} \subset \mathbb{P}^8$ is generated by quadrics and cubics; the eight singular points of $W^8_{BS} \subset \mathbb{P}^8$ are similar and they have a configuration that was excluded by Fano in his paper [23]; the threefold $W^8_{BS} \subset \mathbb{P}^8$ can also be obtained as the image of a certain linear system of septic surfaces of $\mathbb{P}^3$ (see § 6.4);

(iv) the ideal of $W^9_{BS} \subset \mathbb{P}^9$ is generated by quadrics; the embedding of $W^9_{BS}$ in $\mathbb{P}^9$ is the F-EF 3-fold $W^9_F \subset \mathbb{P}^9$ (see § 6.5);

(v) the ideal of $W^{10}_{BS} \subset \mathbb{P}^{10}$ is generated by quadrics and cubics; the eight singular points of $W^{10}_{BS} \subset \mathbb{P}^{10}$ are similar and they have a configuration that was excluded by Fano in his paper [23]; the threefold $W^{10}_{BS} \subset \mathbb{P}^{10}$ can also be obtained as the image of a certain linear system of sextic surfaces of $\mathbb{P}^3$ (see § 6.6);

(vi) the ideal of $W^{13}_{BS} \subset \mathbb{P}^{13}$ is generated by quadrics; the embedding of $W^{13}_{BS}$ in $\mathbb{P}^{13}$ is the F-EF 3-fold $W^{13}_F \subset \mathbb{P}^{13}$ (see § 6.7).

## 6.2 BS-EF 3-fold (VIII) of genus 6

In the following we will often refer to the use of Macaulay2: see Code B.1 of Appendix B for the computational techniques we will use. Let us study the BS-EF 3fold described in [1, §6.2.4]. Let us consider the smooth Fano threefold $X$ given by the intersection of three divisors of bidegree $(1,1)$ in $\mathbb{P}^3_{[x_0:\cdots:x_3]} \times \mathbb{P}^3_{[y_0:\cdots:y_3]}$ with equations

$$\sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x_iy_j = 0, \quad \sum_{i=0}^{3}\sum_{j=0}^{3} b_{ij}x_iy_j = 0, \quad \sum_{i=0}^{3}\sum_{j=0}^{3} c_{ij}x_iy_j = 0$$

where $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$, $c_{ij} = c_{ji}$, for $i, j \in \{0,1,2,3\}$. Let us take the involution $\sigma : X \to X$ defined by the restriction on $X$ of the following map

$$\mathbb{P}^3 \times \mathbb{P}^3 \xrightarrow{\quad\quad \sigma' \quad\quad} \mathbb{P}^3 \times \mathbb{P}^3$$

$$[x_0 : x_1 : x_2 : x_3] \times [y_0 : y_1 : y_2 : y_3] \longmapsto [y_0 : y_1 : y_2 : y_3] \times [x_0 : x_1 : x_2 : x_3]\,.$$

We have that $\sigma$ has eight fixed points $p_1$, $p_2$, $p_3$, $p_4$, $p_5$, $p_6$, $p_7$ and $p_8$ with coordinates $[x_0 : x_1 : x_2 : x_3] \times [x_0 : x_1 : x_2 : x_3]$ such that

$$\begin{cases} \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}x_ix_j = 0 \\ \sum_{i=0}^{3}\sum_{j=0}^{3} b_{ij}x_ix_j = 0 \\ \sum_{i=0}^{3}\sum_{j=0}^{3} c_{ij}x_ix_j = 0. \end{cases}$$

The quotient map $\pi : X \to X/\sigma =: W^6_{BS}$ is given by the restriction on $X$ of the morphism $\varphi : \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^9_{[Z_0:\cdots:Z_9]}$ defined by the $\sigma'$-invariant multihomogeneous polynomials of multidegree $(1,1)$. Thus we have $\varphi : [x_0 : x_1 : x_2 : x_3] \times [y_0 : y_1 : y_2 : y_3] \mapsto [Z_0 : \cdots : Z_9]$, where $Z_0 = x_0y_0$, $Z_1 = x_1y_1$, $Z_2 = x_2y_2$, $Z_3 = x_3y_3$, $Z_4 = x_0y_1 + x_1y_0$, $Z_5 = x_0y_2+x_2y_0$, $Z_6 = x_0y_3+x_3y_0$, $Z_7 = x_1y_2+x_2y_1$, $Z_8 = x_1y_3+x_3y_1$, $Z_9 = x_2y_3+x_3y_2$. By using Macaulay2, one can find that the image of $\mathbb{P}^3 \times \mathbb{P}^3$ via $\varphi$ is a 6-dimensional algebraic variety $F^{10}_6$ of degree 10, whose ideal is generated by the following 10 polynomials

$$-2Z_1Z_5Z_6 + Z_4Z_6Z_7 + Z_4Z_5Z_8 - 2Z_0Z_7Z_8 + 4Z_0Z_1Z_9 - Z_4^2Z_9,$$

$$-2Z_2Z_4Z_6 + Z_5Z_6Z_7 + 4Z_0Z_2Z_8 - Z_5^2Z_8 + Z_4Z_5Z_9 - 2Z_0Z_7Z_9,$$

$$-4Z_1Z_2Z_6 + Z_6Z_7^2 + 2Z_2Z_4Z_8 - Z_5Z_7Z_8 + 2Z_1Z_5Z_9 - Z_4Z_7Z_9,$$

$$-2Z_3Z_4Z_5 + 4Z_0Z_3Z_7 - Z_6^2Z_7 + Z_5Z_6Z_8 + Z_4Z_6Z_9 - 2Z_0Z_8Z_9,$$

$$-4Z_1Z_3Z_5 + 2Z_3Z_4Z_7 - Z_6Z_7Z_8 + Z_5Z_8^2 + 2Z_1Z_6Z_9 - Z_4Z_8Z_9,$$

$$-4Z_2Z_3Z_4 + 2Z_3Z_5Z_7 + 2Z_2Z_6Z_8 - Z_6Z_7Z_9 - Z_5Z_8Z_9 + Z_4Z_9^2,$$

$$-4Z_1Z_2Z_3+Z_3Z_7^2+Z_2Z_8^2-Z_7Z_8Z_9+Z_1Z_9^2, \quad -4Z_0Z_2Z_3+Z_3Z_5^2+Z_2Z_6^2-Z_5Z_6Z_9+Z_0Z_9^2,$$

$$-4Z_0Z_1Z_3+Z_3Z_4^2+Z_1Z_6^2-Z_4Z_6Z_8+Z_0Z_8^2, \quad -4Z_0Z_1Z_2+Z_2Z_4^2+Z_1Z_5^2-Z_4Z_5Z_7+Z_0Z_7^2.$$

We observe that $W^6_{BS} = \varphi(X) = F^{10}_6 \cap H_6$, where $H_6$ is the 6-dimensional projective subspace of $\mathbb{P}^9$ given by the intersection of the following three hyperplanes

$$\{a_{00}Z_0+a_{11}Z_1+a_{22}Z_2+a_{33}Z_3+2a_{01}Z_4+2a_{02}Z_5+2a_{03}Z_6+2a_{12}Z_7+2a_{13}Z_8+2a_{23}Z_9 = 0\},$$

$$\{b_{00}Z_0+b_{11}Z_1+b_{22}Z_2+b_{33}Z_3+2b_{01}Z_4+2b_{02}Z_5+2b_{03}Z_6+2b_{12}Z_7+2b_{13}Z_8+2b_{23}Z_9 = 0\},$$

$$\{c_{00}Z_0+c_{11}Z_1+c_{22}Z_2+c_{33}Z_3+2c_{01}Z_4+2c_{02}Z_5+2c_{03}Z_6+2c_{12}Z_7+2c_{13}Z_8+2c_{23}Z_9 = 0\}.$$

Therefore we have $\pi = \varphi|_X : X \to W^6_{BS} = \varphi(X) \subset H_6 \cong \mathbb{P}^6$.

**Remark 6.1.** The threefold $W^6_{BS}$ is 3-extendable (see Definition 9.1). It would be interesting to understand if this is sharp.

What follows has been proved for fixed values of $a_{ij}$, $b_{ij}$ and $c_{ij}$, in order to simplify the computational analysis.

**Example 6.2.** Let us take

$$(a_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad (b_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad (c_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix},$$

we obtain $X = \{x_0y_0 - 7x_1y_1 + 4x_2y_2 + 2x_3y_3 = 0,\ x_0y_0 - 6x_1y_1 + 2x_2y_2 + 3x_3y_3 = 0,\ x_0y_0 - x_1y_1 - 7x_2y_2 + 7x_3y_3 = 0\}$. Then the eight fixed points of $\sigma : X \to X$ are

$$p_1 = [1:1:1:1] \times [1:1:1:1]\,,\ p_2\,[-1:1:1:1] \times [-1:1:1:1]\,,$$

$$p_3 = [1:-1:1:1] \times [1:-1:1:1]\,,\ p_4 = [-1:-1:1:1] \times [-1:-1:1:1]\,,$$

$$p_5 = [1:1:-1:1] \times [1:1:-1:1]\,,\ p_6 = [-1:1:-1:1] \times [-1:1:-1:1]\,,$$

$$p_7 = [1:-1:-1:1] \times [1:-1:-1:1]\,,\ p_8 = [-1:-1:-1:1] \times [-1:-1:-1:1]\,.$$

Furthermore we have

$$H_6 := \{Z_0 - 7Z_1 + 4Z_2 + 2Z_3 = 0,\ Z_0 - 6Z_1 + 2Z_2 + 3Z_3 = 0,\ Z_0 - Z_1 - 7Z_2 + 7Z_3 = 0\} =$$

$$= \{Z_2 - Z_3 = 0,\ Z_1 - Z_3 = 0,\ Z_0 - Z_3 = 0\},$$

which is the $\mathbb{P}^6_{[w_0:\dots:w_6]}$ embedded in $\mathbb{P}^9_{[Z_0:\dots:Z_9]}$ via the morphism such that

$$Z_i = w_0,\ i = 0,1,2,3, \qquad Z_j = w_{j-3},\ j = 4,\dots,9.$$

By using Macaulay2, we find that the quotient map $\pi : X \to W^6_{BS} \subset H_6 \cong \mathbb{P}^6$ is given by the restriction on $X$ of the morphism $\varphi' : \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^6_{[w_0:\dots:w_6]}$ such that $[x_0 : x_1 : x_2 : y_3 : y_4 : y_5] \mapsto [w_0 : \dots : w_6]$, where $w_0 = x_3y_3$, $w_1 = x_0y_1 + x_1y_0$, $w_2 = x_0y_2 + x_2y_0$, $w_3 = x_0y_3 + x_3y_0$, $w_4 = x_1y_2 + x_2y_1$, $w_5 = x_1y_3 + x_3y_1$, $w_6 = x_2y_3 + x_3y_2$. Thanks to Macaulay2, we obtain that this BS-EF 3-fold $W^6_{BS} \subset \mathbb{P}^6$ has ideal generated by the following 10 polynomials

$$-2w_0w_2w_3 + w_1w_3w_4 + w_1w_2w_5 - 2w_0w_4w_5 + 4w_0^2w_6 - w_1^2w_6,$$

$$-2w_0w_1w_3 + w_2w_3w_4 + 4w_0^2w_5 - w_2^2w_5 + w_1w_2w_6 - 2w_0w_4w_6,$$

$$-4w_0^2w_3 + w_3w_4^2 + 2w_0w_1w_5 - w_2w_4w_5 + 2w_0w_2w_6 - w_1w_4w_6,$$

$$-2w_0w_1w_2 + 4w_0^2w_4 - w_3^2w_4 + w_2w_3w_5 + w_1w_3w_6 - 2w_0w_5w_6,$$

$$-4w_0^2w_2 + 2w_0w_1w_4 - w_3w_4w_5 + w_2w_5^2 + 2w_0w_3w_6 - w_1w_5w_6,$$

$$-4w_0^2w_1 + 2w_0w_2w_4 + 2w_0w_3w_5 - w_3w_4w_6 - w_2w_5w_6 + w_1w_6^2,$$

$$-4w_0^3 + w_0w_4^2 + w_0w_5^2 - w_4w_5w_6 + w_0w_6^2, \quad -4w_0^3 + w_0w_2^2 + w_0w_3^2 - w_2w_3w_6 + w_0w_6^2,$$

$$-4w_0^3 + w_0w_1^2 + w_0w_3^2 - w_1w_3w_5 + w_0w_5^2, \quad -4w_0^3 + w_0w_1^2 + w_0w_2^2 - w_1w_2w_4 + w_0w_4^2.$$

Furthermore this threefold has the following eight singular points

$$P_1 = \pi(p_1) = [1:2:2:2:2:2:2]\,,\ P_2 = \pi(p_2) = [1:-2:-2:-2:2:2:2]\,,$$

$$P_3 = \pi(p_3) = [1:-2:2:2:-2:-2:2]\,,\ P_4 = \pi(p_4) = [1:2:-2:-2:-2:-2:2]\,,$$

$$P_5 = \pi(p_1') = [1:2:-2:2:-2:2:-2]\,,\ P_6 = \pi(p_2') = [1:-2:2:-2:-2:2:-2]\,,$$

$$P_7 = \pi(p_3') = [1:-2:-2:2:2:-2:-2]\,,\ P_8 = \pi(p_4') = [1:2:2:-2:2:-2:-2]\,.$$

One can verify that all the lines joining the points $P_i$ and $P_j$, for $1 \le i < j \le 8$, are contained in $W^6_{BS}$. So we can say that each one of the eight singular points of $W^6_{BS}$ is associated with all the other $m = 7$ points, as in Figure 21 of Appendix A. This is the same configuration of the singularities of the F-EF 3-fold $W^6_F$.

## 6.3 BS-EF 3-fold (X) of genus 7

In the following we will often refer to the use of Macaulay2: see Code B.2 of Appendix B for the computational techniques we will use. Let us study the BS-EF 3fold described in [1, §6.4.1]. Let $X$ be the smooth Fano threefold given by a divisor of type

$$\sum_{i+j+k+l \ odd} a_{ijkl} x_i y_j z_k t_l = 0$$

in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $[x_0 : x_1] \times [y_0 : y_1] \times [z_0 : z_1] \times [t_0 : t_1]$. Let us consider the involution $\sigma : X \to X$ defined by the restriction on $X$ of the map $\sigma' : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such that

$$[x_0 : x_1] \times [y_0 : y_1] \times [z_0 : z_1] \times [t_0 : t_1] \mapsto [x_0 : -x_1] \times [y_0 : -y_1] \times [z_0 : -z_1] \times [t_0 : -t_1]\,.$$

The involution $\sigma : X \to X$ has the following eight fixed points

$$p_1 = [0:1]\times[0:1]\times[0:1]\times[0:1]\,,\quad p_1' = [1:0]\times[1:0]\times[1:0]\times[1:0]\,,$$

$$p_2 = [0:1]\times[1:0]\times[1:0]\times[0:1]\,,\quad p_2' = [1:0]\times[0:1]\times[0:1]\times[1:0]\,,$$

$$p_3 = [1:0]\times[1:0]\times[0:1]\times[0:1]\,,\quad p_3' = [0:1]\times[0:1]\times[1:0]\times[1:0]\,,$$

$$p_4 = [1:0]\times[0:1]\times[1:0]\times[0:1]\,,\quad p_4' = [0:1]\times[1:0]\times[0:1]\times[1:0]\,.$$

The quotient map $\pi : X \to X/\sigma =: W^7_{BS}$ is given by the restriction on $X$ of the morphism $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^7_{[w_0:\cdots:w_7]}$, defined by the $\sigma'$-invariant multihomogeneous polynomials of multidegree $(1,1,1,1)$. In particular we have

$$\varphi : [x_0 : x_1] \times [y_0 : y_1] \times [z_0 : z_1] \times [t_0 : t_1] \mapsto [w_0 : w_1 : w_2 : w_3 : w_4 : w_5 : w_6 : w_7]$$

where $w_0 = x_1y_1z_1t_1$, $w_1 = x_1y_0z_0t_1$, $w_2 = x_0y_0z_1t_1$, $w_3 = x_1y_0z_1t_0$, $w_4 = x_0y_0z_0t_0$, $w_5 = x_0y_1z_1t_0$, $w_6 = x_1y_1z_0t_0$, $w_7 = x_0y_1z_0t_1$.

**Remark 6.3.** By fixing (random) values for $a_{0001}$, $a_{0010}$, $a_{0100}$, $a_{1000}$, $a_{1110}$, $a_{1101}$, $a_{1011}$ and $a_{0111}$, one can verify, with Macaulay2, that the ideal of the BS-EF 3-fold $W^7_{BS}$ is generated by the following 11 polynomials

$$w_2w_6 - w_3w_7, \quad w_1w_5 - w_3w_7, \quad w_0w_4 - w_3w_7,$$

$$a_{1110}w_0w_5w_6 + a_{1011}w_0w_3w_7 + a_{0111}w_0w_5w_7 + a_{0010}w_3w_5w_7 + a_{1101}w_0w_6w_7 + a_{1000}w_3w_6w_7 + a_{0100}w_5w_6w_7 + a_{0001}w_3w_7^2,$$

$$a_{1000}w_1w_4w_6 + a_{1011}w_1w_3w_7 + a_{0001}w_1w_4w_7 + a_{0010}w_3w_4w_7 + a_{1101}w_1w_6w_7 + a_{1110}w_3w_6w_7 + a_{0100}w_4w_6w_7 + a_{0111}w_3w_7^2,$$

$$a_{0010}w_3w_4w_5 + a_{1000}w_3w_4w_6 + a_{1110}w_3w_5w_6 + a_{0100}w_4w_5w_6 + a_{1011}w_3^2w_7 + a_{0001}w_3w_4w_7 + a_{0111}w_3w_5w_7 + a_{1101}w_3w_6w_7,$$

$$a_{0010}w_2w_4w_5 + a_{1011}w_2w_3w_7 + a_{0001}w_2w_4w_7 + a_{1000}w_3w_4w_7 + a_{0111}w_2w_5w_7 + a_{1110}w_3w_5w_7 + a_{0100}w_4w_5w_7 + a_{1101}w_3w_7^2,$$

$$a_{1011}w_1w_2w_3 + a_{0001}w_1w_2w_4 + a_{1000}w_1w_3w_4 + a_{0010}w_2w_3w_4 + a_{1101}w_1w_3w_7 + a_{0111}w_2w_3w_7 + a_{1110}w_3^2w_7 + a_{0100}w_3w_4w_7,$$

$$a_{1011}w_0w_2w_3 + a_{0111}w_0w_2w_5 + a_{1110}w_0w_3w_5 + a_{0010}w_2w_3w_5 + a_{1101}w_0w_3w_7 + a_{0001}w_2w_3w_7 + a_{1000}w_3^2w_7 + a_{0100}w_3w_5w_7,$$

$$a_{1011}w_0w_1w_3 + a_{1101}w_0w_1w_6 + a_{1110}w_0w_3w_6 + a_{1000}w_1w_3w_6 + a_{0111}w_0w_3w_7 + a_{0001}w_1w_3w_7 + a_{0010}w_3^2w_7 + a_{0100}w_3w_6w_7,$$

$$a_{1011}w_0w_1w_2 + a_{1101}w_0w_1w_7 + a_{0111}w_0w_2w_7 + a_{0001}w_1w_2w_7 + a_{1110}w_0w_3w_7 + a_{1000}w_1w_3w_7 + a_{0010}w_2w_3w_7 + a_{0100}w_3w_7^2.$$

Thus the ideal of $W^7_{BS}$ is generated by quadrics and cubics. Since $W^7_{BS}$ is projectively normal in $\mathbb{P}^7$ (see § 3.3), then the ideal of its general hyperplane section $S \subset \mathbb{P}^6$ is generated by quadrics and cubics too. This is consistent with the fact that the $\phi$ of a general hyperplane section of $S$ is $3 < 4$ (see [35, Theorem 1.1 (ii)]), as we will see in the proof of Theorem 9.2.

**Remark 6.4.** Let us consider the eight singular points of $W^7_{BS}$

$$P_1 = \pi(p_1) = [1:0:0:0:0:0:0:0]\,,\; P'_1 = \pi(p'_1) = [0:0:0:0:1:0:0:0]\,,$$

$$P_2 = \pi(p_2) = [0:1:0:0:0:0:0:0]\,,\; P'_2 = \pi(p'_2) = [0:0:0:0:0:1:0:0]\,,$$

$$P_3 = \pi(p_3) = [0:0:1:0:0:0:0:0]\,,\; P'_3 = \pi(p'_3) = [0:0:0:0:0:0:1:0]\,,$$

$$P_4 = \pi(p_4) = [0:0:0:1:0:0:0:0]\,,\; P'_4 = \pi(p'_4) = [0:0:0:0:0:0:0:1]\,.$$

Let $l_{i,j}$ be the line joining the singular points $P_i$ and $P_j$ with $i, j \in \{1, 2, 3, 4, 1', 2', 3', 4'\}$ and $i \neq j$. Then we have $l_{1,2} = \{w_i = 0 | i \neq 0, 1\}$, $l_{1,3} = \{w_i = 0 | i \neq 0, 2\}$, $l_{1,4} = \{w_i = 0 | i \neq 0, 3\}$, $l_{1,1'} = \{w_i = 0 | i \neq 0, 4\}$, $l_{1,2'} = \{w_i = 0 | i \neq 0, 5\}$, $l_{1,3'} = \{w_i = 0 | i \neq 0, 6\}$, $l_{1,4'} = \{w_i = 0 | i \neq 0, 7\}$, $l_{2,3} = \{w_i = 0 | i \neq 1, 2\}$,

$l_{2,4} = \{w_i = 0 | i \neq 1,3\}$, $l_{2,1'} = \{w_i = 0 | i \neq 1,4\}$, $l_{2,2'} = \{w_i = 0 | i \neq 1,5\}$, $l_{2,3'} = \{w_i = 0 | i \neq 1,6\}$, $l_{2,4'} = \{w_i = 0 | i \neq 1,7\}$, $l_{3,4} = \{w_i = 0 | i \neq 2,3\}$, $l_{3,1'} = \{w_i = 0 | i \neq 2,4\}$, $l_{3,2'} = \{w_i = 0 | i \neq 2,5\}$, $l_{3,3'} = \{w_i = 0 | i \neq 2,6\}$, $l_{3,4'} = \{w_i = 0 | i \neq 2,7\}$, $l_{4,1'} = \{w_i = 0 | i \neq 3,4\}$, $l_{4,2'} = \{w_i = 0 | i \neq 3,5\}$, $l_{4,3'} = \{w_i = 0 | i \neq 3,6\}$, $l_{4,4'} = \{w_i = 0 | i \neq 3,7\}$, $l_{1',2'} = \{w_i = 0 | i \neq 4,5\}$, $l_{1',3'} = \{w_i = 0 | i \neq 4,6\}$, $l_{1',4'} = \{w_i = 0 | i \neq 4,7\}$, $l_{2',3'} = \{w_i = 0 | i \neq 5,6\}$, $l_{2',4'} = \{w_i = 0 | i \neq 5,7\}$, $l_{3',4'} = \{w_i = 0 | i \neq 6,7\}$. By Remark 6.3 we have that $W^7_{BS}$ does not contain the lines $l_{1,1'}$, $l_{2,2'}$, $l_{3,3'}$ and $l_{4,4'}$, while it contains the others. So each one of the eight singular points of $W^7_{BS}$ is associated with $m = 6$ of the other singular points, as in Figure 22 of Appendix A. This is the same configuration of the singularities of the F-EF 3-fold $W^7_F$.

## 6.4 BS-EF 3-fold (XI) of genus 8

In the following we will often refer to the use of Macaulay2: see Code B.3 of Appendix B for the computational techniques we will use. Let us study the BS-EF 3fold described in [1, §6.4.2]. Let us take the hyperplane $\{x_4 = 0\} \subset \mathbb{P}^4_{[x_0:x_1:x_2:x_3:x_4]}$ and two quadric surfaces $Q, R \subset \{x_4 = 0\} \cong \mathbb{P}^3_{[x_0:x_1:x_2:x_3]}$, respectively with equations

$$Q(x_0, x_1, x_2, x_3) := q_{00}x_0^2 + q_{11}x_1^2 + q_{22}x_2^2 + q_{33}x_3^2 + q_{01}x_0x_1 + q_{23}x_2x_3 = 0,$$

$$R(x_0, x_1, x_2, x_3) := r_{00}x_0^2 + r_{11}x_1^2 + r_{22}x_2^2 + r_{33}x_3^2 + r_{01}x_0x_1 + r_{23}x_2x_3 = 0.$$

Let $C := Q \cap R$ be the elliptic quartic curve given by the complete intersection of the above quadrics, and let $Y$ be the quadric cone over $Q$ with vertex $v = [0 : 0 : 0 : 0 : 1]$. Obviously, $Y$ has equation $Q(x_0, x_1, x_2, x_3) = 0$ in $\mathbb{P}^4$. Let $X := \mathrm{Bl}_{v \cup C} Y$ be the threefold obtained by blowing-up the point $v$ and the curve $C$ and let us consider the blow-up map $bl : X \to Y$. We have that $X$ is a smooth Fano threefold. Let us explain this. Let us consider the blow-up map $bl' : \mathrm{Bl}_{v\cup C} \mathbb{P}^4 \to \mathbb{P}^4$ with exceptional divisors $E_v = bl'^{-1}(v)$ and $E_C = bl'^{-1}(C)$. By definition we have that $X$ is the strict transform of $Y$ on $\mathrm{Bl}_{v\cup C} \mathbb{P}^4$ and that $bl = bl'|_X$. If $H$ denotes the pullback of the hyperplane class $h$ of $\mathbb{P}^4$, then we have that $X \sim 2H - 2E_v - E_C$. By the adjunction formula we have that $-K_X = -(K_{\mathrm{Bl}_{v\cup C} \mathbb{P}^4} + X)|_X \sim (3H - E_v - E_C)|_X$. We want to show that $-K_X$ is ample. Let us consider the linear system $\mathcal{C}$ of the cubic hypersurfaces of $\mathbb{P}^4$ containing the curve $C$ and passing through the point $v$. Let us fix a general hyperplane $h_v$ passing through $v$. We have that $\mathcal{C}$ contains a sublinear system $\overline{\mathcal{C}} \subset \mathcal{C}$ whose fixed part is given by $h_v \cup \{x_4 = 0\}$. Since the movable part of $\overline{\mathcal{C}}$ is given by the hyperplanes of $\mathbb{P}^4$, then we obtain the ampleness of $\mathcal{C}$ at least outside $v \cup C$. So we have the ampleness of $-K_X$ at least outside $E_v \cap X$ and $E_C \cap X$, since $| - K_X|$ coincides with the restriction on $X$ of the strict transform of $\mathcal{C}$. Furthermore the movable part of $\overline{\mathcal{C}}$ also contains the hyperplanes of $\mathbb{P}^4$ through $v$, whose strict transforms are very ample on $E_v$: indeed we have $|\mathcal{O}_{E_v}(H - E_v)| = |\mathcal{O}_{E_v}(-E_v^2)| \cong |\mathcal{O}_{\mathbb{P}^3}(1)|$ (see [27, Chap 4, §6]). Thus the ampleness of $-K_X$ along $E_v \cap X$ follows by the fact that $E_v \cap X$ is a smooth quadric surface in $E_v \cong \mathbb{P}^3$. It remains to show the ampleness of $-K_X$ along $S' := E_C \cap X$, which is a $\mathbb{P}^1$-bundle over $C$. Since $C$ is the complete intersection of a hyperplane section and a quadric section of $Y$, then $S' = \mathbb{P}(\mathcal{N}_{C|Y}) \cong \mathbb{P}(\mathcal{O}_C(h|_Y) \oplus \mathcal{O}_C(2h|_Y))$. In particular we have that the class $S'|_{S'}$ is the class of the tautological bundle on $S'$ (see [27, Chap 4, §6]). Thus $-E_C|_{S'} = -S'|_{S'}$ is ample on $S'$, and so $(-K_X)|_{S'} = (3H - E_C)|_{S'}$ is ample too.

Let $\sigma : X \to X$ be the morphism defined by the birational map $\sigma' : Y \dashrightarrow Y$ s.t.

$$[x_0 : x_1 : x_2 : x_3 : x_4] \xmapsto{\sigma'} [x_4x_0 : x_4x_1 : -x_4x_2 : -x_4x_3 : R(x_0, x_1, x_2, x_3)]\,.$$

The map $\sigma : X \to X$ is an involution of $X$ with eight fixed points, which are the preimages via $bl : X \to Y$ of the eight points $p_1$, $p_2$, $p_3$, $p_4$, $p_1'$, $p_2'$, $p_3'$, $p_4'$ such that

$$\{p_1, p_1', p_2, p_2'\} = Y \cap \{x_2 = 0,\ x_3 = 0,\ x_4^2 - R(x_0, x_1, x_2, x_3) = 0\},$$
$$\{p_3, p_3', p_4, p_4'\} = Y \cap \{x_0 = 0,\ x_1 = 0,\ x_4^2 + R(x_0, x_1, x_2, x_3) = 0\}.$$

The $\sigma'$-invariant elements of $\mathcal{C}$ define the rational map $\varphi : Y \dashrightarrow \mathbb{P}^9$ such that $[x_0 : \cdots : x_4] \mapsto [Z_0 : \cdots : Z_9]$, where

$$\begin{gathered} Z_0 = x_4^2x_0 + x_0R(x_0, x_1, x_2, x_3), \quad Z_1 = x_4^2x_1 + x_1R(x_0, x_1, x_2, x_3), \\ Z_2 = x_4^2x_2 - x_2R(x_0, x_1, x_2, x_3), \quad Z_3 = x_4^2x_3 - x_3R(x_0, x_1, x_2, x_3), \\ Z_4 = x_4x_0^2,\ \ Z_5 = x_4x_1^2,\ \ Z_6 = x_4x_2^2,\ \ Z_7 = x_4x_3^2, \quad Z_8 = x_4x_0x_1,\ \ Z_9 = x_4x_2x_3. \end{gathered}$$

We observe that $\varphi(Y)$ is contained in a hyperplane $\mathbb{P}^8_{[w_0:\cdots:w_8]} \cong H_8 \subset \mathbb{P}^9$ with equation $H_8 := \{q_{00}Z_4 + q_{11}Z_5 + q_{22}Z_6 + q_{33}Z_7 + q_{01}Z_8 + q_{23}Z_9 = 0\}$. The rational map $\varphi$ defines the quotient map $\pi : X \to X/\sigma =: W^8_{BS}$, thanks to the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad\pi\quad} & \\ \Big\downarrow{\scriptstyle bl} & & \searrow \\ Y & \overset{\varphi}{\dashrightarrow} & \varphi(Y) = \pi(X) = W^8_{BS} \subset H_8 \cong \mathbb{P}^8. \end{array}$$

What follows has been proved for fixed values of $q_{ij}$ and $r_{ij}$, in order to simplify the computational analysis.

**Example 6.5.** Let us take

$$Q(x_0, x_1, x_2, x_3) = x_0^2 - x_1^2 - x_2^2 + x_3^2 \quad \text{and} \quad R(x_0, x_1, x_2, x_3) = 2x_0^2 - x_1^2 - 3x_2^2 + 2x_3^2.$$

Then $\varphi(Y)$ is contained in the hyperplane $H_8 = \{Z_4 - Z_5 - Z_6 + Z_7\}$, which we can see as the image of the morphism $i : \mathbb{P}^8 \hookrightarrow \mathbb{P}^9$ such that

$$[w_0 : \cdots : w_8] \xmapsto{i} [w_0 : w_1 : w_2 : w_3 : w_4 + w_5 - w_6 : w_5 : w_6 : w_7 : w_8]\,.$$

Thanks to Macaulay2, one can verify that we obtain a BS-EF 3-fold $W^8_{BS} \subset H_8 \cong \mathbb{P}^8$ whose ideal is generated by the following 11 polynomials

$$\begin{gathered} w_5w_6 - w_8^2, \quad w_2w_6 - w_3w_8, \quad w_3w_5 - w_2w_8, \quad w_4^2 + w_4w_5 - w_4w_6 - w_7^2, \\ w_1w_4 + w_1w_5 - w_1w_6 - w_0w_7, \quad w_0w_4 - w_1w_7, \\ w_0^2 - w_1^2 - w_2^2 + w_3^2 - 4w_4w_5 + 4w_5^2 + 4w_4w_6 - 4w_8^2, \\ w_2w_3w_7 - w_0w_1w_8 + 4w_4w_7w_8 - 4w_5w_7w_8, \\ w_0w_1w_6 - w_3^2w_7 - 4w_4w_6w_7 + 4w_7w_8^2, \\ w_3^2w_4 - w_1^2w_6 + 4w_4w_6^2 + 4w_6w_7^2 - 8w_4w_8^2, \\ w_2w_3w_4 - w_1^2w_8 - 8w_4w_5w_8 + 4w_4w_6w_8 + 4w_7^2w_8. \end{gathered}$$

Then the ideal of $W^8_{BS}$ is generated by quadrics and cubics. Since $W^8_{BS}$ is projectively normal in $\mathbb{P}^8$ (see § 3.3), then the ideal of its general hyperplane section $S \subset \mathbb{P}^7$ is generated by quadrics and cubics too. This is consistent with the fact that the $\phi$ of a general hyperplane section of $S$ is $3 < 4$ (see [35, Theorem 1.1 (ii)]), as we will see in the proof of Theorem 9.2.

**Remark 6.6.** The BS-EF 3-fold $W^8_{BS}$ of Example 6.5 has the following eight singular points

$$P_1 = \varphi(p_1) = \varphi([1:1:0:0:1]) = [2:2:0:0:1:0:0:1:0]\,,$$
$$P_2 = \varphi(p_2) = \varphi([-1:-1:0:0:1]) = [-2:-2:0:0:1:0:0:1:0]\,,$$
$$P_3 = \varphi(p_3) = \varphi([0:0:1:1:1]) = [0:0:2:2:0:1:1:0:1]\,,$$
$$P_4 = \varphi(p_4) = \varphi([0:0:-1:1:1]) = [0:0:-2:-2:0:1:1:0:1]\,,$$
$$P'_1 = \varphi(p'_1) = \varphi([-1:1:0:0:1]) = [2:-2:0:0:-1:0:0:1:0]\,,$$
$$P'_2 = \varphi(p'_2) = \varphi([1:-1:0:0:1]) = [-2:2:0:0:-1:0:0:1:0]\,,$$
$$P'_3 = \varphi(p'_3) = \varphi([0:0:-1:-1:1]) = [0:0:2:-2:0:-1:-1:0:1]\,,$$
$$P'_4 = \varphi(p'_4) = \varphi([0:0:1:-1:1]) = [0:0:-2:2:0:-1:-1:0:1]\,.$$

Let $l_{i,j}$ be the line joining $P_i$ and $P_j$ for $i, j \in \{1, 2, 3, 4, 1', 2', 3', 4'\}$ and $i \neq j$, i. e.

$l_{12} = \{w_0 = w_1,\ w_4 = w_7,\ w_2 = w_3 = w_5 = w_6 = w_8 = 0\}$,
$l_{13} = \{w_0 = w_1 = 2w_4 = 2w_7,\ w_2 = w_3 = 2w_5 = 2w_6 = 2w_8\}$,
$l_{14} = \{w_0 = w_1 = 2w_4 = 2w_7,\ -w_2 = -w_3 = 2w_5 = 2w_6 = 2w_8\}$,
$l_{11'} = \{w_0 = 2w_7,\ w_1 = 2w_4,\ w_2 = w_3 = w_5 = w_6 = w_8 = 0\}$,
$l_{12'} = \{w_0 = 2w_4,\ w_1 = 2w_7,\ w_2 = w_3 = w_5 = w_6 = w_8 = 0\}$,
$l_{13'} = \{w_0 = w_1 = 2w_4 = 2w_7,\ w_2 = -w_3 = -w_5 = -2w_6 = 2w_8\}$,
$l_{14'} = \{w_0 = w_1 = 2w_4 = 2w_7,\ -w_2 = w_3 = -2w_5 = -2w_6 = 2w_8\}$,
$l_{23} = \{-w_0 = -w_1 = 2w_4 = 2w_7,\ w_2 = w_3 = 2w_5 = 2w_6 = 2w_8\}$,
$l_{24} = \{-w_0 = -w_1 = 2w_4 = 2w_7,\ -w_2 = -w_3 = 2w_5 = 2w_6 = 2w_8\}$,
$l_{21'} = \{-w_0 = 2w_4,\ -w_1 = 2w_7,\ w_2 = w_3 = w_5 = w_6 = w_8 = 0\}$,
$l_{22'} = \{-w_0 = 2w_7,\ -w_1 = 2w_4,\ w_2 = w_3 = w_5 = w_6 = w_8 = 0\}$,
$l_{23'} = \{-w_0 = -w_1 = 2w_4 = 2w_7,\ w_2 = -w_3 = -2w_5 = -2w_6 = 2w_8\}$,
$l_{24'} = \{-w_0 = -w_1 = 2w_4 = 2w_7,\ -w_2 = w_3 = -2w_5 = -2w_6 = 2w_8\}$,
$l_{34} = \{w_0 = w_1 = w_4 = w_7 = 0,\ w_2 = w_3,\ w_5 = w_6 = w_8\}$,
$l_{31'} = \{w_0 = -w_1 = -2w_4 = 2w_7,\ w_2 = w_3 = 2w_5 = 2w_6 = 2w_8\}$,
$l_{32'} = \{-w_0 = w_1 = -2w_4 = 2w_7,\ w_2 = w_3 = 2w_5 = 2w_6 = 2w_8\}$,
$l_{33'} = \{w_0 = w_1 = w_4 = w_7 = 0,\ w_2 = 2w_8,\ w_3 = 2w_5 = 2w_6\}$,
$l_{34'} = \{w_0 = w_1 = w_4 = w_7 = 0,\ w_2 = 2w_5 = 2w_6,\ w_3 = 2w_8\}$,
$l_{41'} = \{w_0 = -w_1 = -2w_4 = 2w_7,\ -w_2 = -w_3 = -2w_5 = -2w_6 = 2w_8\}$,
$l_{42'} = \{-w_0 = w_1 = -2w_4 = 2w_7,\ -w_2 = -w_3 = 2w_5 = 2w_6 = 2w_8\}$,
$l_{43'} = \{w_0 = w_1 = w_4 = w_7 = 0,\ -w_2 = 2w_5 = 2w_6,\ -w_3 = 2w_8\}$,
$l_{44'} = \{w_0 = w_1 = w_4 = w_7 = 0,\ -w_2 = 2w_8,\ -w_3 = 2w_5 = 2w_6\}$,
$l_{1'2'} = \{w_0 = -w_1,\ w_2 = w_3 = w_5 = w_6 = w_8 = 0,\ -w_4 = w_7\}$,
$l_{1'3'} = \{w_0 = -w_1 = -2w_4 = 2w_7,\ w_2 = -w_3 = -2w_5 = -2w_6 = 2w_8\}$,
$l_{1'4'} = \{w_0 = -w_1 = -2w_4 = 2w_7,\ -w_2 = w_3 = -2w_5 = -2w_6 = 2w_8\}$,
$l_{2'3'} = \{-w_0 = w_1 = -2w_4 = 2w_7,\ w_2 = -w_3 = -2w_5 = -2w_6 = 2w_8\}$,
$l_{2'4'} = \{-w_0 = w_1 = -2w_4 = 2w_7,\ -w_2 = w_3 = -2w_5 = -2w_6 = 2w_8\}$,

$l_{3'4'} = \{w_0 = w_1 = w_4 = w_7 = 0, -w_2 = w_3, -w_5 = -w_6 = w_8\}$. We have that $W^8_{BS}$ does not contain the lines $l_{1,1'}$, $l_{1,2'}$, $l_{2,1'}$, $l_{2,2'}$, $l_{3,3'}$, $l_{3,4'}$, $l_{4,3'}$, $l_{4,4'}$, while it contains the others. So each one of the eight singular points of $W^8_{BS}$ is associated with $m = 5$ of the other singular points, as in Figure 23 of Appendix A. Hence there exist three mutually associated points (for example $P_1$, $P_2$ and $P_3$). This case had been excluded by Fano for $p > 7$, as we said in Remark 4.7 (iv). So this suggests that in Fano's paper there are other gaps to be discovered.

**Theorem 6.7.** Let $T$ be a trihedron with edges $l_0, l_1, l_2$ and vertex $v$ as in Figure 11. Let us choose a general point $q_1 \in l_1$, a general point $q_2 \in l_2$, three distinct points $a_r$, $a_s$, $a_t \in l_0$, a general point $b_1 \in r_1 := \langle q_1, a_r\rangle$ and a general point $b_2 \in r_2 := \langle q_2, a_r\rangle$. Let us take a general conic $C$ through the points $q_1$, $q_2$, $b_1$, $b_2$, in the plane spanned by the three points $a_r$, $q_1$, $q_2$. Finally let us consider the lines $s_1 := \langle q_1, a_s\rangle$, $s_2 := \langle q_2, a_s\rangle$, $t_1 := \langle b_1, a_t\rangle$, $t_2 := \langle b_2, a_t\rangle$ and the lines $l'_1 := \langle q'_1, q_2\rangle$ and $l'_2 := \langle q'_2, q_1\rangle$, where $q'_1$ is a general point on $t_1$ and $q'_2$ a general point on $t_2$. Then the BS-EF 3-fold $W^8_{BS}$ can be obtained as the image of $\mathbb{P}^3$ via the rational map $\nu_{\mathcal{N}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^8$ defined by the linear system $\mathcal{N}$ of the septic surfaces of $\mathbb{P}^3$ which are quadruple at the points $q_1$ and $q_2$, triple at the vertex $v$ and double along the lines $l_0$, $l_1$, $l_2$, $l'_1$, $l'_2$, along the conic $C$ and at the points $c_1 := t_1 \cap s_1$ and $c_2 := t_2 \cap s_2$. Furthermore a general $N \in \mathcal{N}$ contains the lines $t_1$, $t_2$, $r_1$, $r_2$, $s_1$, $s_2$ and $e_0 := \langle q_1, q_2\rangle$.

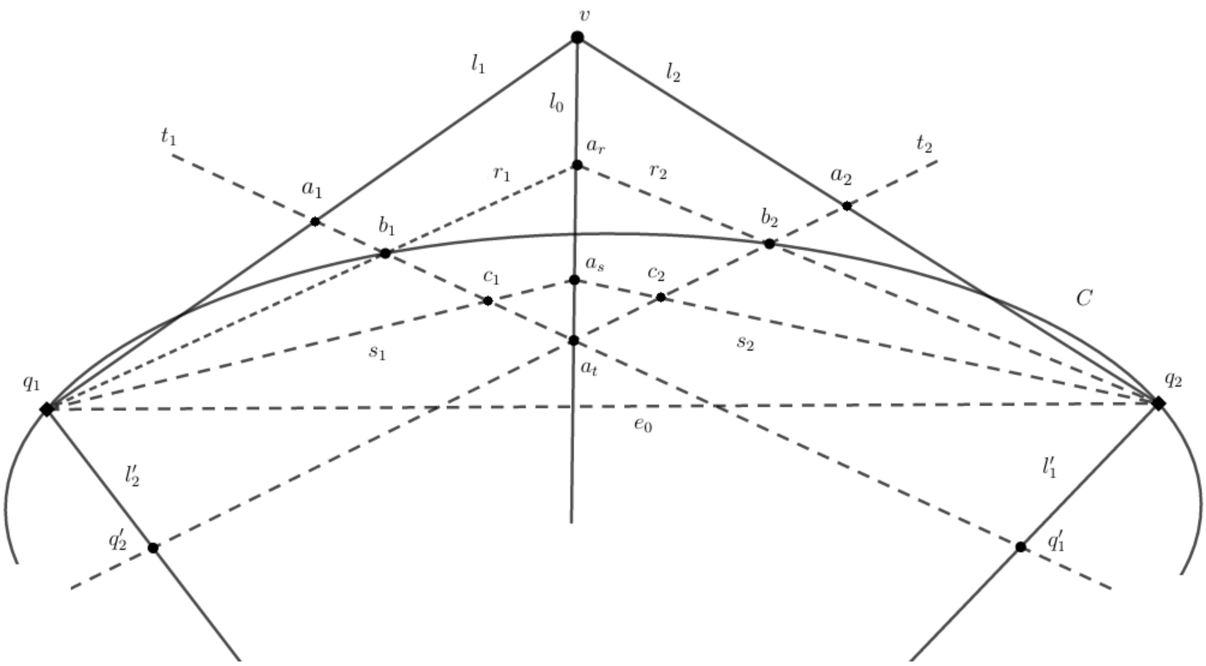


*Figure 11:* *Base locus of the linear system $\mathcal{N}$.*

*Proof.* Let us project $\mathbb{P}^8$ from the $\mathbb{P}^4$ spanned by the singular points $P_1$, $P'_1$, $P_2$, $P_3$, $P'_3$ of the BS-EF 3-fold $W^8_{BS}$ of Example 6.5 (see Remark 6.6). By using Macaulay2, we obtain the rational map

$$\rho : \mathbb{P}^8 \dashrightarrow \mathbb{P}^3, \quad [w_0 : \cdots : w_{13}] \mapsto [w_2 - 2w_8 : w_5 - w_6 : w_3 - 2w_6 : w_0 - w_1 + 2w_4 - 2w_7].$$

One can verify with Macaulay2 that the restriction $\rho|_{W^8_{BS}} : W^8_{BS} \dashrightarrow \mathbb{P}^3$ is birational. Furthermore its inverse map is given by the rational map $\nu : \mathbb{P}^3 \dashrightarrow W^8_{BS} \subset \mathbb{P}^8$ defined by the linear system $\mathcal{N}$ of the septic surfaces

(i) quadruple at $q_1 = [1 : 0 : -2 : 0]$ and $q_2 = [1 : 0 : 2 : 0]$;

(ii) triple at the vertex $v = [0 : 0 : 0 : 1]$ of $T = \{s_1(2s_0 + s_2)(2s_0 - s_2) = 0\}$;

(iii) double at the points $c_1 = [1 : -2 : -2 : 0]$ and $c_2 = [1 : 2 : 2 : 0]$; double along the line $l'_1 = \{s_3 = 2s_0 + 2s_1 - s_2 = 0\} \ni q'_1 = [1 : -2 : -2 : 0]$ and the line $l'_2 = \{s_3 = 2s_0 - 2s_1 + s_2 = 0\} \ni q'_2 = [1 : 2 : 2 : 0]$; double along the edges $l_0 = \{s_0 = s_2 = 0\}$, $l_1 = \{s_1 = 2s_0 + s_2 = 0\}$, $l_2 = \{s_1 = 2s_0 - s_2 = 0\}$; double along the conic $C = \{2s_1 + s_3 = 4s_0^2 - s_2^2 - 2s_2s_3 - 2s_3^2 = 0\}$ passing through $q_1$, $q_2$, $b_1 = [1 : -1 : -2 : 2]$ and $b_2 = [1 : 1 : 2 : -2]$;

(iv) containing the lines $r_1 = \{2s_1+s_3 = 2s_0+s_2 = 0\}$, $r_2 = \{2s_1+s_3 = 2s_0-s_2 = 0\}$, $s_1 = \{s_3 = 2s_0 + s_2 = 0\}$, $s_2 = \{s_3 = 2s_0 - s_2 = 0\}$, $t_1 = \{2s_1 - 2s_2 - s_3 = 2s_0 + s_2 = 0\}$, $t_2 = \{2s_1 - 2s_2 - s_3 = 2s_0 - s_2 = 0\}$.

□

It would be interesting to verify if (the desingularization of) a general $N \in \mathcal{N}$ is actually an Enriques surface.

## 6.5 BS-EF 3-fold (XII) of genus 9

In the following we will often refer to the use of Macaulay2: see Code B.4 of Appendix B for the computational techniques we will use. Let us study the BS-EF 3fold described in[1, §6.1.4]. Let us take two quadric hypersurfaces of $\mathbb{P}^5_{[x_0:x_1:x_2:y_3:y_4:y_5]}$, i.e.

$$Q_1 : s_1(x_0, x_1, x_2) + r_1(y_3, y_4, y_5) = 0, \quad Q_2 : s_2(x_0, x_1, x_2) + r_2(y_3, y_4, y_5) = 0,$$

where $s_1$, $s_2$, $r_1$, $r_2$ are quadratic homogeneous forms:

$$s_1(x_0, x_1, x_2) = \sum_{i,j \in \{0,1,2\}} a_{i,j} x_i x_j, \quad s_2(x_0, x_1, x_2) = \sum_{i,j \in \{0,1,2\}} a'_{i,j} x_i x_j,$$

$$r_1(y_3, y_4, y_5) = \sum_{i,j \in \{3,4,5\}} b_{i,j} y_i y_j, \quad r_2(y_3, y_4, y_5) = \sum_{i,j \in \{3,4,5\}} b'_{i,j} y_i y_j.$$

Let us consider the smooth Fano threefold $X = Q_1 \cap Q_2$ and the involution $\sigma : X \to X$ defined by the restriction on $X$ of the morphism $\sigma' : \mathbb{P}^5 \to \mathbb{P}^5$ such that

$$[x_0 : x_1 : x_2 : y_3 : y_4 : y_5] \longmapsto [x_0 : x_1 : x_2 : -y_3 : -y_4 : -y_5] .$$

The involution $\sigma : X \to X$ has eight fixed points $p_1$, $p_2$, $p_3$, $p_4$, $p'_1$, $p'_2$, $p'_3$, $p'_4$ such that

$\{p_1, p_2, p_3, p_4\} = X \cap \{y_3 = y_4 = y_5 = 0\}$ and $\{p'_1, p'_2, p'_3, p'_4\} = X \cap \{x_0 = x_1 = x_2 = 0\}$.

The quotient map $\pi : X \to X/\sigma =: W^9_{BS}$ is given by the restriction on $X$ of the morphism defined by the linear system of the $\sigma$-invariant quadric hypersurfaces of $\mathbb{P}^5$, that is the morphism $\varphi : \mathbb{P}^5 \to \mathbb{P}^{11}_{[Z_0:\cdots:Z_{11}]}$ such that

$$[x_0 : x_1 : x_2 : y_3 : y_4 : y_5] \mapsto \left[x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_0x_2 : x_1x_2 : y_3^2 : y_4^2 : y_5^2 : y_3y_4 : y_3y_5 : y_4y_5\right].$$

By using Macaulay2, one can find that the image of $\mathbb{P}^5$ via $\varphi$ is a 5-dimensional algebraic variety $F^{16}_5$ of degree 16, whose ideal is generated by the following 12 polynomials

$$Z_9Z_{10} - Z_6Z_{11},\ Z_7Z_{10} - Z_9Z_{11},\ Z_8Z_9 - Z_{10}Z_{11},\ Z_7Z_8 - Z_{11}^2,\ Z_6Z_8 - Z_{10}^2,\ Z_6Z_7 - Z_9^2,$$

$$Z_3Z_4 - Z_0Z_5,\ Z_1Z_4 - Z_3Z_5,\ Z_2Z_3 - Z_4Z_5,\ Z_1Z_2 - Z_5^2,\ Z_0Z_2 - Z_4^2,\ Z_0Z_1 - Z_3^2.$$

We observe that $W^9_{BS} = \varphi(X) = F^{16}_5 \cap H_9$, where $H_9$ is the following 9-dimensional projective subspace of $\mathbb{P}^{11}$

$$H_9 := \{a_{00}Z_0 + a_{11}Z_1 + a_{22}Z_2 + (a_{01} + a_{10})Z_3 + (a_{02} + a_{20})Z_4 + (a_{12} + a_{21})Z_5+$$
$$+b_{33}Z_6 + b_{44}Z_7 + b_{55}Z_8 + (b_{34} + b_{43})Z_9 + (b_{35} + b_{53})Z_{10} + (b_{45} + b_{54})Z_{11} = 0,$$
$$a'_{00}Z_0 + a'_{11}Z_1 + a'_{22}Z_2 + (a'_{01} + a'_{10})Z_3 + (a'_{02} + a'_{20})Z_4 + (a'_{12} + a'_{21})Z_5+$$
$$+b'_{33}Z_6 + b'_{44}Z_7 + b'_{55}Z_8 + (b'_{34} + b'_{43})Z_9 + (a'_{35} + b'_{53})Z_{10} + (b'_{45} + b'_{54})Z_{11} = 0\}.$$

Therefore we have $\pi = \varphi|_X : X \to W^9_{BS} = \varphi(X) \subset H_9 \cong \mathbb{P}^9$.

**Remark 6.8.** The threefold $W^9_{BS}$ is 2-extendable (see Definition 9.1). It would be interesting to understand if this is sharp. We observe that $W^9_{BS}$ can be at most 3-extendable by Theorem 9.2 and [10, Corollary 1.2].

What follows has been proved for fixed values of $a_{ij}$, $b_{ij}$, $a'_{ij}$ and $b'_{ij}$, in order to simplify the computational analysis.

**Example 6.9.** If $(a_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix} = (b'_{ij})$ and $(b_{ij}) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 5 \end{pmatrix} = (a'_{ij})$, then we obtain

$$p_1 = [1 : 1 : 1 : 0 : 0 : 0]\,,\ p'_1\,[0 : 0 : 0 : 1 : 1 : 1]$$
$$p_2 = [-1 : 1 : 1 : 0 : 0 : 0]\,,\ p'_2 = [0 : 0 : 0 : -1 : 1 : 1]$$
$$p_3 = [1 : -1 : 1 : 0 : 0 : 0]\,,\ p'_3 = [0 : 0 : 0 : 1 : -1 : 1]$$
$$p_4 = [1 : 1 : -1 : 0 : 0 : 0]\,,\ p'_4 = [0 : 0 : 0 : 1 : 1 : -1]\,.$$

Furthermore we have

$$H_9 := \{Z_0 - 3Z_1 + 2Z_2 + 3Z_6 - 8Z_7 + 5Z_8 = 0, \quad 3Z_0 - 8Z_1 + 5Z_2 + Z_6 - 3Z_7 + 2Z_8 = 0\} =$$
$$= \{Z_1 - Z_2 - 8Z_6 + 21Z_7 - 13Z_8 = 0, \quad Z_0 - Z_2 - 21Z_6 + 55Z_7 - 34Z_8 = 0\},$$

which is the $\mathbb{P}^9_{[w_0:\cdots:w_9]}$ embedded in $\mathbb{P}^{11}_{[Z_0:\cdots:Z_{11}]}$ via the morphism such that

$$Z_0 = w_0 + 21w_4 - 55w_5 + 34w_6, \quad Z_1 = w_0 + 8w_4 - 21w_5 + 13w_6, \quad Z_{i+2} = w_i,\ i = 0, \dots 9.$$

By using Macaulay2, we find that the quotient map $\pi : X \to W^9_{BS} \subset H_9 \cong \mathbb{P}^9$ is given by the restriction on $X$ of the morphism $\varphi' : \mathbb{P}^5 \to \mathbb{P}^9$ such that

$$[x_0 : x_1 : x_2 : y_3 : y_4 : y_5] \mapsto \left[x_2^2 : x_0x_1 : x_0x_2 : x_1x_2 : y_3^2 : y_4^2 : y_5^2 : y_3y_4 : y_3y_5 : y_4y_5\right].$$

In particular we obtain a BS-EF 3-fold $W^9_{BS} \subset \mathbb{P}^9$ whose ideal is generated by the following 12 polynomials

$$\begin{gathered} w_7w_8 - w_4w_9,\ w_5w_8 - w_7w_9,\ w_6w_7 - w_8w_9,\ w_5w_6 - w_9^2,\ w_4w_6 - w_8^2,\ w_4w_5 - w_7^2, \\ w_2^2 - w_3^2 - 13w_0w_4 + 34w_0w_5 - 21w_0w_6, \\ w_1w_2 - w_0w_3 - 21w_3w_4 + 55w_3w_5 - 34w_3w_6, \\ w_0w_2 - w_1w_3 + 8w_2w_4 - 21w_2w_5 + 13w_2w_6, \\ w_1^2 - w_3^2 - 21w_0w_4 - 168w_4^2 + 55w_0w_5 - 1155w_5^2 - 34w_0w_6 - 442w_6^2 + 881w_7^2 + -545w_8^2 + 1429w_9^2, \\ w_0w_1 - w_2w_3, \\ w_0^2 - w_3^2 + 8w_0w_4 - 21w_0w_5 + 13w_0w_6, \end{gathered}$$

Then the ideal of this BS-EF 3-fold $W^9_{BS}$ is generated by quadrics. Since $W^9_{BS}$ is projectively normal in $\mathbb{P}^9$ (see § 3.3), then the ideal of its general hyperplane section $S \subset \mathbb{P}^8$ is generated by quadrics too. This is consistent with the fact that the $\phi$ of a general hyperplane section of $S$ is 4 (see [35, Theorem 1.1 (ii)]), as we will see in the proof of Theorem 9.2.

**Remark 6.10.** The threefold $W^9_{BS}$ of Example 6.9 has the following eight singular points

$$P_1 = \pi(p_1) = [1:1:1:1:0:0:0:0:0:0]\,,\ P_1' = \pi(p_1') = [0:0:0:0:1:1:1:1:1:1]\,,$$

$$P_2 = \pi(p_2) = [1:-1:-1:1:0:0:0:0:0:0]\,,\ P_2' = \pi(p_2') = [0:0:0:0:1:1:1:-1:-1:1]\,,$$

$$P_3 = \pi(p_3) = [1:-1:1:-1:0:0:0:0:0:0]\,,\ P_3' = \pi(p_3') = [0:0:0:0:1:1:1:-1:1:-1]\,,$$

$$P_4 = \pi(p_4) = [1:1:-1:-1:0:0:0:0:0:0]\,,\ P_4' = \pi(p_4') = [0:0:0:0:1:1:1:1:-1:-1]\,.$$

Let $l_{i,j}$ be the line joining $P_i$ and $P_j$ for $i, j \in \{1, 2, 3, 4, 1', 2', 3', 4'\}$ and $i \neq j$. i.e.

$l_{1,2} = \{w_0 = w_3,\ w_1 = w_2,\ w_4 = w_5 = w_6 = w_7 = w_8 = w_9 = 0\}$,
$l_{1,3} = \{w_0 = w_2,\ w_1 = w_3,\ w_4 = w_5 = w_6 = w_7 = w_8 = w_9 = 0\}$,
$l_{1,4} = \{w_0 = w_1,\ w_2 = w_3,\ w_4 = w_5 = w_6 = w_7 = w_8 = w_9 = 0\}$,
$l_{1,1'} = \{w_0 = w_1 = w_2 = w_3,\ w_4 = w_5 = w_6 = w_7 = w_8 = w_9\}$,
$l_{1,2'} = \{w_0 = w_1 = w_2 = w_3,\ w_4 = w_5 = w_6 = w_9 = -w_7 = -w_8 = w_9\}$,
$l_{1,3'} = \{w_0 = w_1 = w_2 = w_3,\ -w_4 = -w_5 = -w_6 = w_7 = -w_8 = w_9\}$,
$l_{1,4'} = \{w_0 = w_1 = w_2 = w_3,\ -w_4 = -w_5 = -w_6 = -w_7 = w_8 = w_9\}$,
$l_{2,3} = \{w_0 = -w_1, w_2 = -w_3,\ w_4 = w_5 = w_6 = w_7 = w_8 = w_9 = 0\}$,
$l_{2,4} = \{w_0 = -w_2, w_1 = -w_3,\ w_4 = w_5 = w_6 = w_7 = w_8 = w_9 = 0\}$,
$l_{2,1'} = \{w_0 = -w_1 = -w_2 = w_3,\ w_4 = w_5 = w_6 = w_7 = w_8 = w_9\}$,
$l_{2,2'} = \{w_0 = -w_1 = -w_2 = w_3,\ w_4 = w_5 = w_6 = -w_7 = -w_8 = w_9\}$,
$l_{2,3'} = \{w_0 = -w_1 = -w_2 = w_3,\ -w_4 = -w_5 = -w_6 = w_7 = -w_8 = w_9\}$,
$l_{2,4'} = \{w_0 = -w_1 = -w_2 = w_3,\ -w_4 = -w_5 = -w_6 = -w_7 = w_8 = w_9\}$,
$l_{3,4} = \{w_0 = -w_3,\ w_1 = -w_2,\ w_4 = w_5 = w_6 = w_7 = w_8 = w_9 = 0\}$,
$l_{3,1'} = \{-w_0 = w_1 = -w_2 = w_3,\ w_4 = w_5 = w_6 = w_7 = w_8 = w_9\}$,
$l_{3,2'} = \{-w_0 = w_1 = -w_2 = w_3,\ w_4 = w_5 = w_6 = -w_7 = -w_8 = w_9\}$,

$l_{3,3'} = \{-w_0 = w_1 = -w_2 = w_3,\ -w_4 = -w_5 = -w_6 = w_7 = -w_8 = w_9\}$,
$l_{3,4'} = \{-w_0 = w_1 = -w_2 = w_3,\ -w_4 = -w_5 = -w_6 = -w_7 = w_8 = w_9\}$,
$l_{4,1'} = \{-w_0 = -w_1 = w_2 = w_3,\ w_4 = w_5 = w_6 = w_7 = w_8 = w_9\}$,
$l_{4,2'} = \{-w_0 = -w_1 = w_2 = w_3,\ w_4 = w_5 = w_6 = -w_7 = -w_8 = w_9\}$,
$l_{4,3'} = \{-w_0 = -w_1 = w_2 = w_3,\ -w_4 = -w_5 = -w_6 = w_7 = -w_8 = w_9\}$,
$l_{4,4'} = \{-w_0 = -w_1 = w_2 = w_3,\ -w_4 = -w_5 = -w_6 = -w_7 = w_8 = w_9\}$,
$l_{1',2'} = \{w_0 = w_1 = w_2 = w_3 = 0,\ w_4 = w_5 = w_6 = w_9,\ w_7 = w_8\}$,
$l_{1',3'} = \{w_0 = w_1 = w_2 = w_3 = 0,\ w_4 = w_5 = w_6 = w_8,\ w_7 = w_9\}$,
$l_{1',4'} = \{w_0 = w_1 = w_2 = w_3 = 0,\ w_4 = w_5 = w_6 = w_7,\ w_8 = w_9\}$,
$l_{2',3'} = \{w_0 = w_1 = w_2 = w_3 = 0,\ -w_4 = -w_5 = -w_6 = w_7,\ w_8 = -w_9\}$,
$l_{2',4'} = \{w_0 = w_1 = w_2 = w_3 = 0,\ -w_4 = -w_5 = -w_6 = w_8,\ w_7 = -w_9\}$,
$l_{3',4'} = \{w_0 = w_1 = w_2 = w_3 = 0,\ -w_4 = -w_5 = -w_6 = w_9,\ w_7 = -w_8\}$.

It follows that $W^9_{BS}$ contains the lines $l_{1,1'}$, $l_{1,2'}$, $l_{1,3'}$, $l_{1,4'}$, $l_{2,1'}$, $l_{2,3'}$, $l_{2,3'}$, $l_{2,4'}$, $l_{3,1'}$, $l_{3,2'}$, $l_{3,3'}$, $l_{3,4'}$, $l_{4,1'}$, $l_{4,3'}$, $l_{4,3'}$, $l_{4,4'}$, but it does not contain the others. So each one of the eight singular points of $W^9_{BS}$ is associated with $m = 4$ of the other singular points, as in Figure 24 of Appendix A. This is the same configuration of the singularities of the Enriques-Fano threefold $W^9_F$.

**Theorem 6.11.** The embedding of the BS-EF 3-fold $W^9_{BS}$ in $\mathbb{P}^9$ is the F-EF 3-fold $W^9_F$.

*Proof.* Let us project $\mathbb{P}^9$ from the $\mathbb{P}^5$ spanned by the singular points $P_2$, $P_3$, $P_4$, $P_2'$, $P_3'$, $P_4'$ of the BS-EF 3-fold $W^9_{BS}$ of Example 6.9 (see Remark 6.10). By using Macaulay2, we obtain the rational map $\rho : \mathbb{P}^9 \dashrightarrow \mathbb{P}^3$ such that

$$[w_0 : \cdots : w_9] \mapsto [w_0 + w_1 + w_2 + w_3 : -w_4 + w_5 : -w_4 + w_6 : w_4 + w_7 + w_8 + w_9]\,.$$

The restriction $\rho|_{W^9_{BS}} : W^9_{BS} \dashrightarrow \mathbb{P}^3$ is a birational map (one can verify it with Macaulay2), whose inverse map is the rational map $\nu : \mathbb{P}^3 \dashrightarrow W^9_{BS} \subset \mathbb{P}^9$ defined by the linear system $\mathcal{K}$ of the septic surfaces of $\mathbb{P}^3$ double along the six edges of the two trihedra

$$T : (s_0 - 21s_1 + 13s_2)s_0(s_0 - 55s_1 + 34s_2) = 0, \quad T' : (s_2 + s_3)(s_1 + s_3)s_3 = 0,$$

and containing the nine lines given by the intersection of a face of $T$ and one of $T'$. $\square$

## 6.6 BS-EF 3-fold (XIII) of genus 10

In the following we will often refer to the use of Macaulay2: see Code B.5 of Appendix B for the computational techniques we will use. Let us study the BS-EF 3fold described in [1, §6.5.1]. Let us consider the smooth Fano threefold $X = \mathbb{P}^1 \times S_6$, where $S_6$ is a smooth sextic Del Pezzo surface. We recall that $S_6$ is the image of $\mathbb{P}^2$ via the rational map defined by the linear system of the plane cubic curves passing through three fixed points $a_1$, $a_2$, $a_3$ in general position. Up to a change of coordinates, we can consider $a_1 = [1 : 0 : 0], a_2 = [0 : 1 : 0], a_3 = [0 : 0 : 1]$. So we have the rational map $\lambda : \mathbb{P}^2_{[u_0:u_1:u_2]} \dashrightarrow \mathbb{P}^6_{[x_0:x_1:x_2:x_3:x_4:x_5:x_6]}$ such that

$$[u_0 : u_1 : u_2] \mapsto \left[u_1^2u_2 : u_1u_2^2 : u_0^2u_2 : u_0u_2^2 : u_0^2u_1 : u_0u_1^2 : u_0u_1u_2\right]$$

and $S_6 = \lambda(\mathbb{P}^2) \subset \mathbb{P}^6$. Thanks to Macaulay2 (see also [17, Theorem 8.4.1]), we can say that $S_6$ has ideal generated by the following polynomials

$$x_3x_5 - x_6^2, \quad x_2x_5 - x_4x_6, \quad x_1x_5 - x_0x_6, \quad x_3x_4 - x_2x_6, \quad x_1x_4 - x_6^2,$$

$$x_0x_4 - x_5x_6, \quad x_0x_3 - x_1x_6, \quad x_0x_2 - x_3x_6, \quad x_0x_2 - x_6^2.$$

Let us see now how the quadratic transformation $q_{a_1,a_2,a_3} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given by the linear system of conics passing through the three fixed points $a_1, a_2, a_3$, defines an involution of $S_6$. By the following diagram

$$\begin{array}{ccc} [u_0 : u_1 : u_2] & \xrightarrow{q_{a_1,a_2,a_3}} & \left[\frac{1}{u_0} : \frac{1}{u_1} : \frac{1}{u_2}\right] \\ \downarrow \lambda & & \downarrow \lambda \\ [u_1^2u_2 : u_1u_2^2 : u_0^2u_2 : u_0u_2^2 : u_0^2u_1 : u_0u_1^2 : u_0u_1u_2] & \longmapsto & [u_0^2u_2 : u_1u_0^2 : u_1^2u_2 : u_0u_1^2 : u_1u_2^2 : u_0u_2^2 : u_0u_1u_2], \end{array}$$

we obtain an involution $t$ of $\mathbb{P}^6$ given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \xmapsto{t'} [x_2 : x_4 : x_0 : x_5 : x_1 : x_3 : x_6].$$

The locus of $t$-fixed points of $\mathbb{P}^6$ consists of two projective subspaces

$$F_1 = \{x_0 + x_2 = x_1 + x_4 = x_3 + x_5 = x_6 = 0\} \cong \mathbb{P}^2,$$

$$F_2 = \{x_0 - x_2 = x_1 - x_4 = x_3 - x_5 = 0\} \cong \mathbb{P}^3.$$

In particular we have $F_1 \cap S_6 = \emptyset$ and $F_2 \cap S_6 = \{d_1, d_2, d_3, d_4\}$, where

$$d_1 = [1 : 1 : 1 : 1 : 1 : 1 : 1], d_2 = [1 : -1 : 1 : -1 : -1 : -1 : 1],$$

$$d_3 = [-1 : 1 : -1 : -1 : 1 : -1 : 1], d_4 = [-1 : -1 : -1 : 1 : -1 : 1 : 1].$$

Then $\sigma_2 := t|_{S_6}$ is an involution of $S_6$ with four fixed points. We also consider the involution of $\mathbb{P}^1$ with two fixed points $[0 : 1]$ and $[1 : 0]$, that is the map $\sigma_1 : \mathbb{P}^1 \to \mathbb{P}^1$ such that $[y_0 : y_1] \mapsto [y_0 : -y_1]$. Let us take the map $\sigma' := (\sigma_1 \times t) : \mathbb{P}^1 \times \mathbb{P}^6 \to \mathbb{P}^1 \times \mathbb{P}^6$ such that

$$\sigma' : [y_0 : y_1] \times [x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \longmapsto [y_0 : -y_1] \times [x_2 : x_4 : x_0 : x_5 : x_1 : x_3 : x_6].$$

We have that $\sigma := \sigma'|_X = (\sigma_1 \times \sigma_2) : X \to X$ is an involution with eight fixed points

$$p_1 = [0 : 1] \times [1 : 1 : 1 : 1 : 1 : 1 : 1], \quad p_1' = [1 : 0] \times [1 : 1 : 1 : 1 : 1 : 1 : 1],$$

$$p_2 = [0 : 1] \times [1 : -1 : 1 : -1 : -1 : -1 : 1], \quad p_2' = [1 : 0] \times [1 : -1 : 1 : -1 : -1 : -1 : 1],$$

$$p_3 = [0 : 1] \times [-1 : 1 : -1 : -1 : 1 : -1 : 1], \quad p_3' = [1 : 0] \times [-1 : 1 : -1 : -1 : 1 : -1 : 1],$$

$$p_4 = [0 : 1] \times [-1 : -1 : -1 : 1 : -1 : 1 : 1], \quad p_4' = [1 : 0] \times [-1 : -1 : -1 : 1 : -1 : 1 : 1].$$

The quotient map $\pi : X \to X/\sigma =: W_{BS}^{10}$ is given by the restriction on $X$ of the morphism $\varphi : \mathbb{P}^1 \times \mathbb{P}^6 \to \mathbb{P}^{10}_{[w_0:\cdots:w_{10}]}$ defined by the $\sigma'$-invariant multihomogeneous polynomials of multidegree $(2, 1)$, i.e. $\varphi : [y_0 : y_1] \times [x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \mapsto [w_0 : w_1 : w_2 : w_3 : w_4 : w_5 : w_6 : w_7 : w_8 : w_9 : w_{10}]$ where $w_0 = y_0^2x_6$, $w_1 = y_0^2(x_0 + x_2)$, $w_2 = y_0^2(x_1 + x_4)$, $w_3 = y_0^2(x_3 + x_5)$, $w_4 = y_1^2x_6$, $w_5 = y_1^2(x_0 + x_2)$, $w_6 = y_1^2(x_1 + x_4)$, $w_7 = y_1^2(x_3 + x_5)$, $w_8 = y_0y_1(x_0 - x_2)$, $w_9 = y_0y_1(x_1 - x_4)$, $w_{10} = y_0y_1(x_3 - x_5)$.

**Remark 6.12.** Thanks to Macaulay2, one can find that the BS-EF 3-fold $W^{10}_{BS}$ has ideal generated by the following 20 polynomials

$$w_7w_8 - 2w_4w_9 + w_5w_{10}, \quad w_6w_8 - w_5w_9 + 2w_4w_{10}, \quad 2w_4w_8 - w_7w_9 + w_6w_{10},$$
$$w_3w_8 - 2w_0w_9 + w_1w_{10}, \quad w_2w_8 - w_1w_9 + 2w_0w_{10}, \quad 2w_0w_8 - w_3w_9 + w_2w_{10},$$
$$w_3w_6 - w_2w_7, \quad w_2w_6 - w_3w_7 - w_9^2 + w_{10}^2, \quad w_1w_6 - 2w_0w_7 - w_8w_9,$$
$$2w_0w_6 - w_1w_7 - w_8w_{10}, \quad w_3w_5 - w_1w_7, \quad w_2w_5 - 2w_0w_7 - w_8w_9,$$
$$w_1w_5 - w_3w_7 - w_8^2 + w_{10}^2, \quad 2w_0w_5 - w_2w_7 + w_9w_{10}, \quad w_3w_4 - w_0w_7,$$
$$2w_2w_4 - w_1w_7 - w_8w_{10}, \quad 2w_1w_4 - w_2w_7 + w_9w_{10}, \quad 4w_0w_4 - w_3w_7 + w_{10}^2,$$
$$4w_4^3 - w_4w_5^2 - w_4w_6^2 + w_5w_6w_7 - w_4w_7^2, \quad 4w_0^3 - w_0w_1^2 - w_0w_2^2 + w_1w_2w_3 - w_0w_3^2.$$

Then the ideal of $W^{10}_{BS}$ is generated by quadrics and cubics. Since $W^{10}_{BS}$ is projectively normal in $\mathbb{P}^{10}$ (see § 3.3), then the ideal of its general hyperplane section $S \subset \mathbb{P}^9$ is generated by quadrics and cubics. This is consistent with the fact that the $\phi$ of a general hyperplane section of $S$ is $3 < 4$ (see [35, Theorem 1.1 (ii)]), as we will see in the proof of Theorem 9.2.

**Remark 6.13.** Let us consider the eight singular points of $W^{10}_{BS}$. They are

$$P_1 = \pi(p_1) = [0:0:0:0:1:2:2:2:0:0:0],$$
$$P_2 = \pi(p_2) = [0:0:0:0:1:2:-2:-2:0:0:0],$$
$$P_3 = \pi(p_3) = [0:0:0:0:1:-2:2:-2:0:0:0],$$
$$P_4 = \pi(p_4) = [0:0:0:0:1:-2:-2:2:0:0:0],$$
$$P'_1 = \pi(p'_1) = [1:2:2:2:0:0:0:0:0:0:0],$$
$$P'_2 = \pi(p'_2) = [1:2:-2:-2:0:0:0:0:0:0:0],$$
$$P'_3 = \pi(p'_3) = [1:-2:2:-2:0:0:0:0:0:0:0],$$
$$P'_4 = \pi(p'_4) = [1:-2:-2:2:0:0:0:0:0:0:0].$$

Let $l_{i,j}$ be the line joining $P_i$ and $P_j$ for $i,j \in \{1,2,3,4,1',2',3',4'\}$ and $i \neq j$, i.e.

$l_{12} = \{w_0 = w_1 = w_2 = w_3 = 0,\ 2w_4 = w_5,\ w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{13} = \{w_0 = w_1 = w_2 = w_3 = 0,\ 2w_4 = w_6,\ w_5 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{14} = \{w_0 = w_1 = w_2 = w_3 = 0,\ 2w_4 = w_7,\ w_5 = w_6,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{11'} = \{2w_0 = w_1 = w_2 = w_3,\ 2w_4 = w_5 = w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{12'} = \{-2w_0 = -w_1 = w_2 = w_3,\ 2w_4 = w_5 = w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{13'} = \{-2w_0 = w_1 = -w_2 = w_3,\ 2w_4 = w_5 = w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$
$l_{14'} = \{2w_0 = -w_1 = -w_2 = w_3,\ 2w_4 = w_5 = w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{23} = \{w_0 = w_1 = w_2 = w_3 = 0,\ -2w_4 = w_7, -w_5 = w_6,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{24} = \{w_0 = w_1 = w_2 = w_3 = 0,\ -2w_4 = w_6, -w_5 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{21'} = \{2w_0 = w_1 = w_2 = w_3,\ -2w_4 = -w_5 = w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{22'} = \{-2w_0 = -w_1 = w_2 = w_3,\ -2w_4 = -w_5 = w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{23'} = \{-2w_0 = w_1 = -w_2 = w_3,\ -2w_4 = -w_5 = w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{24'} = \{2w_0 = -w_1 = -w_2 = w_3,\ -2w_4 = -w_5 = w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{34} = \{w_0 = w_1 = w_2 = w_3 = 0,\ -2w_4 = w_5,\ -w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{31'} = \{2w_0 = w_1 = w_2 = w_3,\ -2w_4 = w_5 = -w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{32'} = \{-2w_0 = -w_1 = w_2 = w_3,\ -2w_4 = w_5 = -w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,

$l_{33'} = \{-2w_0 = w_1 = -w_2 = w_3,\ -2w_4 = w_5 = -w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{34'} = \{2w_0 = -w_1 = -w_2 = w_3,\ -2w_4 = w_5 = -w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{41'} = \{2w_0 = w_1 = w_2 = w_3,\ 2w_4 = -w_5 = -w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{42'} = \{-2w_0 = -w_1 = w_2 = w_3,\ 2w_4 = -w_5 = -w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{43'} = \{-2w_0 = w_1 = -w_2 = w_3,\ 2w_4 = -w_5 = -w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{44'} = \{2w_0 = -w_1 = -w_2 = w_3,\ 2w_4 = -w_5 = -w_6 = w_7,\ w_8 = w_9 = w_{10} = 0\}$,
$l_{1'2'} = \{2w_0 = w_1,\ w_2 = w_3,\ w_4 = w_5 = w_6 = w_7 = w_8 = w_9 = w_{10} = 0\}$,
$l_{1'3'} = \{2w_0 = w_2,\ w_1 = w_3,\ w_4 = w_5 = w_6 = w_7 = w_8 = w_9 = w_{10} = 0\}$,
$l_{1'4'} = \{2w_0 = w_3,\ w_1 = w_2,\ w_4 = w_5 = w_6 = w_7 = w_8 = w_9 = w_{10} = 0\}$,
$l_{2'3'} = \{-2w_0 = w_3,\ -w_1 = w_2,\ w_4 = w_5 = w_6 = w_7 = w_8 = w_9 = w_{10} = 0\}$,
$l_{2'4'} = \{-2w_0 = w_2,\ -w_1 = w_3,\ w_4 = w_5 = w_6 = w_7 = w_8 = w_9 = w_{10} = 0\}$,
$l_{3'4'} = \{-2w_0 = w_1,\ -w_2 = w_3,\ w_4 = w_5 = w_6 = w_7 = w_8 = w_9 = w_{10} = 0\}$.

By Remark 6.12 we have that $W^{10}_{BS}$ contains the lines $l_{1,2}$, $l_{1,3}$, $l_{1,4}$, $l_{1,1'}$, $l_{2,3}$, $l_{2,4}$, $l_{2,2'}$, $l_{3,4}$, $l_{3,3'}$, $l_{4,4'}$, $l_{1',2'}$, $l_{1',3'}$, $l_{1',4'}$, $l_{2',3'}$, $l_{2',4'}$, $l_{3',4'}$, while it does not contain the others. So each one of the eight singular points of $W^{10}_{BS}$ is associated with $m = 4$ of the other singular points, as in Figure 25 of Appendix A. Hence there exist three mutually associated points (for example $P_1$, $P_2$ and $P_3$). This case had been excluded by Fano for $p > 7$, as we said in Remark 4.7 (iv). So this suggests that in Fano's paper there are other gaps to be discovered.

**Theorem 6.14.** Let $T \subset \mathbb{P}^3$ be a tetrahedron with faces $f_i$ and edges $l_{ij} := f_i \cap f_j$ for $0 \le i < j \le 3$. Let $v_i$ be the vertex opposite to the face $f_i$, for $0 \le i \le 3$. Let $\pi$ be a plane through the vertex $v_0$, which intersects the face $f_i$ along a line $r_i$, for $1 \le i \le 3$, and let us define the point $q_i := r_i \cap l_{0i}$ (see Figure 12). Then $W^{10}_{BS}$ can be obtained as the image of $\mathbb{P}^3$ via the rational map $\nu_{\mathcal{M}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{10}$ defined by the linear system $\mathcal{M}$ of the sextic surfaces quadruple at the vertex $v_0$, triple at the other three vertices $v_1$, $v_2$, $v_3$, and double along the three lines $r_1$, $r_2$, $r_3$. Furthemore a general $M \in \mathcal{M}$ also contains the six edges of $T$.

*Proof.* Let us project $\mathbb{P}^{10}$ from the $\mathbb{P}^6$ spanned by the singular points $P_1$, $P_2$, $P_3$, $P_4$, $P'_1$, $P'_2$, $P'_3$ of $W^{10}_{BS}$ (see Remark 6.13). By using Macaulay2, we obtain the rational map

$$\rho : \mathbb{P}^{10} \dashrightarrow \mathbb{P}^3, \quad [w_0 : \cdots : w_{13}] \mapsto [-2w_0 + w_1 + w_2 - w_3 : w_8 : w_9 : w_{10}]\,.$$

Thanks to Macaulay2, we see that the restriction $\rho|_{W^{10}_{BS}} : W^{10}_{BS} \dashrightarrow \mathbb{P}^3$ is birational. We can also compute its inverse map, which is given by the rational map $\nu : \mathbb{P}^3 \dashrightarrow W^{10}_{BS} \subset \mathbb{P}^{10}$, such that $[s_0 : s_1 : s_2 : s_3] \mapsto [w_0 : \cdots : w_{10}]$, where

$w_0 = s_1^4s_2s_3 + s_1^3s_2^2s_3 - s_1^2s_2^3s_3 - s_1s_2^4s_3 + s_1^3s_2s_3^2 + 2s_1^2s_2^2s_3^2 + s_1s_2^3s_3^2 - s_1^2s_2s_3^3 + s_1s_2^2s_3^3 - s_1s_2s_3^4$,

$w_1 = s_0^2s_1^2s_2s_3 - s_0^2s_1s_2^2s_3 - s_0^2s_1s_2s_3^2$,

$w_2 = s_1^5s_3 - s_1^3s_2^2s_3 + s_1^2s_2^3s_3 - s_2^5s_3 + s_1^4s_3^2 + s_1^3s_2s_3^2 + s_1s_2^3s_3^2 + s_2^4s_3^2 - 2s_1^3s_3^3 + s_1^2s_2s_3^3 - s_1s_2^2s_3^3 + 2s_2^3s_3^3 - 2s_1^2s_3^4 - s_1s_2s_3^4 - 2s_2^2s_3^4 + s_1s_3^5 - s_2s_3^5 + s_3^6$,

$w_3 = s_0s_1^4s_3 - 2s_0s_1^2s_2^2s_3 + s_0s_2^4s_3 - 2s_0s_1^2s_3^3 - 2s_0s_2^2s_3^3 + s_0s_3^5$,

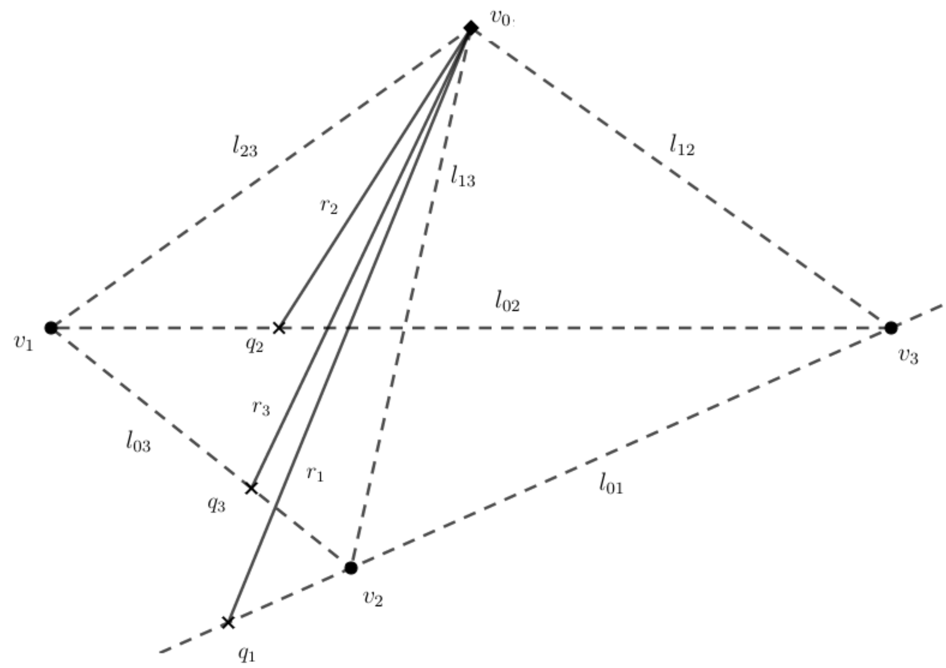


*Figure 12:* *Base locus of the linear system* $\mathcal{M}$.

$$w_4 = s_0^2 s_1^3 s_3 - s_0^2 s_2^3 s_3 - s_0^2 s_1^2 s_3^2 - s_0^2 s_1 s_2 s_3^2 - s_0^2 s_2^2 s_3^2 - s_0^2 s_1 s_3^3 + s_0^2 s_2 s_3^3 + s_0^2 s_3^4,$$

$$w_6 = s_1^5 s_2 + s_1^4 s_2^2 - 2s_1^3 s_2^3 - 2s_1^2 s_2^4 + s_1 s_2^5 + s_2^6 + s_1^3 s_2^2 s_3 + s_1^2 s_2^3 s_3 - s_1 s_2^4 s_3 - s_2^5 s_3 - s_1^3 s_2 s_3^2 - s_1 s_2^3 s_3^2 - 2s_2^4 s_3^2 + s_1^2 s_2 s_3^3 + s_1 s_2^2 s_3^3 + 2s_2^3 s_3^3 + s_2^2 s_3^4 - s_2 s_3^5,$$

$$w_7 = s_0 s_1^4 s_2 - 2s_0 s_1^2 s_2^3 + s_0 s_2^5 - 2s_0 s_1^2 s_2 s_3^2 - 2s_0 s_2^3 s_3^2 + s_0 s_2 s_3^4,$$

$$w_8 = s_0^2 s_1^3 s_2 - s_0^2 s_1^2 s_2^2 - s_0^2 s_1 s_2^3 + s_0^2 s_2^4 - s_0^2 s_1 s_2^2 s_3 + s_0^2 s_2^3 s_3 - s_0^2 s_2^2 s_3^2 - s_0^2 s_2 s_3^3,$$

$$w_9 = s_1^6 - 3s_1^4 s_2^2 + 3s_1^2 s_2^4 - s_2^6 - 2s_1^3 s_2^2 s_3 - 2s_1^2 s_2^3 s_3 + 2s_1 s_2^4 s_3 + 2s_2^5 s_3 - 3s_1^4 s_3^2 - 2s_1^3 s_2 s_3^2 - 2s_1^2 s_2^2 s_3^2 - 2s_1 s_2^3 s_3^2 + s_2^4 s_3^2 - 2s_1^2 s_2 s_3^3 - 2s_1 s_2^2 s_3^3 - 4s_2^3 s_3^3 + 3s_1^2 s_3^4 + 2s_1 s_2 s_3^4 + s_2^2 s_3^4 + 2s_2 s_3^5 - s_3^6,$$

$$w_{10} = s_0 s_1^5 - 2s_0 s_1^3 s_2^2 + s_0 s_1 s_2^4 - 2s_0 s_1^3 s_3^2 - 2s_0 s_1 s_2^2 s_3^2 + s_0 s_1 s_3^4, s_0^2 s_1^4 - 2s_0^2 s_1^2 s_2^2 + s_0^2 s_2^4 - 2s_0^2 s_1^2 s_3^2 - 2s_0^2 s_2^2 s_3^2 + s_0^2 s_3^4.$$

By using Macaulay2, we can study the base locus of $\nu$. We find that $\nu$ is the rational map defined by the linear system of the sextic surfaces of $\mathbb{P}^3$

(i) containing the six edges $l_{23} = \{s_1 = 0, s_2 - s_3 = 0\}$, $l_{13} = \{s_3 = 0, s_1 + s_2 = 0\}$, $l_{12} = \{s_2 = 0, s_1 + s_3 = 0\}$, $l_{01} = \{s_0 = 0, s_1 + s_2 + s_3 = 0\}$, $l_{03} = \{s_0 = 0, s_1 - s_2 + s_3 = 0\}$ and $l_{02} = \{s_0 = 0, s_1 + s_2 - s_3 = 0\}$ of the tetrahedron $T$ with faces $f_0 = \{s_0 = 0\}$, $f_1 = \{s_1 + s_2 + s_3 = 0\}$, $f_2 = \{s_1 - s_2 + s_3 = 0\}$ and $f_3 = \{s_1 + s_2 - s_3 = 0\}$;

(ii) double along the three lines $r_1 = \{s_1 = 0, s_2 + s_3 = 0\}$, $r_2 = \{s_3 = 0, s_1 - s_2 = 0\}$ and $r_3 = \{s_2 = 0, s_1 - s_3 = 0\}$ contained in the plane $\pi = \{s_1 - s_2 - s_3 = 0\}$, and obviously double at the points $q_1 := l'_1 \cap r_1 = [0:0:-1:1]$, $q_2 := l'_2 \cap r_2 = [0:1:1:0]$, $q_3 := l'_3 \cap r_3 = [0:1:0:1]$;

(iii) triple at the vertices $v_1 = [0:0:1:1]$, $v_2 = [0:1:-1:0]$, $v_3 = [0:1:0:-1]$;

(iv) and quadruple at the vertex $v_0 = [1:0:0:0]$.

□

It would be interesting to verify if (the desingularization of) a general $M \in \mathcal{M}$ is actually an Enriques surface.

## 6.7 BS-EF 3-fold (XIV) of genus 13

In the following we will often refer to the use of Macaulay2: see Code B.6 of Appendix B for the computational techniques we will use. Let us study the BS-EF 3fold described in [1, §6.3.2]. Let us consider the smooth Fano threefold $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and the involution $\sigma : X \to X$ such that

$$[x_0 : x_1] \times [y_0 : y_1] \times [z_0 : z_1] \mapsto [x_0 : -x_1] \times [y_0 : -y_1] \times [z_0 : -z_1].$$

This involution has the following eight fixed points

$$p'_1 = [0:1] \times [1:0] \times [1:0], \quad p_1 = [1:0] \times [0:1] \times [0:1],$$

$$p'_2 = [0:1] \times [0:1] \times [0:1], \quad p_2 = [1:0] \times [1:0] \times [1:0],$$
$$p_3 = [0:1] \times [1:0] \times [0:1], \quad p'_3 = [1:0] \times [0:1] \times [1:0],$$
$$p_4 = [0:1] \times [0:1] \times [1:0], \quad p'_4 = [1:0] \times [1:0] \times [0:1].$$

The $\sigma$-invariant multihomogeneous polynomials of multidegree $(2,2,2)$ define the coordinates of the quotient map $\pi : X \to X/\sigma =: W^{13}_{BS} \subset \mathbb{P}^{13}$, i.e.

$$\begin{array}{c}[x_0 : x_1] \times [y_0 : y_1] \times [z_0 : z_1] \\ \downarrow \pi \\ [w_0 : w_1 : w_2 : w_3 : w_4 : w_5 : w_6 : w_7 : w_8 : w_9 : w_{10} : w_{11} : w_{12} : w_{13}]\end{array}$$

where $w_0 = x_0^2 y_0^2 z_0^2$, $w_1 = x_0^2 y_0^2 z_1^2$, $w_2 = x_0^2 y_0 y_1 z_0 z_1$, $w_3 = x_0^2 y_1^2 z_0^2$, $w_4 = x_0^2 y_1^2 z_1^2$, $w_5 = x_0 x_1 y_0^2 z_0 z_1$, $w_6 = x_0 x_1 y_0 y_1 z_0^2$, $w_7 = x_0 x_1 y_0 y_1 z_1^2$, $w_8 = x_0 x_1 y_1^2 z_0 z_1$, $w_9 = x_1^2 y_0^2 z_0^2$, $w_{10} = x_1^2 y_0^2 z_1^2$, $w_{11} = x_1^2 y_0 y_1 z_0 z_1$, $w_{12} = x_1^2 y_1^2 z_0^2$, $w_{13} = x_1^2 y_1^2 z_1^2$.

**Remark 6.15.** Thanks to Macaulay2, we find that the Enriques-Fano threefold $W^{13}_{BS}$ has ideal generated by the following 42 polynomials

$$\begin{array}{cccc}
w_{10}w_{12} - w_9 w_{13}, & w_7 w_{12} - w_6 w_{13}, & w_4 w_{12} - w_3 w_{13}, & w_1 w_{12} - w_0 w_{13}, \\
w_{11}^2 - w_9 w_{13}, & w_8 w_{11} - w_6 w_{13}, & w_7 w_{11} - w_5 w_{13}, & w_6 w_{11} - w_5 w_{12}, \\
w_4 w_{11} - w_2 w_{13}, & w_3 w_{11} - w_2 w_{12}, & w_2 w_{11} - w_0 w_{13}, & w_8 w_{10} - w_5 w_{13}, \\
w_6 w_{10} - w_5 w_{11}, & w_4 w_{10} - w_1 w_{13}, & w_3 w_{10} - w_0 w_{13}, & w_2 w_{10} - w_1 w_{11},
\end{array}$$

$$\begin{gathered}w_8w_9 - w_5w_{12}, \quad w_7w_9 - w_5w_{11}, \quad w_4w_9 - w_0w_{13}, \quad w_3w_9 - w_0w_{12},\\ w_2w_9 - w_0w_{11}, \quad w_1w_9 - w_0w_{10}, \quad w_8^2 - w_3w_{13}, \quad w_7w_8 - w_2w_{13},\\ w_6w_8 - w_2w_{12}, \quad w_5w_8 - w_0w_{13}, \quad w_7^2 - w_1w_{13}, \quad w_6w_7 - w_0w_{13},\\ w_5w_7 - w_1w_{11}, \quad w_3w_7 - w_2w_8, \quad w_2w_7 - w_1w_8, \quad w_6^2 - w_0w_{12},\\ w_5w_6 - w_0w_{11}, \quad w_4w_6 - w_2w_8, \quad w_2w_6 - w_0w_8, \quad w_1w_6 - w_0w_7,\\ w_5^2 - w_0w_{10}, \quad w_4w_5 - w_1w_8, \quad w_3w_5 - w_0w_8, \quad w_2w_5 - w_0w_7,\\ w_1w_3 - w_0w_4, \quad w_2^2 - w_0w_4.\end{gathered}$$

Thus the ideal of $W^{13}_{BS}$ is generated by quadrics. Since $W^{13}_{BS}$ is projectively normal in $\mathbb{P}^{13}$ (see § 3.3), then the ideal of its general hyperplane section $S \subset \mathbb{P}^{12}$ is generated by quadrics too. This is consistent with the fact that the $\phi$ of a general hyperplane section of $S$ is 4 (see [35, Theorem 1.1 (ii)]), as we will see in the proof of Theorem 9.2.

**Remark 6.16.** The above threefold $W^{13}_{BS}$ has the following eight singular points

$$P_1 = \pi(p_1) = [0:0:0:0:1:0:0:0:0:0:0:0:0:0],$$

$$P_2 = \pi(p_2) = [1:0:0:0:0:0:0:0:0:0:0:0:0:0],$$

$$P_3 = \pi(p_3) = [0:0:0:0:0:0:0:0:0:0:1:0:0:0],$$

$$P_4 = \pi(p_4) = [0:0:0:0:0:0:0:0:0:0:0:0:1:0],$$

$$P'_1 = \pi(p'_1) = [0:0:0:0:0:0:0:0:0:1:0:0:0:0],$$

$$P'_2 = \pi(p'_2) = [0:0:0:0:0:0:0:0:0:0:0:0:0:1],$$

$$P'_3 = \pi(p'_3) = [0:0:0:1:0:0:0:0:0:0:0:0:0:0],$$

$$P'_4 = \pi(p'_4) = [0:1:0:0:0:0:0:0:0:0:0:0:0:0].$$

Let $l_{i,j}$ be the line joining $P_i$ and $P_j$ with $i, j \in \{1, 2, 3, 4, 1', 2', 3', 4'\}$ and $i \neq j$. Then we have $l_{1,2} = \{w_i = 0 | i \neq 0, 4\}$, $l_{1,3} = \{w_i = 0 | i \neq 4, 10\}$, $l_{1,4} = \{w_i = 0 | i \neq 4, 12\}$, $l_{1,1'} = \{w_i = 0 | i \neq 4, 9\}$, $l_{1,2'} = \{w_i = 0 | i \neq 4, 13\}$, $l_{1,3'} = \{w_i = 0 | i \neq 3, 4\}$, $l_{1,4'} = \{w_i = 0 | i \neq 1, 4\}$, $l_{2,3} = \{w_i = 0 | i \neq 0, 10\}$, $l_{2,4} = \{w_i = 0 | i \neq 0, 12\}$, $l_{2,1'} = \{w_i = 0 | i \neq 0, 9\}$, $l_{2,2'} = \{w_i = 0 | i \neq 0, 13\}$, $l_{2,3'} = \{w_i = 0 | i \neq 0, 3\}$, $l_{2,4'} = \{w_i = 0 | i \neq 0, 1\}$, $l_{3,4} = \{w_i = 0 | i \neq 10, 12\}$, $l_{3,1'} = \{w_i = 0 | i \neq 9, 10\}$, $l_{3,2'} = \{w_i = 0 | i \neq 10, 13\}$, $l_{3,3'} = \{w_i = 0 | i \neq 3, 10\}$, $l_{3,4'} = \{w_i = 0 | i \neq 1, 10\}$, $l_{4,1'} = \{w_i = 0 | i \neq 9, 12\}$, $l_{4,2'} = \{w_i = 0 | i \neq 12, 13\}$, $l_{4,3'} = \{w_i = 0 | i \neq 3, 13\}$, $l_{4,4'} = \{w_i = 0 | i \neq 1, 12\}$, $l_{1',2'} = \{w_i = 0 | i \neq 9, 13\}$, $l_{1',3'} = \{w_i = 0 | i \neq 3, 9\}$, $l_{1',4'} = \{w_i = 0 | i \neq 1, 9\}$, $l_{2',3'} = \{w_i = 0 | i \neq 2, 3\}$, $l_{2',4'} = \{w_i = 0 | i \neq 1, 2\}$, $l_{3',4'} = \{w_i = 0 | i \neq 1, 3\}$. By Remark 6.15 we see that $W^{13}_{BS}$ contains the lines $l_{1,2'}$, $l_{1,3'}$, $l_{1,4'}$, $l_{2,1'}$, $l_{2,3'}$, $l_{2,4'}$, $l_{3,1'}$, $l_{3,2'}$, $l_{3,4'}$, $l_{4,1'}$, $l_{4,2'}$, $l_{4,3'}$, while it does not contain the others. So each one of the eight singular points of $W^{13}_{BS}$ is associated with $m = 3$ of the other singular points, as in Figure 26 of Appendix A. This is the same configuration of the singularities of the F-EF 3-fold $W^{13}_F$.

**Theorem 6.17.** The embedding of the BS-EF 3-fold $W^{13}_{BS}$ in $\mathbb{P}^{13}$ is the F-EF 3-fold $W^{13}_F \subset \mathbb{P}^{13}$

*Proof.* Let us project $\mathbb{P}^{13}$ from the $\mathbb{P}^7$ spanned by the eight singular points of $W^{13}_{BS}$ (see Remark 6.16). So we obtain the rational map $\rho : \mathbb{P}^{13} \dashrightarrow \mathbb{P}^5$ s.t. $[w_0 : \cdots : w_{13}] \mapsto [w_2 : w_5 : w_6 : w_7 : w_8 : w_{11}]$. Thanks to Macaylay2 we verify that the restriction map $\rho|_{W^{13}_{BS}} : W^{13}_{BS} \dashrightarrow \mathbb{P}^5$ is birational onto the image, which is a quartic threefold $T^4_3 \subset \mathbb{P}^5$ given by the complete intersection of two quadric hypersurfaces of $\mathbb{P}^5_{[t_0:t_1:t_2:t_3:t_4:t_5]}$

$$Q_1 : t_1t_4 - t_0t_5 = 0, \quad Q_2 : t_2t_3 - t_0t_5 = 0.$$

Such a threefold $T^4_3$ is birational to $\mathbb{P}^3$ via the rational map

$$q : \mathbb{P}^3 \dashrightarrow T^4_3 \subset \mathbb{P}^5, \quad [s_0 : s_1 : s_2 : s_3] \mapsto [s_0s_1 : s_1s_2 : s_1s_3 : s_0s_2 : s_0s_3 : s_2s_3],$$

defined by the linear system of the quadric surfaces passing through the four vertices of the tetrahedron $\{s_0s_1s_2s_3 = 0\}$. By using Macaulay2 we can take the inverse map of $q$, which is $q^{-1} : T^4_3 \subset \mathbb{P}^5 \dashrightarrow \mathbb{P}^3$ s.t. $[t_0 : t_1 : t_2 : t_3 : t_4 : t_5] \mapsto [t_3t_4 : t_0t_5 : t_3t_5 : t_4t_5]$. So we can construct the birational map $(q^{-1} \circ \rho|_{W^{13}_{BS}}) : W^{13}_{BS} \subset \mathbb{P}^{13} \dashrightarrow \mathbb{P}^3$ s. t. $[w_0 : \cdots : w_{13}] \mapsto [w_7w_8 : w_2w_{11} : w_7w_{11} : w_8w_{11}]$. Thanks to Macaulay2, we can compute again its inverse map, which is given by $\nu : \mathbb{P}^3 \dashrightarrow W^{13}_{BS} \subset \mathbb{P}^{13}$ s.t. $[s_0 : s_1 : s_2 : s_3] \mapsto [w_0 : \cdots : w_{13}]$, where $w_0 = s_0s_1^3s_2s_3$, $w_1 = s_0^2s_1^2s_2^2$, $w_2 = s_0^2s_1^2s_2s_3$, $w_3 = s_0^2s_1^2s_3^2$, $w_4 = s_0^3s_1s_2s_3$, $w_5 = s_0s_1^2s_2^2s_3$, $w_6 = s_0s_1^2s_2s_3^2$, $w_7 = s_0^2s_1s_2^2s_3$, $w_8 = s_0^2s_1s_2s_3^2$, $w_9 = s_1^2s_2^2s_3^2$, $w_{10} = s_0s_1s_2^3s_3$, $w_{11} = s_0s_1s_2^2s_3^2$, $w_{12} = s_0s_1s_2s_3^3$, $w_{13} = s_0^2s_2^2s_3^2$. We observe that $\nu$ is the rational map defined by the linear system $\mathcal{S}$ of the sextic surfaces of $\mathbb{P}^3$ double along the six edges of the tetrahedron $\{s_0s_1s_2s_3 = 0\}$. □

# 7 Singularities of the KLM-EF 3-fold

## 7.1 Abstract

We recall that the KLM-EF 3-fold $W^9_{KLM} \subset \mathbb{P}^9$ is an Enriques-Fano threefold given by the projection of the classical Enriques-Fano threefold $W^{13}_F \subset \mathbb{P}^{13}$ from the $\mathbb{P}^3$ spanned by a certain curve $E_3 \subset W^{13}_F$ (see [36, §13]). We will computationally analyze the KLM-EF 3-fold and we will find that its ideal in $\mathbb{P}^9$ is generated by quadrics and cubics. We will also study the image of the eight quadruple points of $W^{13}_F$ via the above projection map. We will find that they are five singular points of $W^9_{KLM}$ such that four of them are quadruple points, whose tangent cone is a cone over a Veronese surface (see Proposition 7.4), and the last one is a sextuple point, whose tangent cone is a cone over the union of four planes and a quadric surface (see Theorem 7.5). These five points are so non-similar singular points of $W^9_{KLM}$ and we will see that they have the configuration in Figure 28 of Appendix A.

## 7.2 Construction of the KLM-EF 3-fold (XV)

Let us see how to construct the KLM-EF 3-fold $W^9_{KLM}$. First we consider the F-EF 3-fold $W^{13}_F$, which is the image of $\mathbb{P}^3$ via the rational map $\nu_{\mathcal{S}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{13}$ defined by the linear system $\mathcal{S}$ of the sextic surfaces double along the six edges of a tetrahedron $T$. We need to recall some details. We will use the notations of § 5.2. Let $T \subset \mathbb{P}^3$ be the tetrahedron of Figure 2, with faces $f_i$, vertices $v_i$ and edges $l_{ij} = f_i \cap f_j = \langle v_k, v_h \rangle$, for distinct indices $i, j, k, h \in \{0, 1, 2, 3\}$ with $i < j$. In the following we will denote by $l_{ij}$ the edge $f_i \cap f_j$ even if $i > j$, by abuse of notation. Let us take the smooth rational threefold $Y$ obtained by blowing-up first the vertices of $T$, then the strict transforms of the edges of $T$ and finally certain (twelve) disjoint curves (see proof of Theorem 5.4). We have that an element $\Sigma \in \mathcal{S}$ is isomorphic to a divisor $\widetilde{\Sigma}$ on $Y$ which is linearly equivalent to $6\mathcal{H} - \sum_{i=0}^{3} 3\mathcal{E}_k - \sum_{0 \leq i < j \leq 3} 2\mathcal{F}_{ij} - \sum_{\substack{i,j=0 \\ i \neq j}}^{2} 4\Gamma_{ij}$. Hence $W_F^{13} = \nu_{\widetilde{\mathcal{S}}}(Y) \subset \mathbb{P}^{13}$, where $\nu_{\widetilde{\mathcal{S}}} : Y \to \mathbb{P}^{13}$ is the morphism defined by the linear system $\widetilde{\mathcal{S}} := |\mathcal{O}_Y(\widetilde{\Sigma})|$.

**Proposition 7.1.** Let $\widetilde{\Sigma}$ be a general element of $\widetilde{\mathcal{S}}$. Then we have that

$$(2\mathcal{H} - \mathcal{F}_{ik} - \mathcal{F}_{ih} - \mathcal{F}_{jk} - \mathcal{F}_{jh} - 2\Gamma_{kh} - 2\Gamma_{hk} - 2\Gamma_{ij} - 2\Gamma_{ji} - \Gamma_{ki} - \Gamma_{hi} - \Gamma_{kj} - \Gamma_{hj} - \Gamma_{ik} - \Gamma_{ih} - \Gamma_{jk} - \Gamma_{jh})|_{\widetilde{\Sigma}}$$

$\sim 2(\mathcal{F}_{ij} + \Gamma_{ki} + \Gamma_{hi} + \Gamma_{kj} + \Gamma_{hj})|_{\widetilde{\Sigma}}$ for distinct indices $i, j, k, h \in \{0, 1, 2, 3\}$.

*Proof.* Let us take the reducible quadric surface $Q_{ij} := f_i \cup f_j$ of $\mathbb{P}^3$ given by the union of two faces of $T$. This surface contains doubly the common line $l_{ij}$ of the two faces, simply the edges $l_{ik}$, $l_{ih}$, $l_{jk}$ and $l_{jh}$ and does not contain the edge $l_{kh}$. Its strict transform $\widetilde{Q}_{ij}$ on $Y$ is linearly equivalent to $2\mathcal{H} - 2\mathcal{E}_k - 2\mathcal{E}_h - \mathcal{E}_i - \mathcal{E}_j - 2\mathcal{F}_{ij} - \mathcal{F}_{ik} - \mathcal{F}_{ih} - \mathcal{F}_{jk} - \mathcal{F}_{jh} - 3\Gamma_{ki} - 3\Gamma_{hi} - 3\Gamma_{kj} - 3\Gamma_{hj} - 2\Gamma_{kh} - 2\Gamma_{hk} - 2\Gamma_{ij} - 2\Gamma_{ji} - \Gamma_{ik} - \Gamma_{ih} - \Gamma_{jk} - \Gamma_{jh}$. We recall that $\mathcal{E}_i \cdot \widetilde{\Sigma} = 0$ for all $i$. So we obtain that $K_{\widetilde{\Sigma}} = (K_Y + \widetilde{\Sigma})|_{\widetilde{\Sigma}} \sim (2\mathcal{H} - \sum_{0 \leq i < j \leq 3} \mathcal{F}_{ij} - \sum_{\substack{i,j=0 \\ i \neq j}}^{3} \Gamma_{ij})|_{\widetilde{\Sigma}} \sim (\widetilde{Q}_{ij} + \mathcal{F}_{ij} - \mathcal{F}_{kh} + 2\Gamma_{ki} + 2\Gamma_{hi} + 2\Gamma_{kj} + 2\Gamma_{hj} + \Gamma_{kh} + \Gamma_{hk} + \Gamma_{ij} + \Gamma_{ji})|_{\widetilde{\Sigma}}$. Since $Q_{ij}|_{\widetilde{\Sigma}} \sim 0$, then we have $K_{\widetilde{\Sigma}} + (\mathcal{F}_{ij} + \mathcal{F}_{kh} - \Gamma_{kh} - \Gamma_{hk} - \Gamma_{ij} - \Gamma_{ji})|_{\widetilde{\Sigma}} \sim 2(\mathcal{F}_{ij} + \Gamma_{ki} + \Gamma_{hi} + \Gamma_{kj} + \Gamma_{hj})|_{\widetilde{\Sigma}}$ and so the expression of the statement. □

Let us *fix* now a general $\Sigma \in \mathcal{S}$ and its strict transform $\widetilde{\Sigma}$ on $Y$. Let $i, j, k, h$ be four distinct indices in $\{0, 1, 2, 3\}$ with $i < j$ and $k < h$. The curve $\mathcal{F}_{ij} \cap \widetilde{\Sigma}$ intersects each of the four curves $\Gamma_{ki} \cap \widetilde{\Sigma}$, $\Gamma_{hi} \cap \widetilde{\Sigma}$, $\Gamma_{kj} \cap \widetilde{\Sigma}$ and $\Gamma_{hj} \cap \widetilde{\Sigma}$ at one point (use Remark 5.9). We recall that these four curves are contracted by $\nu_{\widetilde{\mathcal{S}}} : Y \to W_F^{13} \subset \mathbb{P}^{13}$ to points of $\nu_{\widetilde{\mathcal{S}}}(\widetilde{\Sigma})$ (see Remark 5.13). Let us define $\lambda_{ij} := (\mathcal{F}_{ij} \cup \Gamma_{ki} \cup \Gamma_{hi} \cup \Gamma_{kj} \cup \Gamma_{hj}) \cap \widetilde{\Sigma}$. Then $|\mathcal{O}_{\widetilde{\Sigma}}(2\lambda_{ij})|$ is an elliptic pencil on $\widetilde{\Sigma}$. Indeed $|\mathcal{O}_{\widetilde{\Sigma}}(2\lambda_{ij})|$ is isomorphic to the linear system cut out on $\Sigma$ by the quadric surfaces of $\mathbb{P}^3$ containing the lines $l_{ik}$, $l_{ih}$, $l_{jk}$, $l_{jh}$ (see Proposition 7.1), which is an elliptic pencil on $\Sigma$ (see [27, p. 634]). By Proposition 7.1 we also have that

$$\widetilde{\mathcal{S}}|_{\widetilde{\Sigma}} = |\mathcal{O}_{\widetilde{\Sigma}}(6\mathcal{H} - \sum_{0 \leq i < j \leq 3} 2\mathcal{F}_{ij} - \sum_{\substack{i,j=0 \\ i \neq j}}^{3} 4\Gamma_{ij})| = |\mathcal{O}_{\widetilde{\Sigma}}(2\lambda_{12} + 2\lambda_{13} + 2\lambda_{23})|.$$

The linear system $\widetilde{\mathcal{S}}|_{\widetilde{\Sigma}}$ defines the morphism $\nu_{\widetilde{\mathcal{S}}}|_{\widetilde{\Sigma}} : \widetilde{\Sigma} \to \mathbb{P}^{12}$, whose image $S := \nu_{\widetilde{\mathcal{S}}}(\widetilde{\Sigma}) \subset \mathbb{P}^{12}$ is a hyperplane section of $W_F^{13} \subset \mathbb{P}^{13}$. Hence there exists a hyperplane $H \cong \mathbb{P}^{12}$ in $\mathbb{P}^{13}$ such that $S = W_F^{13} \cap H$. Let us define the curves $E_1 := \nu_{\widetilde{\mathcal{S}}}(\lambda_{12})$, $E_2 := \nu_{\widetilde{\mathcal{S}}}(\lambda_{13})$, $E_3 := \nu_{\widetilde{\mathcal{S}}}(\lambda_{23})$, $E_1' := \nu_{\widetilde{\mathcal{S}}}(\lambda_{03})$, $E_2' := \nu_{\widetilde{\mathcal{S}}}(\lambda_{02})$, $E_3' := \nu_{\widetilde{\mathcal{S}}}(\lambda_{01}) \subset S$. They are smooth irreducible elliptic quartic curves such that $E_i^2 = 0$, $E_i \cdot E_i' = 0$, $E_i \cdot E_j = E_i \cdot E_j' = E_i' \cdot E_j = 1$ for $1 \leq i < j \leq 3$ (use Remarks 5.9, 5.11, 5.14).

If $\rho_{\langle E_3\rangle} : \mathbb{P}^{13} \dashrightarrow \mathbb{P}^9$ is the projection of $\mathbb{P}^{13}$ from the three-dimensional linear space $\langle E_3\rangle \cong \mathbb{P}^3$ spanned by $E_3$, then $W^9_{KLM} := \rho_{\langle E_3\rangle}(W^{13}_F) \subset \mathbb{P}^9$ is an Enriques-Fano threefold of genus $p = 9$ (see [36, §13]).

**Remark 7.2.** For the construction of $W^9_{KLM}$, we have fixed a general sextic $\Sigma \in \mathcal{S}$. The hyperplane sections of $W^9_{KLM}$ correspond to the hyperplane sections of $W^{13}_F$ containing $E_3$, which are images via $\nu_{\mathcal{S}}$ of the sextic surfaces $R$ of $\mathcal{S}$ which are tangent to $\Sigma$ along the two branches of $\Sigma$ intersecting at $l_{23}$. Then $W^9_{KLM}$ is the image of $\mathbb{P}^3$ via the rational map defined by the sublinear system $\mathcal{R} \subset \mathcal{S}$ of these sextic surfaces $R$.

## 7.3 Computational analysis of the KLM-EF 3-fold

In the following we will often refer to the use of Macaulay2: see Code B.8 of Appendix B for the computational techniques we will use. Up to a change of coordinates, we can take the tetrahedron $T := \{s_0s_1s_2s_3 = 0\} \subset \mathbb{P}^3_{[s_0:\cdots:s_3]}$ with faces $f_i := \{s_i = 0\}$ and edges $l_{ij} = \{s_i = s_j = 0\}$ for $0 \le i < j \le 3$. Then $\mathcal{S}$ defines the rational map

$$\nu_{\mathcal{S}} : \mathbb{P}^3 \dashrightarrow W^{13}_F \subset \mathbb{P}^{13}, \quad [s_0 : s_1 : s_2 : s_3] \mapsto [w_0 : \cdots : w_{13}],$$

where $w_0 = s_0s_1^3s_2s_3$, $w_1 = s_0^2s_1^2s_2^2$, $w_2 = s_0^2s_1^2s_2s_3$, $w_3 = s_0^2s_1^2s_3^2$, $w_4 = s_0^3s_1s_2s_3$, $w_5 = s_0s_1^2s_2^2s_3$, $w_6 = s_0s_1^2s_2s_3^2$, $w_7 = s_0^2s_1s_2^2s_3$, $w_8 = s_0^2s_1s_2s_3^2$, $w_9 = s_1^2s_2^2s_3^2$, $w_{10} = s_0s_1s_2^3s_3$, $w_{11} = s_0s_1s_2^2s_3^2$, $w_{12} = s_0s_1s_2s_3^3$, $w_{13} = s_0^2s_2^2s_3^2$. By Theorem 6.17 the ideal of $W^{13}_F$ is the one in Remark 6.15. Furthemore $W^{13}_F$ has eight singular points $P_1$, $P_2$, $P_3$, $P_4$, $P'_1$, $P'_2$, $P'_3$, $P'_4$ with coordinates as in Remark 6.16 and configuration as in Figure 26 of Appendix A. Let us take $S = W^{13}_F \cap H$, where $H$ is a general hyperplane in $\mathbb{P}^{13}$ not passing through $P_1$, $P_2$, $P_3$, $P_4$, $P'_1$, $P'_2$, $P'_3$, $P'_4$, and so defined by

$$a_0w_0 + a_1w_1 + a_2w_2 + a_3w_3 + a_4w_4 + a_5w_5 + a_6w_6 +$$

$$+ a_7w_7 + a_8w_8 + a_9w_9 + a_{10}w_{10} + a_{11}w_{11} + a_{12}w_{12} + a_{13}w_{13} = 0,$$

where $a_0, a_1, a_3, a_4, a_9, a_{10}, a_{12}, a_{13} \in \mathbb{C}$ are not equal to zero. Let us consider $a_0 = 1$ and let $\Sigma$ be the corresponding element of $\mathcal{S}$ such that $\nu_{\mathcal{S}}(\Sigma) = S$. The hyperplane sections of $W^{13}_F$ containing $\nu_{\widetilde{\mathcal{S}}}(\mathcal{F}_{23})$ correspond to the divisors on $Y$ linearly equivalent to $6\mathcal{H} - 3\sum_{i=0}^3 \mathcal{E}_i - 3\mathcal{F}_{23} - \sum_{\substack{0\le i<j\le 3\\(i,j)\neq(2,3)}} 2\mathcal{F}_{ij} - \sum_{\substack{i,j=0\\i\neq j}}^3 4\Gamma_{ij}$. Since

$$\left\langle \nu_{\widetilde{\mathcal{S}}}(\mathcal{F}_{23})\right\rangle = \{w_5 = w_6 = w_7 = w_8 = w_9 = w_{10} = w_{11} = w_{12} = w_{13} = 0\} \cong \mathbb{P}^4,$$

then we have $\langle E_3\rangle = H \cap \left\langle \nu_{\widetilde{\mathcal{S}}}(\mathcal{F}_{23})\right\rangle \cong \mathbb{P}^3$ and so $E_3 = S \cap \langle E_3\rangle$, which is defined by the equations

$$\begin{gathered} w_0 + a_1w_1 + a_2w_2 + a_3w_3 + a_4w_4 = 0, \\ w_5 = w_6 = w_7 = w_8 = w_9 = w_{10} = w_{11} = w_{12} = w_{13} = 0, \\ w_1w_3 + a_1w_1w_4 + a_2w_2w_4 + a_3w_3w_4 + a_4w_4^2 = 0, \\ w_2^2 + a_1w_1w_4 + a_2w_2w_4 + a_3w_3w_4 + a_4w_4^2 = 0. \end{gathered}$$

Since $E_3$ is the complete intersection of two quadric surfaces of $\langle E_3 \rangle \cong \mathbb{P}^3$, it is a quartic elliptic curve. By using Macaulay2 and by considering the following projection map

$$\begin{array}{ccc} \mathbb{P}^{13} & & [w_0 : w_1 : w_2 : w_3 : w_4 : w_5 : w_6 : w_7 : w_8 : w_9 : w_{10} : w_{11} : w_{12} : w_{13}] \\ \Big\downarrow \rho_{\langle E_3 \rangle} & & \Big\downarrow \\ \mathbb{P}^9 & & [w_0 + a_1 w_1 + a_2 w_2 + a_3 w_3 + a_4 w_4 : w_5 : w_6 : w_7 : w_8 : w_9 : w_{10} : w_{11} : w_{12} : w_{13}] \end{array}$$

we can compute $W^9_{KLM} = \rho_{\langle E_3 \rangle}(W^{13}_F) \subset \mathbb{P}^9_{[z_0:z_1:z_2:z_3:z_4:z_5:z_6:z_7:z_8:z_9]}$, which has ideal generated by the following 16 polynomials

$$z_6z_8 - z_5z_9, \quad z_3z_8 - z_2z_9, \quad z_7^2 - z_5z_9, \quad z_4z_7 - z_2z_9, \quad z_3z_7 - z_1z_9, \quad z_2z_7 - z_1z_8,$$
$$z_4z_6 - z_1z_9, \quad z_2z_6 - z_1z_7, \quad z_4z_5 - z_1z_8, \quad z_3z_5 - z_1z_7, \quad z_2z_3 - z_1z_4,$$

$$z_1z_2 + a_1z_1z_3 + a_2z_1z_4 + a_3z_2z_4 + a_4z_3z_4 - z_0z_7,$$

$$z_2^2z_9 + a_1z_1z_4z_9 + a_2z_2z_4z_9 + a_3z_4^2z_8 + a_4z_4^2z_9 - z_0z_8z_9,$$

$$z_1^2z_9 + a_1z_3^2z_6 + a_2z_1z_3z_9 + a_3z_1z_4z_9 + a_4z_3^2z_9 - z_0z_6z_9,$$

$$z_2^2z_5 + a_3z_2^2z_8 - a_2a_3z_2z_4z_8 - z_0z_5z_8 + a_2z_0z_7z_8 - a_1^2z_1^2z_9 + (a_4 - a_2^2 - a_1a_3)z_2^2z_9 + \\ +2a_1^2a_2z_1z_3z_9 + a_1(2a_2^2 - a_4)z_1z_4z_9 + (2a_1a_2a_3 - a_2a_4)z_2z_4z_9 + 2a_1a_2a_4z_3z_4z_9 + \\ +a_1z_0z_5z_9 - a_1a_2z_0z_7z_9,$$

$$z_1^2z_5 + a_1z_1^2z_6 - z_0z_5z_6 + a_2z_1^2z_7 + (a_4 - a_1a_2)z_1^2z_9 - a_3^2z_2^2z_9 + a_1a_2a_3z_1z_3z_9 - a_3(a_4 - \\ a_2^2)z_1z_4z_9 + a_2a_3^2z_2z_4z_9 + a_2a_3a_4z_3z_4z_9 + a_3z_0z_5z_9 - a_2a_3z_0z_7z_9.$$

**Remark 7.3.** The ideal of $W^9_{KLM}$ is generated by quadrics and cubics. Since $W^9_{KLM}$ is projectively normal in $\mathbb{P}^9$ (see § 3.3), then the ideal of its general hyperplane section $S_{KLM} \subset \mathbb{P}^8$ is generated by quadrics and cubics too. It is consistent with the fact that the $\phi$ of a general hyperplane section of $S_{KLM}$ is $3 < 4$ (see [35, Theorem 1.1 (ii)]), as we will see in the proof of Theorem 9.2.

Let us take the images of the eight quadruple points of $W^{13}_F$, by denoting them, by abuse of notation, in the following way

$$P_1 := \rho_{\langle E_3 \rangle}(P_1') = [0:0:0:0:0:1:0:0:0:0]\,,$$

$$P_2 := \rho_{\langle E_3 \rangle}(P_2') = [0:0:0:0:0:0:0:0:0:1]\,,$$
$$P_3 := \rho_{\langle E_3 \rangle}(P_3) = [0:0:0:0:0:0:1:0:0:0]\,,$$
$$P_4 := \rho_{\langle E_3 \rangle}(P_4) = [0:0:0:0:0:0:0:0:1:0]\,,$$
$$P_5 := \rho_{\langle E_3 \rangle}(P_1) = \rho_{\langle E_3 \rangle}(P_2) = \rho_{\langle E_3 \rangle}(P_3') = \rho_{\langle E_3 \rangle}(P_4') = [1:0:0:0:0:0:0:0:0:0]\,.$$

**Proposition 7.4.** If $i = 1,2,3,4$, the tangent cone $TC_{P_i}W^9_{KLM}$ to $W^9_{KLM}$ at the point $P_i$ is a cone over a Veronese surface.

*Proof.* Each point $P_i$, $i = 1,2,3,4$, can be viewed as the origin of the open affine set $U_{j(i)} = \{z_{j(i)} \neq 0\}$, where $j(1) = 5$, $j(2) = 9$, $j(3) = 6$, $j(4) = 8$. The ideal of the tangent cone $TC_{P_i}(W^9_{KLM} \cap U_{j(i)})$ is generated by the minimal degree homogeneous parts of all the polynomials in the ideal of $W^9_{KLM} \cap U_{j(i)}$. Thanks to Macaulay2, we obtain the following tangent cones.

$TC_{P_1}(W^9_{KLM} \cap U_5)$ has ideal generated by
$z_9,\ z_4,\ z_3,\ z_7^2 - z_6z_8,\ z_2z_7 - z_1z_8,\ z_2z_6 - z_1z_7,\ z_2^2 - z_0z_8,\ z_1z_2 - z_0z_7,\ z_1^2 - z_0z_6$
Hence $TC_{P_1}W^9_{KLM}$ is a cone with vertex $P_1$ over a Veronese surface in the $\mathbb{P}^5$ given by $\{z_i = 0 | i = 3,4,5,9\}$.

$TC_{P_2}(W^9_{KLM} \cap U_9)$ has ideal generated by
$z_5,\ z_2,\ z_1,\ z_7^2 - z_6z_8,\ z_4z_7 - z_3z_8,\ z_4z_6 - z_3z_7,\ a_4z_4^2 - z_0z_8,\ a_4z_3z_4 - z_0z_7,\ a_4z_3^2 - z_0z_6.$
Hence $TC_{P_2}W^9_{KLM}$ is a cone with vertex $P_2$ over a Veronese surface in the $\mathbb{P}^5$ given by $\{z_i = 0 | i = 1,2,5,9\}$.

$TC_{P_3}(W^9_{KLM} \cap U_6)$ has ideal generated by
$z_8,\ z_4,\ z_2,\ z_7^2 - z_5z_9,\ z_3z_7 - z_1z_9,\ z_3z_5 - z_1z_7,\ a_1z_3^2 - z_0z_9,\ a_1z_1z_3 - z_0z_7,\ a_1z_1^2 - z_0z_5.$
Hence $TC_{P_3}W^9_{KLM}$ is a cone with vertex $P_3$ over a Veronese surface in the $\mathbb{P}^5$ given by $\{z_i = 0 | i = 2,4,6,8\}$.

$TC_{P_4}(W^9_{KLM} \cap U_8)$ has ideal generated by
$z_6,\ z_3,\ z_1,\ z_7^2 - z_5z_9,\ z_4z_7 - z_2z_9,\ z_4z_5 - z_2z_7,\ a_3z_4^2 - z_0z_9,\ a_3z_2z_4 - z_0z_7,\ a_3z_2^2 - z_0z_5.$
Hence $TC_{P_4}W^9_{KLM}$ is a cone with vertex $P_4$ over a Veronese surface in the $\mathbb{P}^5$ given by $\{z_i = 0 | i = 1,3,6,8\}$. $\square$

**Theorem 7.5.** The tangent cone $TC_{P_5}W^9_{KLM}$ to $W^9_{KLM}$ at the point $P_5$ is a cone over a reducible sextic surface $M_6 \subset \mathbb{P}^7 \subset \mathbb{P}^9$, which is given by the union of four planes $\pi_1$, $\pi_2$, $\pi_1'$, $\pi_2'$ and a quadric surface $Q \subset \mathbb{P}^3 \subset \mathbb{P}^7$. In particular each one of the planes $\pi_1$, $\pi_2$, $\pi_1'$, $\pi_2'$ intersects the quadric surface $Q$ respectively along a line $l_1$, $l_2$, $l_1'$, $l_2'$, where $l_1$ is disjoint from $l_1'$, and $l_2$ is disjoint from $l_2'$. In the other cases the intersections of two of these lines identify four points on $Q$, which are $q_{1,2} := l_1 \cap l_2$, $q_{1,2'} := l_1 \cap l_2'$, $q_{1',2} := l_1' \cap l_2$, $q_{1',2'} := l_1' \cap l_2'$.

*Proof.* The point $P_5$ can be viewed as the origin of the open affine set $U_0 = \{z_0 \neq 0\}$. The ideal of the tangent cone $TC_{P_5}(W^9_{KLM} \cap U_0)$ is generated by the minimal degree homogeneous parts of all the polynomials in the ideal of $W^9_{KLM} \cap U_0$. By using Macaulay2 one can find that $TC_{P_5}(W^9_{KLM} \cap U_0)$ has ideal generated by the following polynomials

$$z_7,\ z_8z_9,\ z_6z_9,\ z_5z_9,\ z_2z_9,\ z_1z_9,\ z_6z_8,\ z_5z_8,\ z_3z_8,\ z_1z_8,\ z_5z_6,\ z_4z_6,\ z_2z_6,\ z_4z_5,\ z_3z_5,$$

$$z_2z_3 - z_1z_4.$$

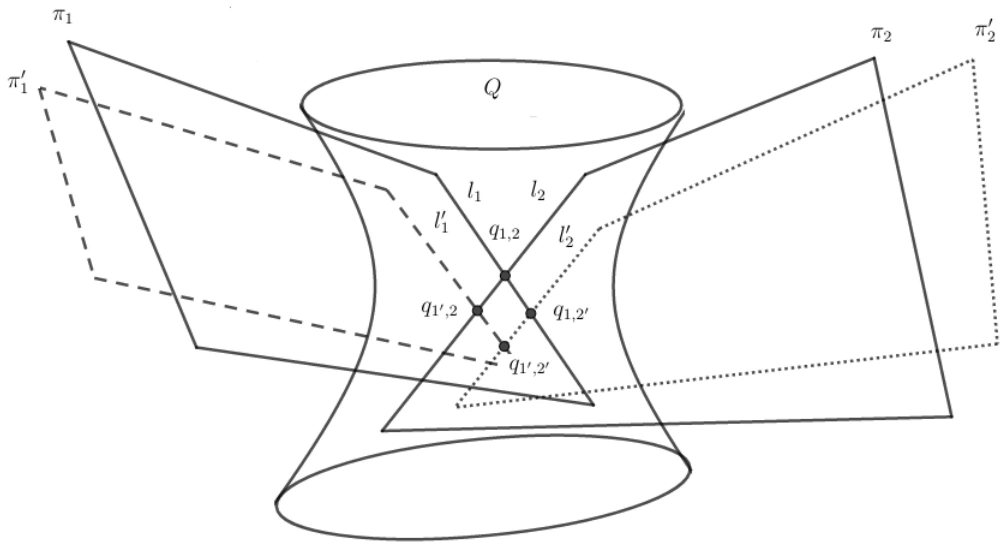


*Figure 13: The reducible sextic surface $M_6 \subset \mathbb{P}^7$ given by the union of four planes $\pi_1$, $\pi_2$, $\pi_1'$, $\pi_2'$ and a quadric surface $Q \subset \mathbb{P}^3 \subset \mathbb{P}^7$, which intersect as in the statement of Theorem 7.5.*

Hence $TC_{P_5}W^9_{KLM}$ is a cone with vertex $P_5$ over a surface $M_6$ contained in the $\mathbb{P}^7$ given by $\{z_i = 0 | i = 0, 7\}$. The surface $M_6$ is the union of four planes $\pi_1, \pi_2, \pi_1', \pi_2'$ and a quadric surface $Q$, where

$$\pi_1 := \{z_i = 0 | i = 0,1,2,5,6,7,8\}, \quad \pi_2 := \{z_i = 0 | i = 0,1,3,5,6,7,9\},$$
$$\pi_1' := \{z_i = 0 | i = 0,3,4,6,7,8,9\}, \quad \pi_2' := \{z_i = 0 | i = 0,2,4,5,7,8,9\},$$
$$Q := \{z_i = 0 | i = 0,5,6,7,8,9\} \cap \{z_2z_3 - z_1z_4 = 0\}.$$

We give an idea of $M_6$ in Figure 13. □

**Remark 7.6.** The point $P_5$ is a canonical non-terminal singularity of $W^9_{KLM}$ (see Remark 8.15). This is consistent with [36, Proposition 12.1(b)].

**Remark 7.7.** Since $P_1$, $P_2$, $P_3$, $P_4$, $P_5$ are singular points of $W^9_{KLM}$, let us see their configuration. Let $l_{i,j}$ be the line joining the singular points $P_i$ and $P_j$ for $1 \leq i < j \leq 5$. Then we have $l_{1,2} = \{z_i = 0 | i \neq 5, 9\}$, $l_{1,3} = \{z_i = 0 | i \neq 5, 6\}$, $l_{1,4} = \{z_i = 0 | i \neq 5, 8\}$, $l_{1,5} = \{z_i = 0 | i \neq 0, 5\}$, $l_{2,3} = \{z_i = 0 | i \neq 6, 9\}$, $l_{2,4} = \{z_i = 0 | i \neq 8, 9\}$, $l_{2,5} = \{z_i = 0 | i \neq 0, 9\}$, $l_{3,4} = \{z_i = 0 | i \neq 6, 8\}$, $l_{3,5} = \{z_i = 0 | i \neq 0, 6\}$, $l_{4,5} = \{z_i = 0 | i \neq 0, 8\}$. The lines $l_{1,3}$, $l_{1,4}$, $l_{1,5}$, $l_{2,3}$, $l_{2,4}$, $l_{2,5}$, $l_{3,5}$, $l_{4,5}$ are contained in $W^9_{KLM}$, while $l_{1,4}$ and $l_{2,3}$ are not. So the five singular points $P_1$, $P_2$, $P_3$, $P_4$, $P_5$ of $W^9_{KLM}$ are associated as in Figure 28 of Appendix A. Furthermore in Figure 14 we can see how the projection $\rho_{\langle E_3 \rangle}$ changes the configuration of the singularities of $W^{13}_F$ in the configuration of the singularities of $W^9_{KLM}$.

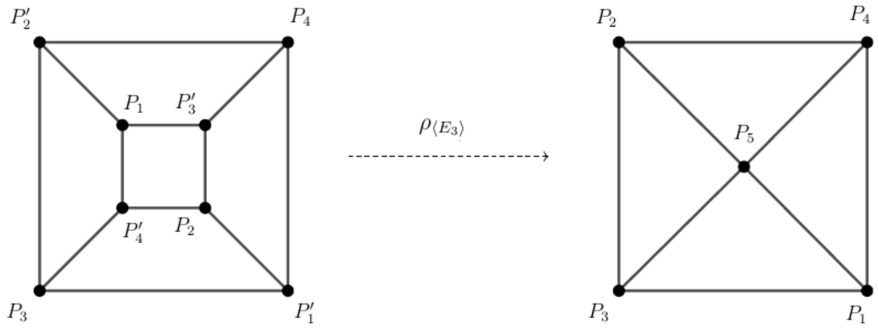


*Figure 14: Comparison between the configurations of the singularities of $W_F^{13}$ and $W_{KLM}^9$.*

# 8 Singularities of the P-EF 3-folds

## 8.1 Abstract

We recall that the P-EF 3-fold $W_P^{17}$ is an Enriques-Fano threefold given by the quotient $\pi : V \to V/\tau =: W$ of a singular Fano threefold $V$ under an involution $\tau : V \to V$ with five fixed points (see [46, Proposition 3.2]). Similarly the P-EF 3-fold $W_P^{13}$, which was mentioned very briefly by Prokhorov in [46, Remark 3.3] and which we will study in more detail. We will computationally analyze both the P-EF 3-folds, by finding the following facts:

(i) the P-EF 3-fold $W_P^{13}$ can be embedded in $\mathbb{P}^{13}$ and its ideal is generated by quadrics; the threefold $W_P^{13} \subset \mathbb{P}^{13}$ has five non-similar singular points such that four of them are quadruple points, whose tangent cone is a cone over a Veronese surface, and the last one is a quintuple point, whose tangent cone is a cone over the union of five planes (see § 8.2);

(ii) the P-EF 3-fold $W_P^{17}$ can be embedded in $\mathbb{P}^{17}$ and its ideal is generated by quadrics; the threefold $W_P^{17} \subset \mathbb{P}^{13}$ has five non-similar singular points such that four of them are quadruple points, whose tangent cone is a cone over a Veronese surface, and the last one is a sextuple point, whose tangent cone is a cone over the union of four planes and a quadric surface (see § 8.3).

## 8.2 P-EF 3-fold (XVI) of genus 13

In the following we will often refer to the use of Macaulay2: see Code B.9 of Appendix B for the computational techniques we will use. Let us consider the linear system of the plane cubic curves passing through three fixed points $a_1$, $a_2$, $a_3$ in general position. Up to a change of coordinates, we may assume $a_1 = [1 : 0 : 0]$, $a_2 = [0 : 1 : 0]$ and $a_3 = [0 : 0 : 1]$ in $\mathbb{P}^2_{[u_0:u_1:u_2]}$. The above linear system so defines the rational map $\lambda : \mathbb{P}^2 \dashrightarrow \mathbb{P}^6$ given by $[u_0 : u_1 : u_2] \mapsto [u_1^2u_2 : u_1u_2^2 : u_0^2u_2 : u_0u_2^2 : u_0^2u_1 : u_0u_1^2 : u_0u_1u_2]$, whose image is a smooth sextic Del Pezzo surface $S_6 \subset \mathbb{P}^6$. If $bl : \mathrm{Bl}_{a_1,a_2,a_3}\mathbb{P}^2 \to \mathbb{P}^2$ denotes the blow-up of the plane at the three fixed points, then we have that $S_6$ is isomorphic to $\mathrm{Bl}_{a_1,a_2,a_3}\mathbb{P}^2$ and that it is anticanonically embedded in $\mathbb{P}^6$. Let $\ell$ be the pullback of the line class on $\mathbb{P}^2$ and let $e_i := bl^{-1}(a_i)$ be the exceptional divisors, for $1 \leq i \leq 3$; then we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Bl}_{a_1,a_2,a_3}\mathbb{P}^2 & & \\ \downarrow bl & \overset{\widetilde{\lambda}_{|3\ell-e_1-e_2-e_3|}=\widetilde{\lambda}_{|-K_{S_6}|}}{\searrow \cong} & \\ \mathbb{P}^2 & \overset{\lambda}{\dashrightarrow} & S_6 \subset \mathbb{P}^6. \end{array}$$

Let us consider $\mathbb{P}^6_{[x_0:x_1:x_2:x_3:x_4:x_5:x_6]}$ as the hyperplane $\{y_0 = 0\} \subset \mathbb{P}^7_{[x_0:x_1:x_2:x_3:x_4:x_5:x_6:y]}$ and let us take the cone $V$ over $S_6$ with vertex $v := [0 : 0 : 0 : 0 : 0 : 0 : 0 : 1]$.

**Remark 8.1.** Since the ideal of $S_6$ is generated by the following polynomials

$$x_3x_5 - x_6^2, \quad x_2x_5 - x_4x_6, \quad x_1x_5 - x_0x_6, \quad x_3x_4 - x_2x_6, \quad x_1x_4 - x_6^2,$$

$$x_0x_4 - x_5x_6, \quad x_0x_3 - x_1x_6, \quad x_1x_2 - x_3x_6, \quad x_0x_2 - x_6^2,$$

in $\mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5, x_6]$, then the ideal of $V$ is generated by the same polynomials as polynomials in $\mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5, x_6, y]$.

**Lemma 8.2.** The variety $V$ is a Gorenstein Fano threefold with canonical singularities. Moreover, $-K_V = 2M$ where $M$ is the class of the hyperplane sections.

*Proof.* Since $S_6 \subset \mathbb{P}^6$ is projectively normal (see [17, Theorem 8.3.4]), then $V$ is normal. Let $\sigma : \mathrm{Bl}_v V \to V$ be the blow-up of $v$ with exceptional divisor $E = \sigma^{-1}(v)$. Then $\mathrm{Bl}_v V$ is a $\mathbb{P}^1$-bundle over $S_6$ and $\sigma$ contracts its negative section $E$ to $v$. In particular we have $\mathrm{Bl}_v V = \mathbb{P}(\mathcal{O}_{S_6} \oplus \mathcal{O}_{S_6}(-K_{S_6}))$ (see [29, V, Ex. 2.11.4]). Since the map $\sigma : \mathrm{Bl}_v V \to V \subset \mathbb{P}^7$ is given by the tautological linear system $|\mathcal{O}_{\mathrm{Bl}_v V}(1)|$, then $\mathcal{O}_{\mathrm{Bl}_v V}(1) \sim \sigma^* M$. A priori we have that $K_{\mathrm{Bl}_v V} = \sigma^* K_V + aE$ for $a \in \mathbb{Q}$. Since $K_{\mathrm{Bl}_v V} \sim \mathcal{O}_{\mathrm{Bl}_v V}(-2)$ (see [47, p. 349 (d)]) and $K_{\mathrm{Bl}_v V} \cdot E = -2(\sigma^* M) \cdot E = 0$, then $a = 0$ and $K_V$ is a Cartier divisor. Thus $V$ has a canonical singularity at the vertex $v$. Finally, since $\sigma^*(-2M) \sim \mathcal{O}_{\mathrm{Bl}_v V}(-2) \sim K_{\mathrm{Bl}_v V} = \sigma^* K_V$, we have that $K_V = -2M$. $\square$

The quadratic transformation $q_{a_1,a_2,a_3} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, given by the linear system of the conics passing through $a_1$, $a_2$ and $a_3$, defines an involution of the sextic Del Pezzo $S_6 \subset \mathbb{P}^6$. Indeed we have

$$\begin{array}{ccc} [u_0 : u_1 : u_2] & \overset{q}{\longmapsto} & \left[\frac{1}{u_0} : \frac{1}{u_1} : \frac{1}{u_2}\right] \\ \downarrow \lambda & & \downarrow \lambda \\ [u_1^2u_2 : u_1u_2^2 : u_0^2u_2 : u_0u_2^2 : u_0^2u_1 : u_0u_1^2 : u_0u_1u_2] & \longmapsto & [u_0^2u_2 : u_1u_0^2 : u_1^2u_2 : u_0u_1^2 : u_1u_2^2 : u_0u_2^2 : u_0u_1u_2] \end{array}$$

and then we obtain the involution $t'$ of $S_6 \subset \mathbb{P}^6$ given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \overset{t'}{\longmapsto} [x_2 : x_4 : x_0 : x_5 : x_1 : x_3 : x_6].$$

Let us take the involution of $\mathbb{P}^7$ defined by $t : \mathbb{P}^7 \to \mathbb{P}^7$ such that

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : y] \longmapsto [x_2 : x_4 : x_0 : x_5 : x_1 : x_3 : x_6 : -y].$$

The locus of $t$-fixed points in $\mathbb{P}^7$ consists of two projective subspaces

$$F_1 = \{x_0 + x_2 = x_1 + x_4 = x_3 + x_5 = x_6 = 0\} \cong \mathbb{P}^3,$$

$$F_2 = \{x_0 - x_2 = x_1 - x_4 = x_3 - x_5 = y = 0\} \cong \mathbb{P}^3.$$

In particular we have that $F_1 \cap V = \{v\}$ and $F_2 \cap V = \{v_1, v_2, v_3, v_4\}$, where

$$v_1 := [1 : 1 : 1 : 1 : 1 : 1 : 1 : 0], \quad v_2 := [1 : -1 : 1 : -1 : -1 : -1 : 1 : 0],$$

$$v_3 := [-1 : 1 : -1 : -1 : 1 : -1 : 1 : 0], \quad v_4 := [-1 : -1 : -1 : 1 : -1 : 1 : 1 : 0].$$

Thus $t$ induces an involution $\tau := t|_V$ of $V$ with five fixed points.

**Theorem 8.3.** The quotient of $V$ by the involution $\tau$ is an Enriques-Fano threefold of genus $p = 13$, which we will denote by $W_P^{13}$.

*Proof.* Let $\mathcal{Q}_V$ be the linear system that is cut out on $V$ by the linear system $\mathcal{Q}$ of the quadric hypersurfaces of $\mathbb{P}^7$ of type

$$q_1(x_0 + x_2, x_1 + x_4, x_3 + x_5, x_6) + q_2(x_0 - x_2, x_1 - x_4, x_3 - x_5, y) = 0,$$

where $q_1$ and $q_2$ are quadratic homogeneus forms. By construction, we have that $\mathcal{Q}_V$ is base point free and each member of $\mathcal{Q}_V$ is $\tau$-invariant. In particular a general member $\widetilde{S} \in \mathcal{Q}_V$ is smooth and does not contains any of $v, v_1, v_2, v_3, v_4$. Then the action of $\tau$ on $\widetilde{S}$ is fixed point free. Moreover $\widetilde{S}$ is a K3 surface, since $\mathcal{Q}_V \subset |2M| = |-K_V|$. Let $\pi : V \to W_P^{13} := V/\tau$ be the quotient morphism and let $S := \pi(\widetilde{S}) = \widetilde{S}/\tau$. Then $S$ is a smooth Enriques surface. Since $\widetilde{S} = \pi^*S$, we have $2p - 2 = S^3 = \frac{1}{2}\widetilde{S}^3 = \frac{1}{2}(2M)^3 = 4 \cdot \deg V = 24$, whence $p = 13$. Furthermore $W_P^{13}$ is normal, since it is the quotient of the normal threefold $V$ under the action of a finite group defined by the involution $\tau$ (see [19, Proposition 2.15]). Thus, by setting $\mathcal{L} := |\mathcal{O}_{W_P^{13}}(S)|$, we have that $(W_P^{13}, \mathcal{L})$ is an Enriques-Fano threefold of genus 13. □

The linear system $\mathcal{Q}$, introduced in the proof of Theorem 8.3, defines a morphism $\varphi : \mathbb{P}^7 \to \mathbb{P}^{19}$ such that $[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : y] \mapsto [Z_0 : Z_1 : \cdots : Z_{18} : Z_{19}]$ where $Z_0 = x_6^2$, $Z_1 = x_0^2 + x_2^2$, $Z_2 = x_1^2 + x_4^2$, $Z_3 = x_3^2 + x_5^2$, $Z_4 = (x_0 + x_2)x_6$, $Z_5 = (x_1+x_4)x_6$, $Z_6 = (x_3+x_5)x_6$, $Z_7 = x_0x_1+x_2x_4$, $Z_8 = x_2x_3+x_0x_5$, $Z_9 = x_1x_3+x_4x_5$, $Z_{10} = (x_0 - x_2)y$, $Z_{11} = (x_1 - x_4)y$, $Z_{12} = (x_3 - x_5)y$, $Z_{13} = y^2$, $Z_{14} = 2x_0x_2$, $Z_{15} = 2x_1x_4$, $Z_{16} = 2x_3x_5$, $Z_{17} = x_4x_3 + x_1x_5$, $Z_{18} = x_0x_3 + x_2x_5$, $Z_{19} = x_1x_2 + x_0x_4$. Hence we have $\pi = \varphi|_V : V \to W_P^{13} \subset \mathbb{P}^{19}$. Furthermore the threefold $W_P^{13}$ is contained in a 13-dimensional projective subspace of $\mathbb{P}^{19}$ given by

$$H_{13} := \{Z_{14} = 2Z_0,\ Z_{15} = 2Z_0,\ Z_{16} = 2Z_0,\ Z_{17} = Z_4,\ Z_{18} = Z_5,\ Z_{19} = Z_6\}$$

(Remark 8.1). Thus the quotient map $\pi : V \to W_P^{13} \subset H_{13} \cong \mathbb{P}^{13}$ is given by

$$[x_0 : x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : y] \mapsto [z_0 : z_1 : \cdots : z_{12} : z_{13}]$$

where $z_0 = x_6^2$, $z_1 = x_0^2+x_2^2$, $z_2 = x_1^2+x_4^2$, $z_3 = x_3^2+x_5^2$, $z_4 = (x_0+x_2)x_6$, $z_5 = (x_1+x_4)x_6$, $z_6 = (x_3+x_5)x_6$, $z_7 = x_0x_1+x_2x_4$, $z_8 = x_2x_3+x_0x_5$, $z_9 = x_1x_3+x_4x_5$, $z_{10} = (x_0-x_2)y$, $z_{11} = (x_1 - x_4)y$, $z_{12} = (x_3 - x_5)y$, $z_{13} = y^2$.

**Remark 8.4.** By using Macaulay2 we find that the P-EF 3-fold $W_P^{13}$ has ideal generated by the following 42 polynomials

$$\begin{gathered}
z_4z_5-2z_0z_6-z_2z_6+z_5z_9, \quad z_5^2-z_6^2-z_6z_7+z_5z_8, \quad 2z_0z_5+z_3z_5-z_4z_6-z_6z_9,\\
z_4^2-z_6^2-z_6z_7+z_4z_9, \quad z_4z_5-2z_0z_6-z_1z_6+z_4z_8, \quad -2z_0z_5-z_1z_5+z_4z_6+z_4z_7,\\
2z_0z_4+z_3z_4-z_5z_6-z_6z_8, \quad 2z_0z_4+z_2z_4-z_5z_6-z_5z_7, \quad 4z_0^2-z_4^2-z_5^2+z_6z_7,\\
z_5z_{10}-z_4z_{11}+2z_0z_{12}, \quad -z_6z_{10}+2z_0z_{11}-z_4z_{12}, \quad 2z_0z_{10}-z_6z_{11}+z_5z_{12},\\
2z_0z_4-2z_5z_6+2z_0z_9, \quad 2z_0z_5-2z_4z_6+2z_0z_8, \quad -2z_4z_5+2z_0z_6+2z_0z_7,\\
2z_0z_3+z_4^2+z_5^2-2z_6^2-z_6z_7, \quad 2z_0z_2+z_4^2-z_5^2-z_6z_7, \quad 2z_0z_1-z_4^2+z_5^2-z_6z_7,\\
z_{12}^2+2z_0z_{13}-z_3z_{13}, \quad z_{11}z_{12}+z_4z_{13}-z_9z_{13}, \quad z_{10}z_{12}-z_5z_{13}+z_8z_{13},\\
z_4z_{10}-z_5z_{11}+z_7z_{12}, \quad z_{11}^2+2z_0z_{13}-z_2z_{13}, \quad z_{10}z_{11}+z_6z_{13}-z_7z_{13},\\
-z_5z_{10}+z_9z_{11}-z_2z_{12}, \quad -z_4z_{10}+z_8z_{11}-z_6z_{12}, \quad -z_6z_{10}+z_3z_{11}-z_9z_{12},\\
z_{10}^2+2z_0z_{13}-z_1z_{13}, \quad z_9z_{10}-z_5z_{11}+z_6z_{12}, \quad z_8z_{10}-z_4z_{11}+z_1z_{12},\\
z_7z_{10}-z_1z_{11}+z_4z_{12}, \quad z_3z_{10}-z_6z_{11}+z_8z_{12}, \quad z_2z_{10}-z_7z_{11}+z_5z_{12},\\
-z_4z_5+2z_0z_6-z_3z_6+z_8z_9, \quad 2z_0z_5-z_2z_5-z_4z_6+z_7z_9, \quad 2z_0z_4-z_5z_7-z_6z_8+z_1z_9,\\
2z_0z_4-z_1z_4-z_5z_6+z_7z_8, \quad -z_1z_5+z_4z_6+z_2z_8-z_6z_9,\\
2z_4z_5-2z_0z_6-z_1z_6-z_2z_6+z_3z_7, \quad z_2z_3+z_5^2-z_6z_7-z_9^2,\\
z_1z_3+z_4^2-z_6z_7-z_8^2, \quad z_1z_2+z_4^2+z_5^2-z_6^2-z_6z_7-z_7^2.
\end{gathered}$$

Then the ideal of $W_P^{13} \subset \mathbb{P}^{13}$ is generated by quadrics. Since $W_P^{13}$ is projectively normal in $\mathbb{P}^{13}$ (see § 3.3), then the ideal of its general hyperplane section $S \subset \mathbb{P}^{12}$ is generated by quadrics too. It is consistent with the fact that the $\phi$ of a general hyperplane section of $S$ is 4 (see [35, Theorem 1.1 (ii)]), as we will prove in the proof of Theorem 9.2.

**Remark 8.5.** The P-EF 3-fold $W_P^{13}$ has the following five singular points

$$\begin{gathered}
P_1 = \pi(v_1) = [1:2:2:2:2:2:2:2:2:2:0:0:0:0]\,,\\
P_2 = \pi(v_2) = [1:2:2:2:2:-2:-2:-2:-2:2:0:0:0:0]\,,\\
P_3 = \pi(v_3) = [1:2:2:2:-2:2:-2:-2:2:-2:0:0:0:0]\,,\\
P_4 = \pi(v_4) = [1:2:2:2:-2:-2:2:2:-2:-2:0:0:0:0]\,,\\
P_5 = \pi(v) = [0:0:0:0:0:0:0:0:0:0:0:0:0:1]\,.
\end{gathered}$$

Let $l_{i,j}$ be the line joining the singular points $P_i$ and $P_j$ for $1 \le i < j \le 5$. Then

$$\begin{gathered}
l_{1,2} = \{2z_0=z_1=z_2=z_3=z_4=z_9,\, z_5=z_6=z_7=z_8,\, z_{10}=z_{11}=z_{12}=z_{13}=0\},\\
l_{1,3} = \{2z_0=z_1=z_2=z_3=z_5=z_8,\, z_4=z_6=z_7=z_9,\, z_{10}=z_{11}=z_{12}=z_{13}=0\},\\
l_{1,4} = \{2z_0=z_1=z_2=z_3=z_6=z_7,\, z_4=z_5=z_8=z_9,\, z_{10}=z_{11}=z_{12}=z_{13}=0\},\\
l_{1,5} = \{2z_0=z_1=z_2=z_3=z_4=z_5=z_6=z_7=z_8=z_9,\, z_{10}=z_{11}=z_{12}=0\},\\
l_{2,3} = \{2z_0=z_1=z_2=z_3=-z_6=-z_7,\, z_4=-z_5=-z_8=z_9, z_{10}=z_{11}=z_{12}=z_{13}=0\},\\
l_{2,4} = \{2z_0=z_1=z_2=z_3=-z_5=-z_8,\, z_4=-z_6=-z_7=z_9, z_{10}=z_{11}=z_{12}=z_{13}=0\},\\
l_{2,5} = \{2z_0=z_1=z_2=z_3=z_4=-z_5=-z_6=-z_7=-z_8=z_9, z_{10}=z_{11}=z_{12}=0\},\\
l_{3,4} = \{2z_0=z_1=z_2=z_3=-z_4=-z_9,\, -z_5=-z_6=-z_7=z_8, z_{10}=z_{11}=z_{12}=z_{13}=0\},\\
l_{3,5} = \{-2z_0=-z_1=-z_2=-z_3=z_4=-z_5=z_6=z_7=-z_8=z_9, z_{10}=z_{11}=z_{12}=0\},\\
l_{4,5} = \{-2z_0=-z_1=-z_2=-z_3=z_4=z_5=-z_6=-z_7=z_8=z_9, z_{10}=z_{11}=z_{12}=0\}.
\end{gathered}$$

By Remark 8.4 we have that the lines $l_{1,5}$, $l_{2,5}$, $l_{3,5}$, $l_{4,5}$ are contained in $W_P^{13}$, while the lines $l_{1,2}$, $l_{1,3}$, $l_{1,4}$, $l_{2,3}$, $l_{2,4}$, $l_{3,4}$ are not. Hence the five singular points $P_1$, $P_2$, $P_3$, $P_4$, $P_5$ of $W_P^{13}$ are associated as in Figure 27 of Appendix A.

**Proposition 8.6.** If $i = 1, 2, 3, 4$, the tangent cone $TC_{P_i}W_P^{13}$ to $W_P^{13}$ at the point $P_i$ is a cone over a Veronese surface.

*Proof.* Let us consider the following change of coordinates of $\mathbb{P}^{13}$

$$z_0 = w_0, \quad z_i = w_i + 2w_0, \quad z_j = w_j, \quad i = 1, \ldots, 9,\ j = 10, \ldots, 13.$$

With respect to the new system of coordinates $[w_0 : \cdots : w_{13}]$ of $\mathbb{P}^{13}$, the point $P_1$ has coordinates $[1:0:0:0:0:0:0:0:0:0:0:0:0:0]$ and, by Remark 8.4, the Enriques-Fano threefold $W_P^{13}$ has ideal generated by

$$\begin{gathered}
-2w_0w_2 + 2w_0w_4 + 4w_0w_5 + w_4w_5 - 4w_0w_6 - w_2w_6 + 2w_0w_9 + w_5w_9,\\
6w_0w_5 + w_5^2 - 6w_0w_6 - w_6^2 - 2w_0w_7 - w_6w_7 + 2w_0w_8 + w_5w_8,\\
2w_0w_3 - 2w_0w_4 + 4w_0w_5 + w_3w_5 - 4w_0w_6 - w_4w_6 - 2w_0w_9 - w_6w_9,\\
6w_0w_4 + w_4^2 - 6w_0w_6 - w_6^2 - 2w_0w_7 - w_6w_7 + 2w_0w_9 + w_4w_9,\\
-2w_0w_1 + 4w_0w_4 + 2w_0w_5 + w_4w_5 - 4w_0w_6 - w_1w_6 + 2w_0w_8 + w_4w_8,\\
-2w_0w_1 + 4w_0w_4 - 4w_0w_5 - w_1w_5 + 2w_0w_6 + w_4w_6 + 2w_0w_7 + w_4w_7,\\
2w_0w_3 + 4w_0w_4 + w_3w_4 - 2w_0w_5 - 4w_0w_6 - w_5w_6 - 2w_0w_8 - w_6w_8,\\
2w_0w_2 + 4w_0w_4 + w_2w_4 - 4w_0w_5 - 2w_0w_6 - w_5w_6 - 2w_0w_7 - w_5w_7,\\
-4w_0w_4 - w_4^2 - 4w_0w_5 - w_5^2 + 2w_0w_6 + 2w_0w_7 + w_6w_7,\\
2w_0w_{10} + w_5w_{10} - 2w_0w_{11} - w_4w_{11} + 2w_0w_{12},\\
-2w_0w_{10} - w_6w_{10} + 2w_0w_{11} - 2w_0w_{12} - w_4w_{12},\\
2w_0w_{10} - 2w_0w_{11} - w_6w_{11} + 2w_0w_{12} + w_5w_{12},\\
2w_0w_4 - 4w_0w_5 - 4w_0w_6 - 2w_5w_6 + 2w_0w_9,\\
-4w_0w_4 + 2w_0w_5 - 4w_0w_6 - 2w_4w_6 + 2w_0w_8,\\
-4w_0w_4 - 4w_0w_5 - 2w_4w_5 + 2w_0w_6 + 2w_0w_7,\\
2w_0w_3 + 4w_0w_4 + w_4^2 + 4w_0w_5 + w_5^2 - 10w_0w_6 - 2w_6^2 - 2w_0w_7 - w_6w_7,\\
2w_0w_2 + 4w_0w_4 + w_4^2 - 4w_0w_5 - w_5^2 - 2w_0w_6 - 2w_0w_7 - w_6w_7,\\
2w_0w_1 - 4w_0w_4 - w_4^2 + 4w_0w_5 + w_5^2 - 2w_0w_6 - 2w_0w_7 - w_6w_7,\\
w_{12}^2 - w_3w_{13}, \quad w_{11}w_{12} + w_4w_{13} - w_9w_{13}, \quad w_{10}w_{12} - w_5w_{13} + w_8w_{13},\\
2w_0w_{10} + w_4w_{10} - 2w_0w_{11} - w_5w_{11} + 2w_0w_{12} + w_7w_{12},\\
w_{11}^2 - w_2w_{13}, \quad w_{10}w_{11} + w_6w_{13} - w_7w_{13},\\
-2w_0w_{10} - w_5w_{10} + 2w_0w_{11} + w_9w_{11} - 2w_0w_{12} - w_2w_{12},\\
-2w_0w_{10} - w_4w_{10} + 2w_0w_{11} + w_8w_{11} - 2w_0w_{12} - w_6w_{12},\\
-2w_0w_{10} - w_6w_{10} + 2w_0w_{11} + w_3w_{11} - 2w_0w_{12} - w_9w_{12},\\
w_{10}^2 - w_1w_{13},\\
2w_0w_{10} + w_9w_{10} - 2w_0w_{11} - w_5w_{11} + 2w_0w_{12} + w_6w_{12},\\
2w_0w_{10} + w_8w_{10} - 2w_0w_{11} - w_4w_{11} + 2w_0w_{12} + w_1w_{12},\\
2w_0w_{10} + w_7w_{10} - 2w_0w_{11} - w_1w_{11} + 2w_0w_{12} + w_4w_{12},\\
2w_0w_{10} + w_3w_{10} - 2w_0w_{11} - w_6w_{11} + 2w_0w_{12} + w_8w_{12},\\
2w_0w_{10} + w_2w_{10} - 2w_0w_{11} - w_7w_{11} + 2w_0w_{12} + w_5w_{12},\\
-2w_0w_3 - 2w_0w_4 - 2w_0w_5 - w_4w_5 - w_3w_6 + 2w_0w_8 + 2w_0w_9 + w_8w_9,\\
-2w_0w_2 - 2w_0w_4 - w_2w_5 - 2w_0w_6 - w_4w_6 + 2w_0w_7 + 2w_0w_9 + w_7w_9,\\
2w_0w_1 + 2w_0w_4 - 2w_0w_5 - 2w_0w_6 - 2w_0w_7 - w_5w_7 - 2w_0w_8 - w_6w_8 + 2w_0w_9 + w_1w_9,\\
-2w_0w_1 - w_1w_4 - 2w_0w_5 - 2w_0w_6 - w_5w_6 + 2w_0w_7 + 2w_0w_8 + w_7w_8,\\
-2w_0w_1 + 2w_0w_2 + 2w_0w_4 - 2w_0w_5 - w_1w_5 + w_4w_6 + 2w_0w_8 + w_2w_8 - 2w_0w_9 - w_6w_9,
\end{gathered}$$

$$-2w_0w_1 - 2w_0w_2 + 2w_0w_3 + 4w_0w_4 + 4w_0w_5 + 2w_4w_5 - 6w_0w_6 - w_1w_6 - w_2w_6 + \\ +2w_0w_7 + w_3w_7,$$
$$2w_0w_2 + 2w_0w_3 + w_2w_3 + 4w_0w_5 + w_5^2 - 2w_0w_6 - 2w_0w_7 - w_6w_7 - 4w_0w_9 - w_9^2,$$
$$2w_0w_1 + 2w_0w_3 + w_1w_3 + 4w_0w_4 + w_4^2 - 2w_0w_6 - 2w_0w_7 - w_6w_7 - 4w_0w_8 - w_8^2,$$
$$2w_0w_1 + 2w_0w_2 + w_1w_2 + 4w_0w_4 + w_4^2 + 4w_0w_5 + w_5^2 - 6w_0w_6 - w_6^2 - 6w_0w_7 - w_6w_7 - w_7^2.$$

Furthermore $P_1$ can be viewed as the origin of the open affine set $U_0 = \{w_0 \neq 0\}$ in $\mathbb{P}^{13}_{[w_0:\cdots:w_{13}]}$. The ideal of the tangent cone $TC_{P_1}(W_P^{13} \cap U_0)$ is generated by the minimal degree homogeneous parts of all the polynomials in the ideal of $W_P^{13} \cap U_0$. One can find, with Macaulay2, that $TC_{P_1}(W_P^{13} \cap U_0)$ has ideal generated by

$$-9w_1 + 8w_7 + 8w_8 - 4w_9, \quad -9w_2 + 8w_7 - 4w_8 + 8w_9, \quad -9w_3 - 4w_7 + 8w_8 + 8w_9,$$
$$-9w_4 + 2w_7 + 2w_8 - w_9, \quad -9w_5 + 2w_7 - w_8 + 2w_9, \quad -9w_6 - w_7 + 2w_8 + 2w_9,$$
$$w_{10} - w_{11} + w_{12},$$
$$9w_{11}w_{12} + 2w_7w_{13} + 2w_8w_{13} - 10w_9w_{13},$$
$$2w_7w_{11} - 10w_8w_{11} + 2w_9w_{11} - 10w_7w_{12} + 2w_8w_{12} + 2w_9w_{12},$$
$$6w_7w_{11} - 6w_8w_{11} - 18w_9w_{11} + 6w_7w_{12} - 6w_8w_{12} + 18w_9w_{12},$$
$$9w_{12}^2 + 4w_7w_{13} - 8w_8w_{13} - 8w_9w_{13},$$
$$9w_{11}^2 - 8w_7w_{13} + 4w_8w_{13} - 8w_9w_{13},$$
$$w_7^2 - 2w_7w_8 + w_8^2 - 2w_7w_9 - 2w_8w_9 + w_9^2.$$

Hence $TC_{P_1}W_P^{13}$ is a cone with vertex at $P_1$ over a Veronese surface in the $\mathbb{P}^5$ given by

$$\{-9w_1 + 8w_7 + 8w_8 - 4w_9 = 0,\ -9w_2 + 8w_7 - 4w_8 + 8w_9 = 0,$$
$$-9w_3 - 4w_7 + 8w_8 + 8w_9 = 0,\ -9w_4 + 2w_7 + 2w_8 - w_9 = 0,$$
$$-9w_5 + 2w_7 - w_8 + 2w_9 = 0,\ -9w_6 - w_7 + 2w_8 + 2w_9 = 0,\ w_{10} - w_{11} + w_{12} = 0\}.$$

Similar analysis for the points $P_2$, $P_3$ and $P_4$. $\square$

**Theorem 8.7.** The tangent cone $TC_{P_5}W_P^{13}$ to $W_P^{13}$ at the point $P_5$ is a cone over a reducible quintic surface $M_5$, which is given by the union of five planes $\pi_0, \pi_1, \pi_2, \pi_3, \pi_4$, such that the four planes $\pi_1, \pi_2, \pi_3, \pi_4$ intersect the plane $\pi_0$ along the four edges of a quadrilateral. We give an idea of $M_5$ in Figure 15.

*Proof.* The point $P_5$ can be viewed as the origin of the open affine set $U_{13} = \{z_{13} \neq 0\}$. The ideal of the tangent cone $TC_{P_5}(W_P^{13} \cap U_{13})$ is generated by the minimal degree homogeneous parts of all the polynomials in the ideal of $W_P^{13} \cap U_{13}$. By using Macaulay2, we find that $TC_{P_5}(W_P^{13} \cap U_{13})$ has ideal generated by the following polynomials

$$z_6 - z_7, \quad z_5 - z_8, \quad z_4 - z_9, \quad z_2 - z_3, \quad z_1 - z_3, \quad 2z_0 - z_3,$$
$$z_9z_{10} - z_8z_{11} + z_7z_{12}, \quad z_8z_{10} - z_9z_{11} + z_3z_{12}, \quad z_7z_{10} - z_3z_{11} + z_9z_{12}, \quad z_3z_{10} - z_7z_{11} + z_8z_{12},$$
$$z_8^2 - z_9^2, \quad z_7^2 - z_9^2, \quad z_3^2 - z_9^2, \quad z_7z_8 - z_3z_9, \quad z_3z_8 - z_7z_9, \quad z_3z_7 - z_8z_9.$$

Hence $TC_{P_5}W_P^{13}$ is a cone with vertex at $P_5$ over a surface $M_5$ contained in the $\mathbb{P}^6$ given by $\{z_6 = z_7,\ z_5 = z_8,\ z_4 = z_9,\ z_2 = z_3,\ z_1 = z_3,\ 2z_0 = z_3\}$. This surface $M_5$ is the union of the following five planes:

$$\pi_0 := \{z_i = 0 | i \neq 10, 11\},$$
$$\pi_1 := \{2z_0 = z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = z_7 = z_8 = z_9,\ z_{10} = z_{11} - z_{12},\ z_{13} = 0\},$$
$$\pi_2 := \{2z_0 = z_1 = z_2 = z_3 = z_4 = -z_5 = -z_6 = -z_7 = -z_8 = z_9,\ z_{10} = z_{12} - z_{11},\ z_{13} = 0\},$$
$$\pi_3 := \{2z_0 = z_1 = z_2 = z_3 = -z_4 = z_5 = -z_6 = -z_7 = z_8 = -z_9,\ z_{10} = -z_{11} - z_{12},\ z_{13} = 0\},$$

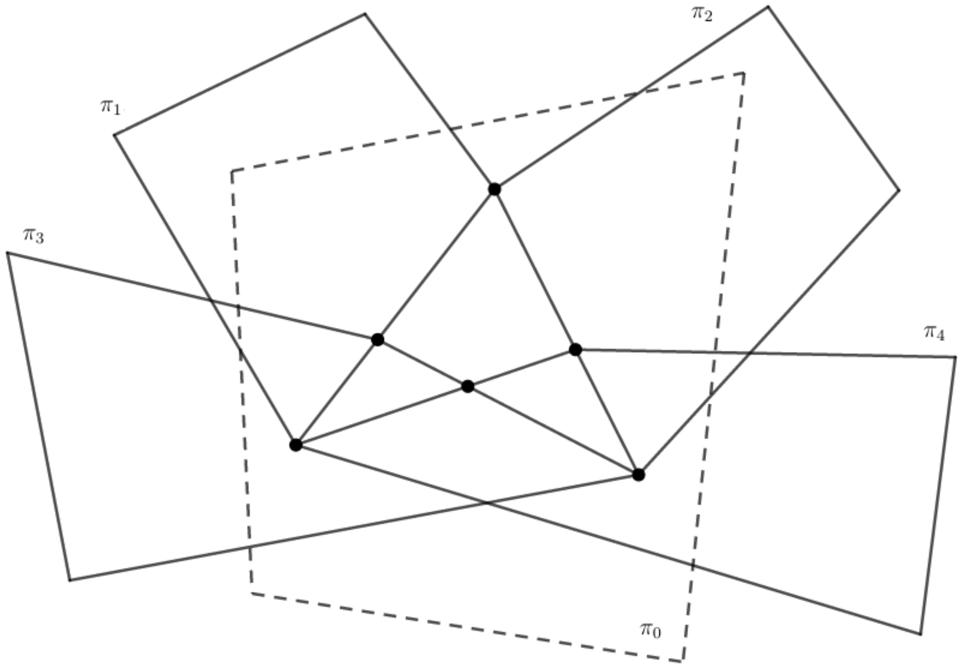


*Figure 15: The reducible quintic surface $M_5 \subset \mathbb{P}^6$ given by the union of five planes $\pi_0$, $\pi_1$, $\pi_2$, $\pi_3$, $\pi_4$, which intersect as in the statement of Theorem 8.7.*

$$\pi_4 := \{2z_0 = z_1 = z_2 = z_3 = -z_4 = -z_5 = z_6 = z_7 = -z_8 = -z_9,\ z_{10} = z_{11} + z_{12},\ z_{13} = 0\}.$$

□

**Remark 8.8.** Prokhorov says that the P-EF 3-fold $W_P^{13}$ has canonical singularities (see [46, Remark 3.3]), but he does not actually go into detail and does not say whether they are terminal or not. Since the singular points $P_1$, $P_2$, $P_3$, $P_4$ of $W_P^{13}$ are terminal (see Proposition 8.6 and [47, Example 1.3]), it remains to understand if $P_5$ is terminal or not (see Theorem 8.7). We recall that if all the singularities of $W_P^{13}$ were terminal, then $W_P^{13}$ would be limit of the classical Enriques-Fano threefold $W_F^{13}$ (see [44, Main Theorem 2]). So if we showed that the *simple isotropic decomposition* of the curve section $H$ of $W_P^{13}$ on a smooth hyperplane section $S$ is $H \sim 2E_1 + 2E_2 + 2E_3 + K_S$ (and not $H \sim 2E_1 + 2E_2 + 2E_3$, which is the *simple isotropic decomposition* of the curve section of $W_F^{13}$), we would obtain the non-terminality of $P_5$ (see § 9 and proof of Theorem 9.2 for more details). For now, this is an open question.

**Remark 8.9.** Since $W_P^{13}$ is projectively normal in $\mathbb{P}^{13}$ (see § 3.3), then it satisfies Assumption CM1 of § 4. By Remark 8.5 we have that it cannot verify Assumption CM3. Let us see that $W_P^{13}$ does not even satisfy Assumption CM2. Let $bl : \mathrm{Bl}_{P_i=1,\dots 5}\, \mathbb{P}^{13} \to \mathbb{P}^{13}$ be the blow-up of $\mathbb{P}^{13}$ at the five singular points of $W_P^{13}$ and let $\widetilde{W}$ be the strict transform of $W_P^{13}$. Then $\widetilde{W}$ intersects the exceptional divisor $bl^{-1}(P_5)$ along a surface isomorphic to $M_5$, which has six singular points locally given by the intersection of three planes of $\mathbb{P}^4$, such that two of them intersect the third along two lines and intersect each other at a point which is intersection of these two lines. Therefore $\widetilde{W}$ is not a desingularization of $W_P^{13}$, since there are six singular points infinitely near to $P_5$.

## 8.3 P-EF 3-fold (XVII) of genus 17

In the following we will often refer to the use of Macaulay2: see Code B.10 of Appendix B for the computational techniques we will use. Let us consider the anticanonical embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^8$, that is the morphism $\lambda : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^8$ such that

$$[u_0 : u_1]\times[v_0 : v_1] \mapsto \left[u_1^2v_1^2 : u_1^2v_0v_1 : u_1^2v_0^2 : u_1u_0v_1^2 : u_1u_0v_0v_1 : u_1u_0v_0^2 : u_0^2v_1^2 : u_0^2v_0v_1 : u_0^2v_0^2\right].$$

The image $P := \lambda(\mathbb{P}^1\times\mathbb{P}^1)$ is an octic surface in $\mathbb{P}^8_{[y_{0,0}:y_{0,1}:y_{0,2},y_{1,0}:y_{1,1}:y_{1,2}:y_{2,0}:y_{2,1}:y_{2,2}]}$, which we can consider as the hyperplane $\{x = 0\}$ in $\mathbb{P}^9_{[y_{0,0}:y_{0,1}:y_{0,2}:y_{1,0}:y_{1,1}:y_{1,2}:y_{2,0}:y_{2,1}:y_{2,2}:x]}$. Let $V$ be the cone over $P$ with vertex $v = [0:0:0:0:0:0:0:0:0:1]$.

**Remark 8.10.** By using Macaulay2 we can see that the ideal of $P$ is generated by the following polynomials

$$\begin{array}{llll} y_{2,1}^2 - y_{2,0}y_{2,2}, & y_{1,2}y_{2,1} - y_{1,1}y_{2,2}, & y_{1,1}y_{2,1} - y_{1,0}y_{2,2}, & y_{0,2}y_{2,1} - y_{0,1}y_{2,2}, \\ y_{0,1}y_{2,1} - y_{0,0}y_{2,2}, & y_{1,2}y_{2,0} - y_{1,0}y_{2,2}, & y_{1,1}y_{2,0} - y_{1,0}y_{2,1}, & y_{0,2}y_{2,0} - y_{0,0}y_{2,2}, \\ y_{0,1}y_{2,0} - y_{0,0}y_{2,1}, & y_{1,2}^2 - y_{0,2}y_{2,2}, & y_{1,1}y_{1,2} - y_{0,1}y_{2,2}, & y_{1,0}y_{1,2} - y_{0,0}y_{2,2}, \\ y_{1,1}^2 - y_{0,0}y_{2,2}, & y_{1,0}y_{1,1} - y_{0,0}y_{2,1}, & y_{0,2}y_{1,1} - y_{0,1}y_{1,2}, & y_{0,1}y_{1,1} - y_{0,0}y_{1,2}, \\ y_{1,0}^2 - y_{0,0}y_{2,0}, & y_{0,2}y_{1,0} - y_{0,0}y_{1,2}, & y_{0,1}y_{1,0} - y_{0,0}y_{1,1}, & y_{0,1}^2 - y_{0,0}y_{0,2}, \end{array}$$

in $\mathbb{C}[y_{0,0}, y_{0,1}, y_{0,2}, y_{1,0}, y_{1,1}, y_{1,2}, y_{2,0}, y_{2,1}, y_{2,2}]$. Then the ideal of $V$ is generated by the same polynomials as polynomials in $\mathbb{C}[y_{0,0}, y_{0,1}, y_{0,2}, y_{1,0}, y_{1,1}, y_{1,2}, y_{2,0}, y_{2,1}, y_{2,2}, x]$.

Let us take the involution $t$ of $\mathbb{P}^9_{[y_{0,0}:y_{0,1}:y_{0,2},y_{1,0}:y_{1,1}:y_{1,2}:y_{2,0}:y_{2,1}:y_{2,2}]}$ defined by

$$[y_{0,0} : \cdots : y_{2,2} : x] \mapsto [y_{0,0} : -y_{0,1} : y_{0,2} : -y_{1,0} : y_{1,1} : -y_{1,2} : y_{2,0} : -y_{2,1} : y_{2,2} : -x].$$

The locus of $t$-fixed points in $\mathbb{P}^9$ consists of two projective subspaces

$$F_1 = \{y_{0,0} = y_{0,2} = y_{1,1} = y_{2,0} = y_{2,2} = 0\} \cong \mathbb{P}^4,$$

$$F_2 = \{y_{0,1} = y_{1,0} = y_{1,2} = y_{2,1} = x = 0\} \cong \mathbb{P}^4.$$

We have that $F_1 \cap V = \{v\}$ and $F_2 \cap V = \{v_{0,0}, v_{0,2}, v_{2,0}, v_{2,2}\}$, where

$$v_{0,0} = [1:0:0:0:0:0:0:0:0:0], v_{0,2} = [0:0:1:0:0:0:0:0:0:0],$$

$$v_{2,0} = [0:0:0:0:0:0:1:0:0], v_{2,2} = [0:0:0:0:0:0:0:0:0:1].$$

Then $t$ defines an involution $\tau := t|_V : V \to V$ of $V$ with five fixed points. The quotient of $V$ via the involution $\tau$ is an Enriques-Fano threefold of genus 17 (see [46, Proposition 3.2]). The quotient map $\pi : V \to V/\tau =: W_P^{17}$ is defined by the restriction on $V$ of the linear system $\mathcal{Q}$ of the quadric hypersurfaces of $\mathbb{P}^9$ of type

$$q_1(y_{0,0}, y_{0,2}, y_{1,1}, y_{2,0}, y_{2,2}) + q_2(y_{0,1}, y_{1,0}, y_{1,2}, y_{2,1}, x) = 0,$$

where $q_1$ and $q_2$ are quadratic homogeneus forms. The linear system $\mathcal{Q}$ defines a morphism $\varphi : \mathbb{P}^9 \to \mathbb{P}^{29}$ such that $[y_{0,0} : \cdots : y_{2,2} : x] \mapsto [Z_0 : \cdots : Z_{29}]$, where

$$Z_0 = y_{1,1}^2,\ Z_1 = y_{0,0}^2,\ Z_2 = y_{0,2}^2,\ Z_3 = y_{2,0}^2,\ Z_4 = y_{2,2}^2,\ Z_5 = x^2,\ Z_6 = y_{0,1}^2,\ Z_7 = y_{1,0}^2,$$
$$Z_8 = y_{1,2}^2,\ Z_9 = y_{2,1}^2,\ Z_{10} = y_{0,1}x,\ Z_{11} = y_{1,0}x,\ Z_{12} = y_{1,2}x,\ Z_{13} = y_{2,1}x,$$
$$Z_{14} = y_{0,0}y_{1,1},\ Z_{15} = y_{0,2}y_{1,1},\ Z_{16} = y_{2,0}y_{1,1},\ Z_{17} = y_{2,2}y_{1,1},$$
$$Z_{18} = y_{0,1}y_{1,0},\ Z_{19} = y_{0,1}y_{1,2},\ Z_{20} = y_{1,0}y_{2,1},\ Z_{21} = y_{1,2}y_{2,1},$$
$$Z_{22} = y_{0,0}y_{0,2},\ Z_{23} = y_{0,0}y_{2,0},\ Z_{24} = y_{0,2}y_{2,2},\ Z_{25} = y_{2,0}y_{2,2},$$
$$Z_{26} = y_{0,1}y_{2,1},\ Z_{27} = y_{0,0}y_{2,2},\ Z_{28} = y_{0,2}y_{2,0},\ Z_{29} = y_{1,0}y_{1,2}.$$

Thus we have $\pi = \varphi|_V : V \to W_P^{17} \subset \mathbb{P}^{29}$. By the expression of $\lambda$, we have that $W_P^{17}$ is contained in a 17-dimensional projective subspace $H_{17}$ of $\mathbb{P}^{29}$ given by

$$H_{17} := \{Z_{18} = Z_{14},\ Z_{19} = Z_{15},\ Z_{20} = Z_{16},\ Z_{21} = Z_{17},\ Z_{22} = Z_6,\ Z_{23} = Z_7,$$
$$Z_{24} = Z_8,\ Z_{25} = Z_9,\ Z_{26} = Z_0,\ Z_{27} = Z_0,\ Z_{28} = Z_0,\ Z_{29} = Z_0\}$$

(see also Remark 8.10). Hence the quotient $\pi : V \to W_P^{17} \subset H_{17} \cong \mathbb{P}^{17}$ is defined by

$$[y_{0,0} : y_{0,1} : y_{0,2} : y_{1,0} : y_{1,1} : y_{1,2} : y_{2,0} : y_{2,1} : y_{2,2} : x] \mapsto [z_0 : z_1 : \cdots : z_{16} : z_{17}]$$

where $z_0 = y_{1,1}^2$, $z_1 = y_{0,0}^2$, $z_2 = y_{0,2}^2$, $z_3 = y_{2,0}^2$, $z_4 = y_{2,2}^2$, $z_5 = x^2$, $z_6 = y_{0,1}^2$, $z_7 = y_{1,0}^2$, $z_8 = y_{1,2}^2$, $z_9 = y_{2,1}^2$, $z_{10} = y_{0,1}x$, $z_{11} = y_{1,0}x$, $z_{12} = y_{1,2}x$, $z_{13} = y_{2,1}x$, $z_{14} = y_{0,0}y_{1,1}$, $z_{15} = y_{0,2}y_{1,1}$, $z_{16} = y_{2,0}y_{1,1}$, $z_{17} = y_{2,2}y_{1,1}$.

The P-EF 3-fold $W_P^{17}$ has the following five singular points

$$P_1 = \pi(v_{0,0}) = [0:1:0:0:0:0:0:0:0:0:0:0:0:0:0:0:0:0],$$
$$P_2 = \pi(v_{0,2}) = [0:0:1:0:0:0:0:0:0:0:0:0:0:0:0:0:0:0],$$
$$P_3 = \pi(v_{2,0}) = [0:0:0:1:0:0:0:0:0:0:0:0:0:0:0:0:0:0],$$
$$P_4 = \pi(v_{2,2}) = [0:0:0:0:1:0:0:0:0:0:0:0:0:0:0:0:0:0],$$
$$P_5 = \pi(v) = [0:0:0:0:0:1:0:0:0:0:0:0:0:0:0:0:0:0].$$

**Remark 8.11.** Thanks to Macaulay2 we can see that the P-EF threefold $W_P^{17}$ has ideal generated by the following 88 polynomials

$z_{15}z_{16} - z_{14}z_{17}$, $z_{12}z_{16} - z_{11}z_{17}$, $z_9z_{16} - z_3z_{17}$, $z_8z_{16} - z_0z_{17}$, $z_6z_{16} - z_1z_{17}$,
$z_4z_{16} - z_9z_{17}$, $z_2z_{16} - z_6z_{17}$, $z_0z_{16} - z_7z_{17}$, $z_{13}z_{15} - z_{10}z_{17}$, $z_9z_{15} - z_0z_{17}$,
$z_8z_{15} - z_2z_{17}$, $z_7z_{15} - z_1z_{17}$, $z_4z_{15} - z_8z_{17}$, $z_3z_{15} - z_7z_{17}$, $z_0z_{15} - z_6z_{17}$,
$z_{13}z_{14} - z_{10}z_{16}$, $z_{12}z_{14} - z_{11}z_{15}$, $z_9z_{14} - z_7z_{17}$, $z_8z_{14} - z_6z_{17}$, $z_7z_{14} - z_1z_{16}$,
$z_6z_{14} - z_1z_{15}$, $z_4z_{14} - z_0z_{17}$, $z_3z_{14} - z_7z_{16}$, $z_2z_{14} - z_6z_{15}$, $z_0z_{14} - z_1z_{17}$,
$z_{12}z_{13} - z_5z_{17}$, $z_{11}z_{13} - z_5z_{16}$, $z_8z_{13} - z_{12}z_{17}$, $z_7z_{13} - z_{11}z_{16}$, $z_6z_{13} - z_{11}z_{15}$,
$z_2z_{13} - z_{12}z_{15}$, $z_1z_{13} - z_{11}z_{14}$, $z_0z_{13} - z_{11}z_{17}$, $z_{11}z_{12} - z_{10}z_{13}$, $z_{10}z_{12} - z_5z_{15}$,
$z_9z_{12} - z_{13}z_{17}$, $z_7z_{12} - z_{10}z_{16}$, $z_6z_{12} - z_{10}z_{15}$, $z_3z_{12} - z_{13}z_{16}$, $z_1z_{12} - z_{10}z_{14}$,
$z_0z_{12} - z_{10}z_{17}$, $z_{10}z_{11} - z_5z_{14}$, $z_9z_{11} - z_{13}z_{16}$, $z_8z_{11} - z_{10}z_{17}$, $z_6z_{11} - z_{10}z_{14}$,
$z_4z_{11} - z_{13}z_{17}$, $z_2z_{11} - z_{10}z_{15}$, $z_0z_{11} - z_{10}z_{16}$, $z_9z_{10} - z_{11}z_{17}$, $z_8z_{10} - z_{12}z_{15}$,
$z_7z_{10} - z_{11}z_{14}$, $z_4z_{10} - z_{12}z_{17}$, $z_3z_{10} - z_{11}z_{16}$, $z_0z_{10} - z_{11}z_{15}$, $z_8z_9 - z_{17}^2$,
$z_7z_9 - z_{16}^2$, $z_6z_9 - z_{14}z_{17}$, $z_5z_9 - z_{13}^2$, $z_2z_9 - z_{15}z_{17}$, $z_1z_9 - z_{14}z_{16}$,
$z_0z_9 - z_{16}z_{17}$, $z_7z_8 - z_{14}z_{17}$, $z_6z_8 - z_{15}^2$, $z_5z_8 - z_{12}^2$, $z_3z_8 - z_{16}z_{17}$,
$z_1z_8 - z_{14}z_{15}$, $z_0z_8 - z_{15}z_{17}$, $z_6z_7 - z_{14}^2$, $z_5z_7 - z_{11}^2$, $z_4z_7 - z_{16}z_{17}$,

$z_2z_7 - z_{14}z_{15}$, $z_0z_7 - z_{14}z_{16}$, $z_5z_6 - z_{10}^2$, $z_4z_6 - z_{15}z_{17}$, $z_3z_6 - z_{14}z_{16}$, $z_0z_6 - z_{14}z_{15}$, $z_0z_5 - z_{10}z_{13}$, $z_3z_4 - z_9^2$, $z_2z_4 - z_8^2$, $z_1z_4 - z_{14}z_{17}$, $z_0z_4 - z_{17}^2$, $z_2z_3 - z_{14}z_{17}$, $z_1z_3 - z_7^2$, $z_0z_3 - z_{16}^2$, $z_1z_2 - z_6^2$, $z_0z_2 - z_{15}^2$, $z_0z_1 - z_{14}^2$, $z_0^2 - z_{14}z_{17}$.

Thus the ideal of $W_P^{17}$ is generated by quadrics. Since $W_P^{17}$ is projectively normal in $\mathbb{P}^{17}$ (see § 3.3), then the ideal of its general hyperplane section $S \subset \mathbb{P}^{16}$ is generated by quadrics too. It is consistent with the fact that the $\phi$ of a general hyperplane section of $S$ is 4 (see [35, Theorem 1.1 (ii)]), as we will see in the proof of Theorem 9.2.

**Remark 8.12.** Let $l_{i,j} := \{z_k = 0 | i \neq i, j\}$ be the line joining the singular points $P_i$ and $P_j$ with $1 \leq i < j \leq 5$. By Remark 8.11 we have that the lines $l_{1,5}$, $l_{2,5}$, $l_{3,5}$, $l_{4,5}$ are contained in $W_P^{17}$, while the lines $l_{1,2}$, $l_{1,3}$, $l_{1,4}$, $l_{2,3}$, $l_{2,4}$, $l_{3,4}$ are not. Hence the five singular points $P_1$, $P_2$, $P_3$, $P_4$, $P_5$ of $W_P^{17}$ are associated as in Figure 27 of Appendix A.

**Proposition 8.13.** If $i = 1, 2, 3, 4$, the tangent cone $TC_{P_i}W_P^{17}$ to $W_P^{17}$ at the point $P_i$ is a cone over a Veronese surface.

*Proof.* Each point $P_i$, $i = 1, 2, 3, 4$, can be viewed as the origin of the open affine set $U_i = \{z_i \neq 0\}$. The ideal of the tangent cone $TC_{P_i}(W_P^{17} \cap U_i)$ is generated by the minimal degree homogeneous parts of all the polynomials in the ideal of $W_P^{17} \cap U_i$. By using Macaulay2 we obtain the following tangent cones.

$TC_{P_1}(W_P^{17} \cap U_1)$ has ideal generated by $z_{17}$, $z_{16}$, $z_{15}$, $z_{13}$, $z_{12}$, $z_9$, $z_8$, $z_4$, $z_3$, $z_2$, $z_0$, $z_{10}z_{11} - z_5z_{14}$, $z_6z_{11} - z_{10}z_{14}$, $z_7z_{10} - z_{11}z_{14}$, $z_6z_7 - z_{14}^2$, $z_5z_7 - z_{11}^2$, $z_5z_6 - z_{10}^2$.

Hence $TC_{P_1}W_P^{17}$ is a cone with vertex $P_1$ over a Veronese surface in the $\mathbb{P}^5$ given by $\{z_i = 0 | i = 0, 1, 2, 3, 4, 8, 9, 12, 13, 15, 16, 17\}$.

$TC_{P_2}(W_P^{17} \cap U_2)$ has ideal generated by $z_{17}$, $z_{16}$, $z_{14}$, $z_{13}$, $z_{11}$, $z_9$, $z_7$, $z_4$, $z_3$, $z_1$, $z_0$, $z_{10}z_{12} - z_5z_{15}$, $z_6z_{12} - z_{10}z_{15}$, $z_8z_{10} - z_{12}z_{15}$, $z_6z_8 - z_{15}^2$, $z_5z_8 - z_{12}^2$, $z_5z_6 - z_{10}^2$.

Hence $TC_{P_2}W_P^{17}$ is a cone with vertex $P_2$ over a Veronese surface in the $\mathbb{P}^5$ given by $\{z_i = 0 | i = 0, 1, 2, 3, 4, 7, 9, 11, 13, 14, 16, 17\}$.

$TC_{P_3}(W_P^{17} \cap U_3)$ has ideal generated by $z_{17}$, $z_{15}$, $z_{14}$, $z_{12}$, $z_{10}$, $z_8$, $z_6$, $z_4$, $z_2$, $z_1$, $z_0$, $z_{11}z_{13} - z_5z_{16}$, $z_7z_{13} - z_{11}z_{16}$, $z_9z_{11} - z_{13}z_{16}$, $z_7z_9 - z_{16}^2$, $z_5z_9 - z_{13}^2$, $z_5z_7 - z_{11}^2$.

Hence $TC_{P_3}W_P^{17}$ is a cone with vertex $P_3$ over a Veronese surface in the $\mathbb{P}^5$ given by $\{z_i = 0 | i = 0, 1, 2, 3, 4, 6, 8, 10, 12, 14, 15, 17\}$.

$TC_{P_4}(W_P^{17} \cap U_4)$ has ideal generated by $z_{16}$, $z_{15}$, $z_{14}$, $z_{11}$, $z_{10}$, $z_7$, $z_6$, $z_3$, $z_2$, $z_1$, $z_0$, $z_{12}z_{13} - z_5z_{17}$, $z_8z_{13} - z_{12}z_{17}$, $z_9z_{12} - z_{13}z_{17}$, $z_8z_9 - z_{17}^2$, $z_5z_9 - z_{13}^2$, $z_5z_8 - z_{12}^2$.

Hence $TC_{P_4}W_P^{17}$ is a cone with vertex $P_4$ over a Veronese surface in the $\mathbb{P}^5$ given by $\{z_i = 0 | i = 0, 1, 2, 3, 4, 6, 7, 10, 11, 14, 15, 16\}$. □

**Theorem 8.14.** The tangent cone $TC_{P_5}W_P^{17}$ to $W_P^{17}$ at the point $P_5$ is a cone over a reducible sextic surface $M_6 \subset \mathbb{P}^7 \subset \mathbb{P}^{17}$, which is given by the union of four planes $\pi_1, \pi_2, \pi'_1, \pi'_2$ and a quadric surface $Q \subset \mathbb{P}^3 \subset \mathbb{P}^7$. In particular each one of the planes $\pi_1, \pi_2, \pi'_1, \pi'_2$ intersects the quadric $Q$ respectively along a line $l_1, l_2, l'_1, l'_2$, where $l_1$ is disjoint from $l'_1$ and $l_2$ is disjoint from $l'_2$. In the other cases the intersections of two of these lines identify four points on $Q$: $q_{1,2} := l_1 \cap l_2$, $q_{1,2'} := l_1 \cap l'_2$, $q_{1',2} := l'_1 \cap l_2$, $q_{1',2'} := l'_1 \cap l'_2$.

*Proof.* The point $P_5$ can be viewed as the origin of the open affine set $U_5 = \{z_5 \neq 0\}$. The ideal of the tangent cone $TC_{P_5}(W_P^{17} \cap U_5)$ is generated by the minimal degree homogeneous parts of all the polynomials in the ideal of $W_P^{17} \cap U_5$. By using Macaulay2 we can find that $TC_{P_5}(W_P^{17} \cap U_5)$ has ideal generated by the following polynomials

$$z_{17},\ z_{16},\ z_{15},\ z_{14},\ z_9,\ z_8,\ z_7,\ z_6,\ z_0,$$

$$z_{11}z_{12} - z_{10}z_{13},$$

$$z_2z_{13},\ z_1z_{13},\ z_3z_{12},\ z_1z_{12},\ z_4z_{11},\ z_2z_{11},\ z_4z_{10},\ z_3z_{10},\ z_3z_4,\ z_2z_4,\ z_1z_4,\ z_2z_3,\ z_1z_3,\ z_1z_2.$$

Hence $TC_{P_5}W_P^{17}$ is a cone with vertex $P_5$ over a surface $M_6$ contained in the $\mathbb{P}^7$ given by $\{z_i = 0 | i = 0, 5, 6, 7, 8, 9, 14, 15, 16, 17\}$. This surface $M_6$ is the union of four planes $\pi_1, \pi_2, \pi'_1, \pi'_2$ and a quadric surface $Q$, where

$$\begin{aligned}
\pi_1 &:= \{z_i = 0 | i = 0, 1, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15, 16, 17\},\\
\pi_2 &:= \{z_i = 0 | i = 0, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17\},\\
\pi'_1 &:= \{z_i = 0 | i = 0, 1, 2, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 17\},\\
\pi'_2 &:= \{z_i = 0 | i = 0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 14, 15, 16, 17\},\\
Q &:= \{z_i = 0 | i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 14, 15, 16, 17\} \cap \{z_{11}z_{12} - z_{10}z_{13} = 0\}.
\end{aligned}$$

We obtain the same situation described in Theorem 7.5 and so a sextic surface $M_6$ as in Figure 13. $\square$

**Remark 8.15.** We recall that $W_P^{17}$ has canonical non-terminal singularities. Indeed if it had terminal singularities, it would be limit of some BS-EF 3-fold and therefore would have genus $p$ with $2 \leq p \leq 10$ or $p = 13$ (see [44, Main Theorem 2]). Since the singular points $P_1$, $P_2$, $P_3$, $P_4$ are terminal (see Proposition 8.13 and [47, Example 1.3]), then $P_5$ is a canonical non-terminal singularity.

**Remark 8.16.** Since $W_P^{17}$ is projectively normal in $\mathbb{P}^{17}$ (§ 3.3), then it satisfies Assumption CM1 of § 4. By Remark 8.12 we have that it cannot verify Assumption CM3. Let us show that $W_P^{17}$ does not even satisfy Assumption CM2. Let $bl : \mathrm{Bl}_{P_{i=1,\dots 5}} \mathbb{P}^{17} \to \mathbb{P}^{17}$ be the blow-up of $\mathbb{P}^{17}$ at the five singular points of $W_P^{17}$ and let $\widetilde{W}$ be the strict transform of $W_P^{17}$. Then $\widetilde{W}$ intersects the exceptional divisor $bl^{-1}(P_5)$ along a surface which is isomorphic to $M_6$ and which has four singular points locally given by the intersection of three planes of $\mathbb{P}^4$, such that two of them intersect the third along two lines and intersect each other at a point which is intersection of these two lines. Thus $\widetilde{W}$ is not a desingularization of $W_P^{17}$, since there are four singular points infinitely near to $P_5$.

# 9 Simple isotropic decompositions of the curve sections of the Enriques-Fano threefolds

## 9.1 Abstract

Let $(W, \mathcal{L})$ be an Enriques-Fano threefold and let us denote by $H$ the class of a curve section on a smooth hyperplane section $S \in \mathcal{L}$. It is known that there are 10 primitive effective isotropic divisors $E_1, \dots, E_{10}$ such that $H \sim a_0E_{1,2}+a_1E_1+\dots+a_{10}E_{10}+\epsilon K_S$ where $E_{1,2} \sim \frac{1}{3}(E_1+\dots+E_{10})-E_1-E_2$, $\epsilon=0,1$ and $a_0, a_1, \dots a_{10}$ are nonnegative integers (see [9, Corollary 4.7]). This expression is called simple isotropic decomposition (simply, SID) of $H$. We will describe the SID of the curve sections of the known Enriques-Fano threefolds (see Theorem 9.2). Generally the SID allow us to identify the various components of the moduli space of the polarized Enriques surfaces. Thus our analysis suggests which families the hyperplane sections of the Enriques-Fano threefolds belong to.

## 9.2 Preliminaries on simple isotropic decompositions

We recall that any irreducible curve $C$ on an Enriques surface satisfies $C^2 = 2p_a(C) - 2 \geq -2$, with equality occurring if and only if $C \cong \mathbb{P}^1$. An Enriques surface is called *unnodal* if it does not contain any smooth rational curve, otherwise it is called *nodal*. We recall that the general Enriques surface is unnodal (see [18]). Let $\mathcal{E}$ be the smooth irreducible 10-dimensional moduli space parametrizing the Enriques surfaces and let $\mathcal{E}_{g,\phi}$ be the moduli space of the pairs $(S, H)$ such that $[S] \in \mathcal{E}$ and $H \in \mathrm{Pic}(S)$ is an ample divisor on $S$ satisfying $H^2 = 2g-2$ and $\phi(H) = \phi$, where

$$\phi(H) := \min\{E \cdot H | E \in \mathrm{NS}(S), E^2 = 0, E > 0\}.$$

The spaces $\mathcal{E}_{g,\phi}$ are in general reducible. We refer to [9] and [33] for more details. Let us consider now an Enriques-Fano threefold $(W, \mathcal{L})$ of genus $p$. We will denote by $H$ the class of a curve section of $W$ on a general (smooth) hyperplane section $S$. Hence we have $|H| = \mathcal{L}|_S$ (see [10, Lemma 4.1 (i)]). We set $\phi := \phi(H)$ and we recall that $\phi^2 \leq H^2 = 2p-2$ (see [16, Cor. 2.7.1]). We say that the rational map associated with $|H|$ is *hyperelliptic* if $p=2$ or if it is of degree 2 onto a surface of degree $p-2$ in $\mathbb{P}^{p-1}$; we say that it is *superelliptic* if $p=2$ or if it is of degree 2 onto a surface of degree $p-1$ in $\mathbb{P}^{p-1}$ (see [16, p. 229]). We have the following results which we will use later:

(a) by [16, Proposition 4.5.1]

$$\phi = 1 \Leftrightarrow |H| \text{ has 2 simple base points } \Leftrightarrow \phi_{\mathcal{L}} \text{ is hyperelliptic on } S;$$

(b) by [10, Lemma 4.1] and [16, Theorem 4.4.1]

$$\phi \geq 2 \Leftrightarrow |H| \text{ base point free } \Leftrightarrow \mathcal{L} \text{ base point free;}$$

(c) by [10, Lemma 4.1 (i)] and [16, Theorems 4.4.1, 4.6.1] (since $H$ ample)

$$\phi \geq 3 \Leftrightarrow \phi_{\mathcal{L}} \text{ is an isomorphism on } S.$$

In the last case (c) we get that $\phi_{\mathcal{L}}(W) \subset \mathbb{P}^p$ is a threefold whose general hyperplane section is a smooth Enriques surface.

We recall now that a divisor $E$ on $S$ is said to be *isotropic* if $E^2 = 0$ and $E \not\equiv 0$, and it is said to be *primitive* if it is non-divisible in $\mathrm{Num}(S)$. On an unnodal Enriques

| $p$ | $\phi$ | comp. | SID |
|---|---|---|---|
| 2 | 1 | $\mathcal{E}_{2,1}$ | $E_1+E_2$ |
| 3 | 1 | $\mathcal{E}_{3,1}$ | $2E_1+E_2$ |
| 3 | 2 | $\mathcal{E}_{3,2}$ | $E_1+E_{1,2}$ |
| 4 | 1 | $\mathcal{E}_{4,1}$ | $3E_1+E_2$ |
| 4 | 2 | $\mathcal{E}_{4,2}$ | $E_1+E_2+E_3$ |
| 5 | 1 | $\mathcal{E}_{5,1}$ | $4E_1+E_2$ |
| 5 | 2 | $\mathcal{E}_{5,2}^{(I)}$ | $2E_1+E_{1,2}$ |
| 5 | 2 | $\mathcal{E}_{5,2}^{(II)+}$ | $2(E_1+E_2)$ |
| 5 | 2 | $\mathcal{E}_{5,2}^{(II)-}$ | $2(E_1+E_2)+K_S$ |
| 6 | 1 | $\mathcal{E}_{6,1}$ | $5E_1+E_2$ |
| 6 | 2 | $\mathcal{E}_{6,2}$ | $2E_1+E_2+E_3$ |
| 6 | 3 | $\mathcal{E}_{6,3}$ | $E_1+E_2+E_{1,2}$ |
| 7 | 1 | $\mathcal{E}_{7,1}$ | $6E_1+E_2$ |
| 7 | 2 | $\mathcal{E}_{7,2}^{(I)}$ | $3E_1+E_{1,2}$ |
| 7 | 2 | $\mathcal{E}_{7,2}^{(II)}$ | $3E_1+2E_2$ |
| 7 | 3 | $\mathcal{E}_{7,3}$ | $E_1+E_2+E_3+E_4$ |
| 8 | 1 | $\mathcal{E}_{8,1}$ | $7E_1+E_2$ |
| 8 | 2 | $\mathcal{E}_{8,2}$ | $3E_1+E_2+E_3$ |
| 8 | 3 | $\mathcal{E}_{8,3}$ | $2E_1+E_3+E_{1,2}$ |
| 9 | 1 | $\mathcal{E}_{9,1}$ | $8E_1+E_2$ |
| 9 | 2 | $\mathcal{E}_{9,2}^{(I)}$ | $4E_1+E_{1,2}$ |
| 9 | 2 | $\mathcal{E}_{9,2}^{(II)+}$ | $2(2E_1+E_2)$ |
| 9 | 2 | $\mathcal{E}_{9,2}^{(II)-}$ | $2(2E_1+E_2)+K_S$ |
| 9 | 3 | $\mathcal{E}_{9,3}^{(I)}$ | $2E_1+E_2+E_{1,2}$ |
| 9 | 3 | $\mathcal{E}_{9,3}^{(II)}$ | $2E_1+2E_2+E_3$ |
| 9 | 4 | $\mathcal{E}_{9,4}^{+}$ | $2(E_1+E_{1,2})$ |
| 9 | 4 | $\mathcal{E}_{9,4}^{-}$ | $2(E_1+E_{1,2})+K_S$ |

| $p$ | $\phi$ | comp. | SID |
|---|---|---|---|
| 10 | 1 | $\mathcal{E}_{10,1}$ | $9E_1+E_2$ |
| 10 | 2 | $\mathcal{E}_{10,2}$ | $4E_1+E_2+E_3$ |
| 10 | 3 | $\mathcal{E}_{10,3}^{(I)}$ | $2E_1+E_2+E_3+E_4$ |
| 10 | 3 | $\mathcal{E}_{10,3}^{(II)}$ | $3(E_1+E_2)$ |
| 10 | 4 | $\mathcal{E}_{10,4}$ | $2E_{1,2}+E_1+E_2$ |
| 13 | 1 | $\mathcal{E}_{13,1}$ | $12E_1+E_2$ |
| 13 | 2 | $\mathcal{E}_{13,2}^{(I)}$ | $6E_1+E_{1,2}$ |
| 13 | 2 | $\mathcal{E}_{13,2}^{(II)+}$ | $2(3E_1+E_2)$ |
| 13 | 2 | $\mathcal{E}_{13,2}^{(II)-}$ | $2(3E_1+E_2)+K_S$ |
| 13 | 3 | $\mathcal{E}_{13,3}^{(I)}$ | $3E_1+E_2+E_3+E_4$ |
| 13 | 3 | $\mathcal{E}_{13,3}^{(II)}$ | $4E_1+3E_2$ |
| 13 | 4 | $\mathcal{E}_{13,4}^{(I)}$ | $2E_1+2E_2+E_{1,2}$ |
| 13 | 4 | $\mathcal{E}_{13,4}^{(II)+}$ | $2(E_1+E_2+E_3)$ |
| 13 | 4 | $\mathcal{E}_{13,4}^{(II)-}$ | $2(E_1+E_2+E_3)+K_S$ |
| 13 | 4 | $\mathcal{E}_{13,4}^{(III)}$ | $3E_1+2E_{1,2}$ |
| 17 | 1 | $\mathcal{E}_{17,1}$ | $16E_1+E_2$ |
| 17 | 2 | $\mathcal{E}_{17,2}^{(I)}$ | $8E_1+E_{1,2}$ |
| 17 | 2 | $\mathcal{E}_{17,2}^{(II)+}$ | $2(4E_1+E_2)$ |
| 17 | 2 | $\mathcal{E}_{17,2}^{(II)-}$ | $2(4E_1+E_2)+K_S$ |
| 17 | 3 | $\mathcal{E}_{17,3}$ | $5E_1+E_3+E_{1,2}$ |
| 17 | 4 | $\mathcal{E}_{17,4}^{(I)}$ | $3E_1+2E_2+2E_3$ |
| 17 | 4 | $\mathcal{E}_{17,4}^{(II)}$ | $3E_1+2E_2+E_{1,2}$ |
| 17 | 4 | $\mathcal{E}_{17,4}^{(III)+}$ | $2(2E_1+E_{1,2})$ |
| 17 | 4 | $\mathcal{E}_{17,4}^{(III)-}$ | $2(2E_1+E_{1,2})+K_S$ |
| 17 | 4 | $\mathcal{E}_{17,4}^{(IV)+}$ | $4(E_1+E_2)$ |
| 17 | 4 | $\mathcal{E}_{17,4}^{(IV)-}$ | $4(E_1+E_2)+K_S$ |
| 17 | 5 | $\mathcal{E}_{17,5}$ | $2E_1+E_3+E_4+E_5+E_{1,2}$ |

*Table 1: All irreducible components of $\mathcal{E}_{p,\phi}$ for $2 \le p \le 10$ and $p = 13, 17$.*

surface, any effective primitive isotropic divisor $E$ is represented by an irreducible curve of arithmetic genus one. By [9, Corollary 4.7] there are 10 primitive effective isotropic divisors $E_1, \dots, E_{10}$ such that $E_i \cdot E_j = 1$ for $i \neq j$ and such that

$$H \sim a_0 E_{1,2} + a_1 E_1 + \cdots + a_{10} E_{10} + \epsilon K_S \tag{2}$$

where $E_{1,2} \sim \frac{1}{3}(E_1 + \cdots + E_{10}) - E_1 - E_2$ and $a_0, a_1, \dots a_{10}$ are nonnegative integers with

$$\begin{cases} \text{either } a_0 = 0 \text{ and } \#\{i | i \in \{1, \dots, 10\}, a_i > 0\} \neq 9, \\ \text{or } a_{10} = 0, \end{cases}$$

and

$$\epsilon = \begin{cases} 0, & \text{if } H + K_S \text{ is not 2-divisible on } \operatorname{Pic}(S), \\ 1, & \text{if } H + K_S \text{ is 2-divisible on } \operatorname{Pic}(S). \end{cases}$$

We call (2) a *simple isotropic decomposition (SID)* of $H$. We also recall that $E_{1,2} \cdot E_1 = E_{1,2} \cdot E_2 = 2$ and $E_{1,2} \cdot E_i = 1$ for $i = 3, \ldots, 10$. For later reference, we list in Table 1 all the irreducible components of $\mathcal{E}_{p,\phi}$ for $2 \leq p \leq 10$ and $p = 13, 17$ (see [9, Appendix]).

**Definition 9.1.** A projective variety $X \subset \mathbb{P}^N$ is said to be *$k$-extendable* if there exists a projective variety $V \subset \mathbb{P}^{N+k}$, that is not a cone, such that $X = V \cap \mathbb{P}^N$ (transversely) and $\dim V = \dim X + k$.

The question of $k$-extendability of Enriques surfaces is still open. It is known that if $S \subset \mathbb{P}^N$ is a 1-extendable Enriques surface, then $h^1(\mathcal{T}_S(-1)) > 0$ (see [10, proof of Corollary 1.2]) and $\phi(\mathcal{O}_S(1)) \geq 3$ (see [16, Theorem 4.6.1]). Moreover, if $S \subset \mathbb{P}^N$ is an unnodal Enriques surface (i.e. not containing any smooth rational curve) which is 1-extendable, then $(S, \mathcal{O}_S(1))$ belongs to the following list: $\mathcal{E}_{17,4}^{(IV)^+}$, $\mathcal{E}_{13,4}^{(II)^+}$, $\mathcal{E}_{13,3}^{(II)}$, $\mathcal{E}_{10,3}^{(II)}$, $\mathcal{E}_{9,4}^{+}$, $\mathcal{E}_{9,3}^{(II)}$, $\mathcal{E}_{7,3}$ (see [10, Corollary 1.2]).

## 9.3 SID of the curve sections of the known EF-3folds

Let us describe the SID of the curve sections of the known Enriques-Fano threefolds.

**Theorem 9.2.** Let $(W, \mathcal{L})$ be an Enriques-Fano threefold in the list (I)-(XVII) of § 3.2. Let $S \in \mathcal{L}$ be a general hyperplane section of $W$ and let $H$ be a general curve section of $W$ on $S$. Then $H$ has the $\phi$ and the SID described in Table 2.

*Proof.* Let us study the known Enriques-Fano threefolds case by case. If $(W, \mathcal{L})$ is a fixed Enriques-Fano threefold of genus $p$, we will denote each time by $S$ a general element of $\mathcal{L}$, by $H$ a general curve section of $W$ on $S$ satisfying $H^2 = 2p - 2$, and by $\phi$ the value $\phi(H)$ defined in § 9.2.

(I) $W = W_{BS}^2$. The map $\phi_{\mathcal{L}} : W \dashrightarrow \mathbb{P}^2$ is a rational map (see [1, §6.1.6]). Since $p = 2$ and $\phi^2 \leq 2p - 2$, then we have $\phi = 1$. So the SID is $H \sim E_1 + E_2$ (see Table 1).

(II) $W = W_{BS}^3$. Since $p = 3$ and $\phi^2 \leq 2p - 2$, then we have $1 \leq \phi \leq 2$. The map $\phi_{\mathcal{L}} : W \to \mathbb{P}^3$ is a morphism and a quadruple cover (see [1, §6.1.5]). This implies $\phi = 2$, because if $\phi = 1$ the map would be a double cover (see § 9.2 (a)). Then the SID is $H \sim E_1 + E_{1,2}$ (see Table 1).

(III) $W = \overline{W}_{BS}^3$. Since $p = 3$ and $\phi^2 \leq 2p - 2$, then we have $1 \leq \phi \leq 2$. The map $\phi_{\mathcal{L}} : W \dashrightarrow \mathbb{P}^3$ is a rational map and a double cover (see [1, §6.2.7]). We have that $\phi_{\mathcal{L}}|_S$ is hyperelliptic, since it is of degree 2 onto a plane. This implies $\phi = 1$ (see § 9.2 (a)) and the SID must be $H \sim 2E_1 + E_2$ (see Table 1).

| Marking | EF 3-fold | SID of $H$ | $\phi(H)$ | $(S,H)\in$ |
|---|---|---|---|---|
| (I) | $W^2_{BS}$ | $E_1+E_2$ | 1 | $\mathcal{E}_{2,1}$ |
| (II) | $W^3_{BS}$ | $E_1+E_{1,2}$ | 2 | $\mathcal{E}_{3,2}$ |
| (III) | $\overline{W}^3_{BS}$ | $2E_1+E_2$ | 1 | $\mathcal{E}_{3,1}$ |
| (IV) | $W^4_{BS}$, $W^4_F$ | $E_1+E_2+E_3$ | 2 | $\mathcal{E}_{4,2}$ |
| (V) | $\overline{W}^4_{BS}$ | $3E_1+E_2$ | 1 | $\mathcal{E}_{4,1}$ |
| (VI) | $W^5_{BS}$ | $2E_1+E_{1,2}$ | 2 | $\mathcal{E}^{(I)}_{5,2}$ |
| (VII) | $\overline{W}^5_{BS}$ | $2(E_1+E_2)$ | 2 | $\mathcal{E}^{(II)-}_{5,2}$ |
| (VIII) | $W^6_{BS}$, $W^6_F$ | $E_1+E_2+E_{1,2}$ | 3 | $\mathcal{E}_{6,3}$ |
| (IX) | $\overline{W}^7_{BS}$ | $3E_1+E_{1,2}$ | 2 | $\mathcal{E}^{(I)}_{7,2}$ |
| (X) | $W^7_{BS}$, $W^7_F$ | $E_1+E_2+E_3+E_4$ | 3 | $\mathcal{E}_{7,3}$ |
| (XI) | $W^8_{BS}$ | $2E_1+E_3+E_{1,2}$ | 3 | $\mathcal{E}_{8,3}$ |
| (XII) | $W^9_{BS}$, $W^9_F$ | $2(E_1+E_{1,2})$ | 4 | $\mathcal{E}^+_{9,4}$ |
| (XIII) | $W^{10}_{BS}$ | $2E_1+E_2+E_3+E_4$ | 3 | $\mathcal{E}^{(I)}_{10,3}$ |
| (XIV) | $W^{13}_{BS}$, $W^{13}_F$ | $2(E_1+E_2+E_3)$ | 4 | $\mathcal{E}^{(II)+}_{13,4}$ |
| (XV) | $W^9_{KLM}$ | $2E_1+2E_2+E_3$ | 3 | $\mathcal{E}^{(II)}_{9,3}$ |
| (XVI) | $W^{13}_P$ | $2(E_1+E_2+E_3)$ or $2(E_1+E_2+E_3)+K_S$ | 4 | $\mathcal{E}^{(II)+}_{13,4}$ or $\mathcal{E}^{(II)-}_{13,4}$ |
| (XVII) | $W^{17}_P$ | $4(E_1+E_2)$ | 4 | $\mathcal{E}^{(IV)+}_{17,4}$ |

*Table 2: SID of the curve section $H$ of an Enriques-Fano threefold $(W,\mathcal{L})$ on a general $S\in\mathcal{L}$.*

(IV) $W=W^4_{BS}$. Since $p=4$ and $\phi^2\le 2p-2$, then we have $1\le\phi\le 2$. The map $\phi_{\mathcal{L}}: W \dashrightarrow \mathbb{P}^4$ is a rational map birational onto the image (see [1, §6.3.3]), which is the Enriques threefold $W^4_F$ of [23, §10]. Since $S$ is mapped by $\phi_{\mathcal{L}}$ to a general sextic surface of $\mathbb{P}^3$ double along the edges of a tetrahedron, then the SID is $H\sim E_1+E_2+E_3$ (see [9, §5]) and $\phi=2$ (see Table 1).

(V) $W=\overline{W}^4_{BS}$. Since $p=4$ and $\phi^2\le 2p-2$, then we have $1\le\phi\le 2$. The map $\phi_{\mathcal{L}}: W \dashrightarrow \mathbb{P}^4$ is a rational map and a double cover over its image which is a quadric cone (see [1, §6.6.2]). We have that $\phi_{\mathcal{L}}|_S$ is hyperelliptic, since it is of degree 2 onto a quadric surface of $\mathbb{P}^3$. Then we have $\phi=1$ (see § 9.2 (a)) and the SID is $H\sim 3E_1+E_2$ (see Table 1).

(VI) $W=W^5_{BS}$. Since $p=5$ and $\phi^2\le 2p-2$, then we have $1\le\phi\le 2$. The map $\phi_{\mathcal{L}}: W\to\mathbb{P}^5$ is a morphism birational onto its image (see [1, §6.2.2]). So we have that $\phi=2$ (see § 9.2 (b)). The SID is $H\sim 2(E_1+E_2)+K_S$ or $H\sim 2E_1+E_{1,2}$ or $H\sim 2(E_1+E_2)$ (see Table 1). The case $H\sim 2(E_1+E_2)$ is excluded, otherwise the map $\phi_{\mathcal{L}}|_S$ would be superelliptic (see [16, Theorem 4.7.1]). Let us consider now a smooth intersection $B_4 := Q_1\cap Q_2$ of two quadric hypersurfaces of $\mathbb{P}^5$ and an elliptic curve $e\subset B_4$ given by the intersection of two hyperplane sections of $B_4$. In Bayle's description, an Enriques-Fano threefold

$W$ of this type is given by the quotient $\pi : X \to X/\sigma =: W$ of $X := \mathrm{Bl}_e B_4$, that is the blow-up of $B_4$ along the curve $e$, where $\sigma$ is an involution of $X$ with eight fixed points. Let us denote the above blow-up by the map $bl : X \to B_4$ and let $E := bl^{-1}(e)$ be the exceptional divisor. If $h$ denotes the hyperplane class of $\mathbb{P}^5$, then $K_{Q_1} = (K_{\mathbb{P}^5} + Q_1)|_{Q_1} = (-4h)|_{Q_1}$ and $K_{B_4} = (K_{Q_1} + B_4)|_{B_4} = (K_{Q_1} + Q_2|_{Q_1})|_{B_4} = (-2h)|_{B_4}$ by the adjunction formula. So we obtain $-K_X = -bl^* K_{B_4} - (\mathrm{codim}(e, B_4) - 1)E = 2bl^*(h) - E$ (see [27, p.187]). Furthermore if $\widetilde{S}$ is the K3-surface $\pi^* S$, then $\pi|_{\widetilde{S}}^* H \sim -K_X|_{\widetilde{S}} = \big(2bl^*(h) - E\big)|_{\widetilde{S}}$. Let us see that $E|_{\widetilde{S}}$ is not 2-divisible. We observe that $\tilde{S}$ is isomorphic to the complete intersection of three quadric hypersurfaces of $\mathbb{P}^5$ and that $E|_{\widetilde{S}}$ is a quartic elliptic curve $C$. If $E|_{\widetilde{S}}$ were 2-divisible, we would have a divisor $D$ on $\widetilde{S}$ such that $C \sim 2D$ and $D^2 = 0$. We observe that $-D$ couldn't be effective, otherwise $-2D \sim -C$ would be effective and this is a contradiction; so by Serre Duality we would have $h^2(\mathcal{O}_{\widetilde{S}}(D)) = 0$. Furthermore by Riemann-Roch we would obtain

$$h^0(\mathcal{O}_{\widetilde{S}}(D)) \geq h^0(\mathcal{O}_{\widetilde{S}}(D)) - h^1(\mathcal{O}_{\widetilde{S}}(D)) = 2 > 0.$$

Thus $D$ would be effective, elliptic (by the adjunction formula) and with degree 2, which is a contradiction. This implies that $H$ is not numerically divisible by 2, so the only possible SID is $H \sim 2E_1 + E_{1,2}$.

(VII) $W = \overline{W}^5_{BS}$. Since $p = 5$ and $\phi^2 \leq 2p - 2$, then we have $1 \leq \phi \leq 2$. The map $\phi_{\mathcal{L}} : W \to \mathbb{P}^5$ is a morphism and it is a double cover of the image, which is the complete intersection of two quadric hypersurfaces (see [1, §6.1.2]). We observe that $\phi_{\mathcal{L}}|_S$ is superelliptic, because it is of degree 2 onto a quartic surface of $\mathbb{P}^4$. Hence we have $\phi = 2$, because if $\phi = 1$ the map would be hyperelliptic (see § 9.2 (a)). Then the SID is $H \sim 2(E_1 + E_2)$ (see Table 1), since $H$ has to be 2-divisible in $\mathrm{Pic}(S)$ (see [16, Theorem 4.7.1]).

(VIII) $W = W^6_{BS}$. Since $p = 6$ and $\phi^2 \leq 2p - 2$, then we have $1 \leq \phi \leq 3$. The map $\phi_{\mathcal{L}} : W \hookrightarrow \mathbb{P}^6$ is a morphism and an isomorphism to its image (see [1, §6.2.4]). Therefore we have $\phi = 3$, otherwise $\phi_{\mathcal{L}}|_S$ would not be an isomorphism to its image (see § 9.2 (c)). Then the SID is $H \sim E_1 + E_2 + E_{1,2}$ (see Table 1). We also recall that the F-EF threefold $W^6_F$ of [23, §3] is a limit of $W^6_{BS}$ (see [44, Main Theorem 2]).

(IX) $W = \overline{W}^7_{BS}$. Since $p = 7$ and $\phi^2 \leq 2p - 2$, then we have $1 \leq \phi \leq 3$. The map $\phi_{\mathcal{L}} : W \to \mathbb{P}^7$ is a morphism, it is birational onto its image but it is not an isomorphism onto its image, since there are points in the image with two preimages (see [1, §6.6.1]). Let us explain this fact better. In Bayle's description, an Enriques-Fano threefold $W$ of this type is given by the quotient $\pi : X \to X/\sigma =: W$ of $X := \mathbb{P}^1 \times S_4$, where $S_4 \subset \mathbb{P}^4$ is a Del Pezzo surface of degree 4 and $\sigma$ is an involution of $X$ with eight fixed points. In his analysis, Bayle introduces a morphism $\varphi : X \to \mathbb{P}^7$ such that we have the following commutative diagram

$$\begin{array}{ccc} X = \mathbb{P}^1 \times S_4 & & \\ \downarrow \pi & \searrow^{\varphi} & \\ W & \xrightarrow{\phi_{\mathcal{L}}} & \varphi(X) = \phi_{\mathcal{L}}(W) \subset \mathbb{P}^7. \end{array}$$

In particular a point $x \in \varphi(X)$ has two preimages in $X$, except in the case in which $x \in \varphi([0:1] \times S_4) \cup \varphi([1:0] \times S_4)$: in this case $\varphi^{-1}(x)$ is given by four points of $X$. Since $\pi : X \to W$ has degree 2, then $\phi_{\mathcal{L}}^{-1}(x)$ is given by one point if $x \in \phi_{\mathcal{L}}(W) \setminus (\varphi([0:1] \times S_4) \cup \varphi([1:0] \times S_4))$, otherwise it is given by two points. This implies $\phi = 2$ (see § 9.2 (b)). For the SID of $H$ we have a priori two possibilities, namely $H \sim 3E_1 + E_{1,2}$ and $H \sim 3E_1 + 2E_2$ (see Table 1).

**Remark 9.3.** In the case in which the SID is $H \sim 3E_1 + E_{1,2}$, the surface $S$ does not contain elliptic cubic curves. Indeed we have that $\deg E_1 = E_1 \cdot H = 2$ and $\deg E_{1,2} = E_{1,2} \cdot H = 6$. Furthermore let $E$ be an elliptic curve in $S$ such that it is not numerically equivalent to $E_1$, $E_{1,2}$, $2E_1$, $2E_{1,2}$. By [34, Lemma 2.1] we have that $E \cdot E_1 > 0$, $E \cdot E_{1,2} > 0$ and so $\deg E = E \cdot H \geq 3 + 1 = 4$.

**Remark 9.4.** In the case in which the SID is $H \sim 3E_1 + 2E_2$, the surface $S$ contains the following elliptic cubic curves: $E_2$ and $E_2 \sim E_2 + K_S$.

It is known that the surface $S_4$ is the image of $\mathbb{P}^2$ via the rational map $\lambda$ defined by the linear system of the plane cubic curves passing through five fixed points $a_1$, $a_2$, $a_3$, $a_4$, $a_5$ in general position. In particular $S_4 \cong \mathrm{Bl}_{a_1,a_2,a_3,a_4,a_5}\mathbb{P}^2$, where $bl : \mathrm{Bl}_{a_1,a_2,a_3,a_4,a_5}\mathbb{P}^2 \to \mathbb{P}^2$ is the blow-up of the plane at these five points. Let $\ell$ be the strict transform of a general line of $\mathbb{P}^2$ and let us consider the exceptional divisors $e_i := bl^{-1}(a_i)$ of $bl : S_4 \to \mathbb{P}^2$, for $1 \leq i \leq 5$. Let us take the K3-surface $\widetilde{S} := \pi^* S$. Then we have that

$$\pi|_{\widetilde{S}}^* H \sim -K_X|_{\widetilde{S}} \sim (2p \times S_4 + \mathbb{P}^1 \times (-K_{S_4}))|_{\widetilde{S}} \sim$$
$$\sim (2p \times S_4 + \mathbb{P}^1 \times (3\ell - e_1 - e_2 - e_3 - e_4 - e_5))\,|_{\widetilde{S}} =$$
$$= 2p \times S_4|_{\widetilde{S}} + \mathbb{P}^1 \times (\ell - e_5)|_{\widetilde{S}} + \mathbb{P}^1 \times (2\ell - e_1 - e_2 - e_3 - e_4)|_{\widetilde{S}}.$$

By setting $\overline{E}_1 := \mathbb{P}^1 \times (\ell - e_5)|_{\widetilde{S}}$, $\overline{E}_2 := \mathbb{P}^1 \times (2\ell - e_1 - e_2 - e_3 - e_4)|_{\widetilde{S}}$ and $\overline{E}_3 := p \times S_4|_{\widetilde{S}}$, we have $\pi|_{\widetilde{S}}^* H \sim \overline{E}_1 + \overline{E}_2 + 2\overline{E}_3$, where $\overline{E}_1^2 = \overline{E}_2^2 = \overline{E}_3^2 = 0$, $\overline{E}_1 \cdot \overline{E}_2 = 4$ and $\overline{E}_1 \cdot \overline{E}_3 = \overline{E}_2 \cdot \overline{E}_3 = 2$. Furthermore, by the adjunction formula, we have that $K_{\overline{E}_i} = 0$ and $p_g(\overline{E}_i) = 1$, for $1 \leq i \leq 3$. Let us suppose that there exists an elliptic cubic curve $E$ on $S$ and let us define $\overline{E} := \pi^{-1}(E)$. Since $E \cdot H = 3$ on $S$, then $\overline{E} \cdot \pi|_{\widetilde{S}}^* H = 6$ on $\widetilde{S}$. Obviously, we have that $\overline{E}$ is not linearly equivalent to $\overline{E}_1$, $\overline{E}_2$, $\overline{E}_3$, because $\overline{E}_1 \cdot \pi|_{\widetilde{S}}^* H = \overline{E}_2 \cdot \pi|_{\widetilde{S}}^* H = 8 \neq 6$ and $\overline{E}_3 \cdot \pi|_{\widetilde{S}}^* H = 4 \neq 6$. Since two elliptic curves on a K3 surface, which are not linearly equivalent, intersect at least in two points, then $\overline{E} \cdot \pi|_{\widetilde{S}}^* H \geq 2 + 2 + 2 \cdot 2 = 8 > 6$, which is a contradiction. Hence $S$ does not contain elliptic cubic curves and so, by Remarks 9.3, 9.4, the SID is $H \sim 3E_1 + E_{1,2}$ with $\phi = 2$.

(X) $W = W^7_{BS}$. Since $p = 7$ and $\phi^2 \leq 2p-2$, then we have $1 \leq \phi \leq 3$. The map $\phi_{\mathcal{L}} : W \hookrightarrow \mathbb{P}^7$ is a morphism and it is an isomorphism onto its image (see [1, §6.4.1]). This implies $\phi = 3$ (see § 9.2 (c)), which yields the SID $H \sim E_1 + E_2 + E_3 + E_4$ (see Table 1). See also [10, Lemma 4.6], where these threefolds are obtained via a *projection* technique from (XIV). The F-EF threefold $W^7_F$ of [23, §4] is a limit of $W^7_{BS}$ (see [44, Main Theorem 2]).

(XI) $W = W^8_{BS}$. Since $p = 8$ and $\phi^2 \leq 2p-2$, then we have $1 \leq \phi \leq 3$. The map $\phi_{\mathcal{L}} : W \hookrightarrow \mathbb{P}^8$ is a morphism and it is an isomorphism onto its image (see [1, §6.4.2]). This implies $\phi = 3$ (see § 9.2 (c)), which yields $H \sim 2E_1 + E_3 + E_{1,2}$ (see Table 1).

(XII) $W = W^9_{BS}$. Since $p = 9$ and $\phi^2 \leq 2p-2$, then we have $1 \leq \phi \leq 4$. The map $\phi_{\mathcal{L}} : W \hookrightarrow \mathbb{P}^9$ is a morphism and it is an isomorphism onto its image (see [1, §6.1.4]), which is the F-EF 3-fold $W^9_F$ of [23, §7] (see Theorem 6.11). Therefore one has $3 \leq \phi \leq 4$. In Bayle's description, an Enriques-Fano threefold $W$ of this type is given by the quotient $\pi : X \to X/\sigma =: W$ of the complete intersection $X$ of two quadric hypersurfaces of $\mathbb{P}^5$, where $\sigma$ is an involution of $X$ with eight fixed points. This implies that $H$ is numerically divisible by 2: indeed if $\widetilde{S}$ is the K3-surface $\pi^*S$, then $\pi|^*_{\widetilde{S}}H \sim -K_X|_{\widetilde{S}}$ where $-K_X$ is a quadric section of $X$. So we have $\phi = 4$ and $H \sim 2(E_1 + E_{1,2})$ or $H \sim 2(E_1 + E_{1,2}) + K_S$ (see Table 1). Furthermore Fano proves that these threefolds are represented on $\mathbb{P}^3$ by the linear system $\mathcal{K}$ of the septic surfaces which are double along the edges of two trihedra $T$ and $T'$. Let us use the notations of § 5.3. Let $\widetilde{K}$ be the divisor in $\widetilde{\mathcal{K}}$ such that $\nu_{\widetilde{\mathcal{K}}}(\widetilde{K}) = S$. There is only one cubic surface in $\mathbb{P}^3$ which is singular along the edges of the trihedron $T'$, that is $T'$ itself. So we have that

$$(3\mathcal{H} - \sum_{i=1}^{3} 2\mathcal{F}'_{ij} - \sum_{i,j=1}^{3} \mathcal{R}_{ij} - \sum_{i=1}^{3} 4\Gamma'_i - \sum_{\substack{i,j,k\in\{1,2,3\}\\ i<j,\ h=i,j}} (\Lambda_{ijk,h} + 3\Lambda'_{ijk,h}))|_{\widetilde{K}} \sim 0.$$

Then we have that $H$ corresponds to the following divisor on $\widetilde{K}$:

$$(7\mathcal{H} - \sum_{1\leq i<j\leq 3} 2(\mathcal{F}_{ij}+\mathcal{F}'_{ij}) - \sum_{i,j=1}^{3} \mathcal{R}_{ij} - \sum_{i=1}^{3} 4(\Gamma_i+\Gamma'_i) - \sum_{\substack{i,j,k=1\\ i<j,\, h=i,j}}^{3} 3(\Lambda_{ijk,h}+\Lambda'_{ijk,h}))|_{\widetilde{K}} \sim$$

$$\sim (4\mathcal{H} - \sum_{1\leq i<j\leq 3} 2\mathcal{F}_{ij} - \sum_{i=1}^{3} 4\Gamma_i - \sum_{\substack{i,j,k=1\\ i<j}}^{3} 2\Lambda_{ijk,i})|_{\widetilde{K}}.$$

This implies that $H$ is 2-divisible and the only possible SID is $H \sim 2(E_1 + E_{1,2})$.

(XIII) $W = W^{10}_{BS}$. Since $p = 10$ and $\phi^2 \leq 2p-2$, then we have $1 \leq \phi \leq 4$. The map $\phi_{\mathcal{L}} : X \hookrightarrow \mathbb{P}^{10}$ is a morphism and it is an isomorphism onto its image (see [1,

§6.5.1]). Therefore one has $3 \leq \phi \leq 4$ (see § 9.2 (c)). The possible cases of the SID of $H$ are $H \sim 2E_1+E_2+E_3+E_4$, $H \sim 3(E_1+E_2)$ and $H \sim 2E_{1,2}+E_1+E_2$ (see Table 1). We recall that an Enriques-Fano threefold $W$ of this type is given by the quotient $\pi : X \to X/\sigma =: W$ of $X := \mathbb{P}^1 \times S_6$, where $S_6$ is a smooth Del Pezzo surface of degree 6 in $\mathbb{P}^6$ and $\sigma$ is an involution of $X$ with eight fixed points. It is known that the surface $S_6$ is the image of $\mathbb{P}^2$ via the rational map $\lambda$ defined by the linear system of the plane cubic curves passing through three fixed points $a_1$, $a_2$, $a_3$ in general position. In particular $S_6 \cong \mathrm{Bl}_{a_1,a_2,a_3}\mathbb{P}^2$, where $bl : \mathrm{Bl}_{a_1,a_2,a_3}\mathbb{P}^2 \to \mathbb{P}^2$ is the blow-up of the plane at these three points. Let $\ell$ be the strict transform of a general line of $\mathbb{P}^2$ and let us consider the exceptional divisors $e_i = bl^{-1}(a_i)$ of $bl : S_6 \to \mathbb{P}^2$, for $1 \leq i \leq 3$. Let us take the K3-surface $\widetilde{S} := \pi^*S$. Then we have that

$$\pi|_{\widetilde{S}}^* H \sim -K_X|_{\widetilde{S}} \sim (2p \times S_6 + \mathbb{P}^1 \times (-K_{S_6}))|_{\widetilde{S}} =$$
$$= (2p \times S_6 + \mathbb{P}^1 \times (3\ell - e_1 - e_2 - e_3))\,|_{\widetilde{S}} =$$
$$= 2p \times S_6|_{\widetilde{S}} + \mathbb{P}^1 \times (\ell - e_1)|_{\widetilde{S}} + \mathbb{P}^1 \times (\ell - e_2)|_{\widetilde{S}} + \mathbb{P}^1 \times (\ell - e_3)|_{\widetilde{S}}.$$

By setting $\overline{E}_1 := p \times S_6|_{\widetilde{S}}$ and $\overline{E_i} := \mathbb{P}^1 \times (\ell - e_i)|_{\widetilde{S}}$, for $2 \leq i \leq 4$, we have $\pi|_{\widetilde{S}}^* H \sim 2\overline{E}_1 + \overline{E}_2 + \overline{E}_3 + \overline{E}_4$, where $\overline{E}_i^2 = 0$ and $\overline{E}_i \cdot \overline{E}_j = 2$, for $1 \leq i < j \leq 4$. Furthermore, by the adjunction formula, we have that $K_{\overline{E}_i} = 0$ and $p_g(\overline{E}_i) = 1$, for $1 \leq i \leq 4$. We will now prove that the SID is $H \sim 2E_1 + E_2 + E_3 + E_4$. If the SID were $H \sim 3(E_1 + E_2)$, then $H$ would be 3-divisible and therefore also $\pi|_{\widetilde{S}}^* H$. But this does not happen because $\pi|_{\widetilde{S}}^* H \cdot \overline{E}_2 = 8$ is not divisible by 3. Now suppose $H \sim 2E_{1,2} + E_1 + E_2$. By setting $\widetilde{E}_{1,2} := \pi|_{\widetilde{S}}^* E_{1,2}$, $\widetilde{E}_1 := \pi|_{\widetilde{S}}^* E_1$ and $\widetilde{E}_2 := \pi|_{\widetilde{S}}^* E_2$, we have $\widetilde{E}_{1,2} \cdot \widetilde{E}_1 = \widetilde{E}_{1,2} \cdot \widetilde{E}_2 = 4$ and $\widetilde{E}_1 \cdot \widetilde{E}_2 = 2$. Hence $\widetilde{E}_{1,2} \cdot \pi|_{\widetilde{S}}^* H = 8$ and $\widetilde{E}_1 \cdot \pi|_{\widetilde{S}}^* H = \widetilde{E}_2 \cdot \pi|_{\widetilde{S}}^* H = 10$. Let $D$ be any elliptic curve on $S$ such that $D^2 = 0$ and that is not linearly equivalent to $\widetilde{E}_{1,2}$, $\widetilde{E}_1$, $\widetilde{E}_2$: then $D \cdot \pi|_{\widetilde{S}}^* H \geq 2 \cdot 2 + 2 + 2 = 8$, since two elliptic curves on a K3 surface, which are not linearly equivalent, intersect at least in two points. But if we took $D = \overline{E}_1$, we would obtain $\overline{E}_1 \cdot \pi|_{\widetilde{S}}^* H = 6 < 8$, which is a contradiction. Then it must be $H \sim 2E_1 + E_2 + E_3 + E_4$ with $\phi = 3$.

(XIV) $W = W_{BS}^{13}$. Since $p = 13$ and $\phi^2 \leq 2p - 2$, then we have $1 \leq \phi \leq 4$. The map $\phi_{\mathcal{L}} : W \hookrightarrow \mathbb{P}^{13}$ is a morphism and it is an isomorphism onto its image (see [1, §6.3.2]), which is the F-EF 3-fold $W_F^{13}$ of [23, §8] (see Theorem 6.17). Therefore one has $3 \leq \phi \leq 4$ (see § 9.2 (c)). According to Bayle, an Enriques-Fano threefold $W$ of this type is given by the quotient $\pi : X \to X/\sigma =: W$ of $X := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ under an involution $\sigma$ of $X$ with eight fixed points. So Bayle's description implies that $H$ is numerically divisible by 2: indeed if $\widetilde{S}$ is the K3-surface $\pi^*S$, then $\pi|_{\widetilde{S}}^* H \sim -K_X|_{\widetilde{S}} \sim (2,2,2)|_{\widetilde{S}}$. Furthermore Fano proves that these threefolds are represented on $\mathbb{P}^3$ by the linear system $\mathcal{S}$ of sextic surfaces singular along the edges of a fixed tetrahedron $T$. Fano's description shows that $H$ is 2-divisible in Pic($S$). Indeed, by using notations of § 5.2 and by taking $\widetilde{\Sigma}$ the divisor in $\widetilde{\mathcal{S}}$ such that $\widetilde{\nu}(\widetilde{\Sigma}) = S$, then we have that $H$ corresponds to the following divisor on $\widetilde{\Sigma}$: $(6\mathcal{H} - \sum_{0\leq i<j\leq 3} 2\mathcal{F}_{ij} - \sum_{\substack{i,j=0\\ i\neq j}}^{2} 4\Gamma_{ij})|_{\widetilde{\Sigma}}$. So the only possible case is $\phi = 4$ and $H \sim 2(E_1 + E_2 + E_3)$ (see Table 1).

(XV) $W = W^9_{KLM}$. These threefolds are obtained by *projection* of the threefolds in (XIV) from, say, the curve $E_3$ (see [36, §13]). Then $H \sim 2(E_1 + E_2) + E_3$ and $\phi = 3$. The general Enriques surface appears as a hyperplane section of these threefolds.

(XVI) $W = W^{13}_P$. Since $p = 13$ and $\phi^2 \leq 2p - 2$, then we have $1 \leq \phi \leq 4$. Let us consider a cone $V \subset \mathbb{P}^7$ over a smooth sextic Del Pezzo surface $S_6$ contained in a hyperplane $\mathbb{P}^6$ of $\mathbb{P}^7$. An Enriques-Fano threefold $W$ of this type is given by the quotient $\pi : V \to V/\tau =: W$ of $V \subset \mathbb{P}^7$ where $\tau : V \to V$ is an involution fixing five points, one of which is the vertex $v$ of the cone (see [46, Remark 3.3]). In particular $V$ is a Gorenstein Fano threefold with canonical singularity at $v$ and with anticanonical divisor $-K_V = 2M$, where $M$ is the class of hyperplane sections (see Lemma 8.2). Furthermore the quotient map $\pi : V \to W$ is defined by the base point free sublinear system $\mathcal{Q} \subset |2M| = |-K_V|$ of the $\tau$-invariant quadric sections of $V$ such that a general member $\widetilde{S} \in \mathcal{Q}$ is a smooth K3 surface not containing the five $\tau$-fixed points and on which the action of $\tau$ is fixed point free. So we have $\mathcal{L} := |\mathcal{O}_W(S)|$, where $S$ is the Enriques surface $\pi(\widetilde{S}) = \widetilde{S}/\tau$. Since $S \subset \mathbb{P}^{12}$ is 1-extendable to $W \subset \mathbb{P}^{13}$ (see § 8.2), then $3 \leq \phi \leq 4$ (see [16, Theorem 4.6.1]). We recall that the surface $S_6$ is isomorphic to the blow-up $bl : \mathrm{Bl}_{a_1,a_2,a_3}\mathbb{P}^2 \to \mathbb{P}^2$ of the plane at three fixed points $a_1$, $a_2$, $a_3$ in general position. Let $\ell$ be the strict transform of a general line of $\mathbb{P}^2$ and let $e_i := bl^{-1}(a_i)$, for $1 \leq i \leq 3$. Since $\widetilde{S}$ is a K3 surface given by a particular quadric section of $V$ and $S_6$ is a hyperplane section of $V$, we have a double cover $f : \widetilde{S} \to S_6$ with ramification locus $R_f = f^*(-K_{S_6}) = f^*(3\ell - e_1 - e_2 - e_3)$ and branch locus $B_f = -2K_{S_6} = 6\ell - 2e_1 - 2e_2 - 2e_3$. Furthermore, by setting $\overline{E}_i := f^*(\ell - e_i)$ for $1 \leq i \leq 3$, we have $M|_{\widetilde{S}} \sim f^*(-K_{S_6}) = f^*(3\ell - e_1 - e_2 - e_3) \sim$
$\sim f^*(\ell - e_1 + \ell - e_2 + \ell - e_3) = f^*(\ell - e_1) + f^*(\ell - e_2) + f^*(\ell - e_3) =$
$= \overline{E}_1 + \overline{E}_2 + \overline{E}_3 =: D$, where $D^2 = 12$, $\overline{E}_i \cdot \overline{E}_j = 2$ for $1 \leq i < j \leq 3$ and $\overline{E}_i^2 = 0$. So $\overline{E}_i$ is an elliptic curve for $1 \leq i \leq 3$. Since $\pi|^*_{\widetilde{S}}H = (\pi^*S)|_{\widetilde{S}} \sim \widetilde{S}|_{\widetilde{S}} \sim$
$\sim 2M|_{\widetilde{S}} \sim 2D = 2(\overline{E}_1 + \overline{E}_2 + \overline{E}_3)$, then $\pi|^*_{\widetilde{S}}H$ is 2-divisible on the K3 surface $\widetilde{S}$ and $H$ is numerically 2-divisible on the Enriques surface $S$, i.e. $H$ or $H + K_S$ is 2-divisible on $S$. Then we have only the following possibile SID: $H \sim 2(E_1+E_2+E_3)$ or $H \sim 2(E_1 + E_2 + E_3) + K_S$ (see Table 1). In both cases, if we consider the elliptic curves $\widetilde{E}_i := \pi^*(E_i)$ for $1 \leq i \leq 3$, we have $\widetilde{H} := \pi|^*_{\widetilde{S}}H = 2(\widetilde{E}_1+\widetilde{E}_2+\widetilde{E}_3)$, where $\widetilde{E}_i \cdot \widetilde{E}_j$ is 2 for $i \neq j$ and it is 0 for $i = j$.

**Remark 9.5.** We have that $\overline{E}_i = \widetilde{E}_i$, for $1 \leq i \leq 3$. Indeed, since $2(\widetilde{E}_1 + \widetilde{E}_2 + \widetilde{E}_3) \sim \pi|^*_{\widetilde{S}}H \sim 2(\overline{E}_1 + \overline{E}_2 + \overline{E}_3)$, we have $\widetilde{E}_1 + \widetilde{E}_2 + \widetilde{E}_3 \sim \overline{E}_1 + \overline{E}_2 + \overline{E}_3$ on the

K3 surface $\widetilde{S}$. Let us suppose $\overline{E}_1 \neq \widetilde{E}_i$ for $1 \leq i \leq 3$, so $\overline{E}_1 \cdot \widetilde{E}_i \geq 2$. Then $4 = \overline{E}_1 \cdot (\overline{E}_1 + \overline{E}_2 + \overline{E}_3) = \overline{E}_1 \cdot (\widetilde{E}_1 + \widetilde{E}_2 + \widetilde{E}_3) \geq 6$, which is a contradiction.

It remains to understand which case really occurs. See § 9.4 for some remarks.

(XVII) $W = W_P^{17}$ - see [46, §3]. By [10, Proposition 4.7] the SID is $H \sim 4(E_1 + E_2)$ and $\phi = 4$.

□

## 9.4 Remarks concerning the SID of the curve sections of the P-EF 3-fold (XVI) of genus 13

The determination of the SID of a general curve section of the P-EF 3-fold $W_P^{13}$ remains an open question. Let us see some results that may be useful in the future.

**Theorem 9.6.** Let $S$ be a general hyperplane section of the P-EF 3-fold $W_P^{13}$ and let $H$ be a general curve section of $W_P^{13}$ on $S$. Let $\pi : \widetilde{S} \to S$ be the K3 double cover and $\widetilde{H} := \pi^* H$. Then $1 \leq h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) \leq 2$.

*Proof.* Let $F_1$ and $F_2$ be two half-fibres on the Enriques surface $S$ such that $F_1 \cdot F_2 = 1$ and let be $\widetilde{R} \sim 2\widetilde{F}_1 + 2\widetilde{F}_2$ where $\widetilde{F}_i := \pi^* F_i$, for $i = 1, 2$. Let us set

$$\alpha := h^1(\mathcal{O}_S(H-2F_1)) + h^1(\mathcal{O}_S(H-2F_1+K_S)) + h^1(\mathcal{O}_S(H-2F_2)) + h^1(\mathcal{O}_S(H-2F_2+K_S))$$

and $\beta := h^0(\mathcal{O}_{\widetilde{R}}(4\widetilde{F}_1 + 4\widetilde{F}_2 - \widetilde{H}))$. By [10, Lemma 5.2] we have that

$$\beta \leq h^0(\mathcal{O}_S(4F_1 + 4F_2 - H)) + h^0(\mathcal{O}_S(4F_1 + 4F_2 - H + K_S)) + \\ + h^1(\mathcal{O}_S(H - 2F_1 - 2F_2)) + h^1(\mathcal{O}_S(H - 2F_1 - 2F_2 + K_S))$$

and $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) \leq \alpha + \beta$, with equality if $\alpha = 0$. Let us use now the same notations of the case (XVI) of the proof of Theorem 9.2. We have $\widetilde{H} \sim 2\widetilde{E}_1 + 2\widetilde{E}_2 + 2\widetilde{E}_3$. Let $E_i := \pi(\widetilde{E}_i)$ for $1 \leq i \leq 3$ and let us take $F_1 = E_1$ and $F_2 = E_2$. Then we have that

$$\alpha = h^1(\mathcal{O}_S(2E_2 + 2E_3 + K_S)) + h^1(\mathcal{O}_S(2E_2 + 2E_3)) + h^1(\mathcal{O}_S(2E_1 + 2E_3 + K_S)) +$$

$$+ h^1(\mathcal{O}_S(2E_1 + 2E_3)) = h^1(\mathcal{O}_{\widetilde{S}}(2\widetilde{E}_2 + 2\widetilde{E}_3)) + h^1(\mathcal{O}_{\widetilde{S}}(2\widetilde{E}_1 + 2\widetilde{E}_3)) = 0.$$

So we obtain the equality $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) = \beta$ where $\beta = h^0(\mathcal{O}_{\widetilde{R}}(2\widetilde{E}_1 + 2\widetilde{E}_2 - 2\widetilde{E}_3))$. Let us consider the following exact sequence

$$0 \to \mathcal{O}_{\widetilde{S}}(2\widetilde{F}_1 + 2\widetilde{F}_2 - \widetilde{H}) \to \mathcal{O}_{\widetilde{S}}(4\widetilde{F}_1 + 4\widetilde{F}_2 - \widetilde{H}) \to \mathcal{O}_{\widetilde{R}}(2\widetilde{R} - \widetilde{H}) \to 0$$

that is

$$0 \to \mathcal{O}_{\widetilde{S}}(-2\widetilde{E}_3) \to \mathcal{O}_{\widetilde{S}}(2\widetilde{E}_1 + 2\widetilde{E}_2 - 2\widetilde{E}_3) \to \mathcal{O}_{\widetilde{R}}(2\widetilde{R} - \widetilde{H}) \to 0.$$

Since $h^0(\mathcal{O}_{\widetilde{S}}(-2\widetilde{E}_3)) = 0$, then $\beta = h^0(\mathcal{O}_{\widetilde{R}}(2\widetilde{R} - \widetilde{H})) \geq h^0(\mathcal{O}_{\widetilde{S}}(2\widetilde{E}_1 + 2\widetilde{E}_2 - 2\widetilde{E}_3))$. Since $\widetilde{E}_i = \overline{E}_i = f^*(\ell - e_i)$ for $1 \leq i \leq 3$ (see Remark 9.5), then $2\widetilde{E}_1 + 2\widetilde{E}_2 -$

$2\widetilde{E}_3 = 2f^*(\ell - e_1 - e_2 + e_3)$ is an effective divisor on $\widetilde{S}$ and in particular we have $h^0(\mathcal{O}_{\widetilde{S}}(2\widetilde{E}_1 + 2\widetilde{E}_2 - 2\widetilde{E}_3)) = 1$. Moreover we have

$$\beta \le h^0(2E_1 + 2E_2 - 2E_3) + h^0(2E_1 + 2E_2 - 2E_3 + K_S) + h^1(2E_3) + h^1(2E_3 + K_S) =$$
$$= h^0(2\widetilde{E}_1 + 2\widetilde{E}_2 - 2\widetilde{E}_3) + h^1(2\widetilde{E}_3) = 1 + 1 = 2.$$

□

**Remark 9.7.** Let $S$ be a general hyperplane section of the P-EF 3-fold $W_P^{13}$ and let $H$ be a general curve section of $W_P^{13}$ on $S$. Let us use the same notations of the case (XVI) of the proof of Theorem 9.2. By setting $\widetilde{H} := \pi|_{\widetilde{S}}^* H$, we have that $h^1(\mathcal{T}_{\widetilde{S}}(-\widetilde{H})) = h^1(\mathcal{T}_S(-H)) + h^1(\mathcal{T}_S(-H + K_S))$ (see [10, (9)]). In our case it must be

$$h^1(\mathcal{T}_{\widetilde{S}}(-2\widetilde{E}_1 - 2\widetilde{E}_2 - 2\widetilde{E}_3)) = h^1(\mathcal{T}_S(-2E_1 - 2E_2 - 2E_3)) + h^1(\mathcal{T}_S(-2E_1 - 2E_2 - 2E_3 + K_S)).$$

Since a general (unnodal) element of $\mathcal{E}_{13,4}^{(II)^+}$ is extendable to the classical Enriques-Fano threefold, then by semicontinuity we have that $h^1(\mathcal{T}_S(-2E_1 - 2E_2 - 2E_3)) \ge 1$. By Theorem 9.6 we obtain the following possibilities:

(i) $h^1(\mathcal{T}_S(-2E_1 - 2E_2 - 2E_3)) = 1$ and $h^1(\mathcal{T}_S(-2E_1 - 2E_2 - 2E_3 + K_S)) = 0$; hence $S$ would be an Enriques surface extendable to $W_P^{13}$ such that $(S,H) \in \mathcal{E}_{13,4}^{(II)^+}$;

(ii) $h^1(\mathcal{T}_S(-2E_1 - 2E_2 - 2E_3)) = 2$ and $h^1(\mathcal{T}_S(-2E_1 - 2E_2 - 2E_3 + K_S)) = 0$; hence $S$ is an Enriques surface extendable to $W_P^{13}$ such that $(S,H) \in \mathcal{E}_{13,4}^{(II)^+}$;

(iii) $h^1(\mathcal{T}_S(-2E_1 - 2E_2 - 2E_3)) = 1$ and $h^1(\mathcal{T}_S(-2E_1 - 2E_2 - 2E_3 + K_S)) = 1$; hence $S$ is an Enriques surface extendable to $W_P^{13}$ such that $(S,H) \in \mathcal{E}_{13,4}^{(II)^+}$ or $\mathcal{E}_{13,4}^{(II)^-}$.

**Remark 9.8.** Ciliberto-Dedieu-Galati-Knutsen show that if $S$ is an unnodal Enriques surface such that $(S, \mathcal{O}_S(1))$ belongs to $\mathcal{E}_{13,4}^{(II)^-}$, then $h^1(\mathcal{T}_S(-1)) = 0$ and so it is not extendable (see [10, proof of Theorem 1]). Let $S$ be a general hyperplane section of the P-EF 3-fold $W_P^{13}$ and let $H$ be a general curve section of $W_P^{13}$ on $S$. By Theorem 9.2 we have that $(S,H) \in \mathcal{E}_{13,4}^{(II)^+}$ or $(S,H) \in \mathcal{E}_{13,4}^{(II)^-}$. Since $(S,H)$ is 1-extendable to $W_P^{13}$ by construction, then it must be $h^1(\mathcal{T}_S(-H)) > 0$. Thus, if $(S,H)$ belonged to $\mathcal{E}_{13,4}^{(II)^-}$, it should be $S$ nodal, that is $S$ should contain some smooth rational curve. Anyway, let us use the same notations of the case (XVI) of the proof of Theorem 9.2. We have that, in both cases $(S,H) \in \mathcal{E}_{13,4}^{(II)^+}$ or $(S,H) \in \mathcal{E}_{13,4}^{(II)^-}$, the K3 surface $\widetilde{S} = \pi^* S$ contains smooth rational curves. For example $\widetilde{S}$ contains the $(-2)$-curves given by $f^*(\ell - e_1 - e_2)$, $f^*(\ell - e_1 - e_3)$, $f^*(\ell - e_2 - e_3)$, $f^*(e_1)$, $f^*(e_2)$, $f^*(e_3)$.

# 10 On Enriques-Fano threefolds and a conjecture of Castelnuovo

## 10.1 Abstract

Let $\mathcal{L}$ be an $r$-dimensional linear system of surfaces on $\mathbb{P}^3$ such that (the desingularization of) the general element has zero geometric genus and zero arithmetic genus.

What happens if we force the surfaces of $\mathcal{L}$ to have a triple point at a general point of $\mathbb{P}^3$? Castelnuovo conjectured in [4, pp. 187-188] that we get an $(r-10)$-dimensional sublinear system $\mathcal{L}_\bullet$ such that the general surface satisfies one of the following three properties: it is an irreducible surface with irregular desingularization, with zero geometric genus and with arithmetic genus equal to $-1$; it is reducible in two rational surfaces intersecting along a rational curve; it has the same genera as a general element of $\mathcal{L}$. First we will apply the arguments of Castelnuovo to (rational) regular smooth irreducible threefolds (see § 10.2), and then to (rational) normal threefolds with isolated singularities and regular desingularization (see § 10.3). In particular we will analyze the sublinear system $\mathcal{L}_\bullet \subset \mathcal{L}$ of the hyperplane sections of the classical Enriques-Fano threefold $(W_F^{13}, \mathcal{L})$ having a triple point at a general point $w \in W_F^{13}$ (see § 10.4). We will find that the general element of this linear system satisfies the first property conjectured by Castelnuovo, since it is birational to an elliptic ruled surface. Furthemore we will prove that the image of $W_F^{13}$ via the rational map defined by $\mathcal{L}_\bullet$ is the Cayley cubic surface (see Theorem 10.25). Finally we will observe that also by imposing a triple point at the general hyperplane section of the P-EF 3-fold $W_P^{17}$, we obtain a surface whose desingularization has $q = 1$, $p_g = 0$ and $p_a = -1$ (see § 10.5).

## 10.2 Castelnuovo's conjecture for smooth threefolds

In [4, pp.187-188], Castelnuovo proposed some ideas about certain irreducible threefolds and particular linear systems of surfaces on them. In order to explain these ideas, let us start by talking about the link between the irregularity of a surface contained in a threefold and the one of the threefold itself, which was studied in [5, §4].

**Proposition 10.1.** Let $W$ be a smooth irreducible threefold endowed with an $r$-dimensional linear system $\mathcal{L}$, where $r \geq 2$, such that the general element is an irreducible surface. If the divisors of $\mathcal{L}$ are big and nef, then $W$ has the same irregularity of a general surface $S \in \mathcal{L}$.

*Proof.* Let $S$ be a general element of $\mathcal{L}$ and let us consider the following exact sequence

$$0 \to \mathcal{O}_W(-S) \to \mathcal{O}_W \to \mathcal{O}_S \to 0.$$

Since $S$ is a big and nef divisor, then we have that $h^{i=1,2}(\mathcal{O}_W(-S)) = 0$ by the Kawamata-Viehweg vanishing theorem and so we obtain $q(W) = q(S)$. □

**Remark 10.2.** Let $W$ be a smooth irreducible threefold and let $\mathcal{L}$ be an $r$-dimensional linear system on $W$ such that $r \geq 2$ and such that the general element is a smooth irreducible surface. Let us suppose that $\mathcal{L}$ has *base curves*. We can consider the appropriate blow-ups of $W$ along these curves in order to obtain a birational morphism $\overline{bl} : \overline{W} \to W$ such that the strict transform $\overline{\mathcal{L}}$ of $\mathcal{L}$ has no base curves. Let $\overline{S}$ be an element of $\overline{\mathcal{L}}$. Since $\mathcal{L}$ could have base points outside the base curves, then $\overline{S}$ is a (semi-ample and hence) nef divisor by the Zariski-Fujita theorem. Furthemore $\overline{S}$ is the strict transform of an element $S \in \mathcal{L}$. If $S$ is a big divisor, then $\overline{S}$ is big too.

Some consequences of Proposition 10.1 are stated in [5, §6] for the three-dimensional projective space $W = \mathbb{P}^3$. We will adapt them to any regular smooth irreducible threefold.

**Proposition 10.3.** Let $W$ be a regular smooth irreducible threefold endowed with an $r$-dimensional linear system $\mathcal{L}$, where $r \geq 3$, such that the general element is a smooth irreducible surface. If the intersection of two general surfaces of $\mathcal{L}$, outside the base locus, is an irreducible curve, then a general element $S \in \mathcal{L}$ is a regular surface.

*Proof.* We may assume that the base locus of $\mathcal{L}$ is empty or at worst a finite set. Indeed, if this were not the case, we could continue the proof with the pair $(\overline{W}, \overline{\mathcal{L}})$ of Remark 10.2 instead of the pair $(W, \mathcal{L})$. This would be possible since a general $\overline{S} \in \overline{\mathcal{L}}$ is a smooth surface isomorphic to a general $S \in \mathcal{S}$ and such that $q(\overline{S}) = q(S)$.

Let us fix now a general $S \in \mathcal{L}$, which is a nef divisor on $W$ by the Zariski-Fujita theorem. Let us suppose that $S$ is an irregular surface. Let $\Delta \subseteq \mathbb{P}^r$ be the image of the rational map $\phi_{\mathcal{L}} : W \dashrightarrow \mathbb{P}^r$ defined by $\mathcal{L}$. Since $r \geq 3 > 1$ and $S$ is an irreducible surface by hypothesis, then $\dim \Delta > 1$. Since $S$ is not a big divisor (see Proposition 10.1), then $\dim \Delta < 3$. So $\Delta$ is a surface and the general fibre of $\phi_{\mathcal{L}}$ is a curve. Let $S'$ be another general element of $\mathcal{L}$. The intersection curve $S \cap S'$ is sent by $\phi_{\mathcal{L}}$ to the intersection of two general hyperplane sections of $\Delta$, that is a set of $d := \deg \Delta$ points of $\Delta$. We observe that $\Delta$ cannot be a plane, since $r \geq 3 > 2$. Hence we have that $S \cap S'$ is a reducible curve given by $d \geq 2$ fibres of $\phi_{\mathcal{L}}$. Since this is a contradiction with the hypothesis, then $S$ must be regular. □

We recall that a one-dimensional linear system on a variety $X$ is called *pencil*. In the following we will extend the use of this term. Let $S$ be a smooth surface and let $B$ be a smooth curve of genus $b \geq 0$. A surjective rational map $f : S \dashrightarrow B$ with connected fibres is called a *pencil of genus* $b$ of curves on $S$. All the curves of such a pencil are linearly equivalent if and only if $b = 0$. In this case we will refer to it as *rational pencil*. If $b > 0$ we will talk about *irrational pencil* and, in this case, $f$ is a morphism (see [2, p.114]). In particular, an irrational pencil of genus one is called *elliptic pencil*.

**Definition 10.4.** A *congruence of curves* of a threefold $W$ is a two-dimensional irreducible family $\mathcal{V}$ of curves contained in $W$ such that through a general point of $W$ passes only one curve of the family.

**Proposition 10.5.** Let $W$ be a regular smooth irreducible threefold endowed with an $r_\bullet$-dimensional linear system $\mathcal{L}_\bullet$, where $r_\bullet \geq 3$, such that the general element is an irregular smooth irreducible surface. Then two general elements $S_\bullet$ and $S'_\bullet$ of $\mathcal{L}_\bullet$ intersect each other (outside the base locus) along reducible curves. In particular on a fixed $S_\bullet$ the components of these curves are fibres of a pencil of genus $b$ where $0 \leq b \leq q(S_\bullet)$. Furthermore, by varying the surface $S_\bullet$, these component curves give a congruence $\mathcal{V}$ of curves of $W$.

*Proof.* We may assume that $\mathcal{L}_\bullet$ is base point free. Indeed if $\mathcal{L}_\bullet$ had base curves, then we would take the pair $(\overline{W}, \overline{\mathcal{L}}_\bullet)$ as in Remark 10.2, where $\overline{\mathcal{L}}_\bullet$ has no base curves. If $\overline{\mathcal{L}}_\bullet$ still had a finite set of base points, then we would consider the blow-ups necessary to have a birational morphism $\widetilde{bl} : \widetilde{W} \to \overline{W}$ such that the strict transform $\widetilde{\mathcal{L}}_\bullet$ of $\overline{\mathcal{L}}_\bullet$ is base point free. Thus we continue the proof by denoting the pair $(\widetilde{W}, \widetilde{\mathcal{L}}_\bullet)$ by $(W, \mathcal{L}_\bullet)$: this is possible since a general surface $\widetilde{S}_\bullet \in \widetilde{\mathcal{L}}_\bullet$ is birational to a general surface $S_\bullet \in \mathcal{L}_\bullet$ and they have same irregularity. Since the divisors of $\mathcal{L}_\bullet$ are not big (see Proposition 10.1), then the image of the morphism $\phi_\bullet : W \to \mathbb{P}^{r_\bullet}$ defined by $\mathcal{L}_\bullet$ is not a threefold. Moreover, since $r_\bullet \geq 3 > 1$ and the elements of $\mathcal{L}_\bullet$ are generically irreducible, then $\phi_\bullet(W)$ is not even a curve. The image of $W$ via $\phi_\bullet$ is thus a surface $\Delta$ and a general $S_\bullet \in \mathcal{L}_\bullet$ is sent via $\phi_\bullet$ to a curve $\Gamma$, which is a general hyperplane section of $\Delta$. Since $S_\bullet$ is smooth, the morphism $\phi_\bullet|_{S_\bullet} : S_\bullet \to \Gamma$ factorizes via the normalization $n : B \to \Gamma$ of $\Gamma$, i.e. there exist a morphism $\psi : S_\bullet \to B$ such that $\phi_\bullet|_{S_\bullet} = n \circ \psi$. Furthermore the fibres of $\psi : S_\bullet \to B$ are generically equal to the ones of $\phi_\bullet|_{S_\bullet} : S_\bullet \to \Gamma$. The curves on $S_\bullet$ given by the intersection with another general element of $\mathcal{L}_\bullet$ are reducible (see Proposition 10.3) and they are fibres of the map $\phi_\bullet|_{S_\bullet} : S_\bullet \to \Gamma$. We observe that $0 \leq b := p_g(B) = p_g(\Gamma) \leq q(S_\bullet)$, since we have the injection $H^0(\Omega^1_\Gamma) \hookrightarrow H^0(\Omega^1_{S_\bullet})$. Finally, by varying the surface $S_\bullet$, we obtain that the fibres of the morphism $\phi_\bullet : W \to \Delta \subset \mathbb{P}^{r_\bullet}$ give a two dimensional family $\mathcal{V}$ such that through a general point $w \in W$ passes only one curve of the family, that is $\phi_\bullet^{-1}(\phi_\bullet(w))$. □

If we take $W = \mathbb{P}^3$ as in [5, §6], or more in general a *rational* smooth irreducible threefold, instead of any regular smooth irreducible threefold, we obtain an additional property. Let us see which one.

**Remark 10.6.** Let $(W, \mathcal{L}_\bullet)$ be a pair given by a threefold and a linear system satisfying the hypothesis of Proposition 10.5. If $W$ is rational, the congruence $\mathcal{V}$ of curves of $W$ is parametrized by a rational surface $R$. Let us explain why. Through a general point $w \in W$ passes only one curve $\gamma_w \in \mathcal{V}$ (see Definition 10.4). If $R$ is the surface parametrizing the curves of $\mathcal{V}$, let $r_w$ be the point of $R$ corresponding to the curve $\gamma_w$. We have a dominant rational map $W \dashrightarrow R$ such that $w \mapsto r_w$. Since $W$ is rational, then $R$ is unirational, and so, as consequence of the Castelnuovo Rationality criterion, $R$ is rational.

**Castelnuovo's conjecture.** Let us take a rational smooth irreducible threefold $W$ and an $r$-dimensional linear system $\mathcal{L}$ on $W$ such that a general $S \in \mathcal{L}$ is a smooth irreducible surface with zero geometric genus $p_g(S) = 0$ and zero arithmetic genus $p_a(S) = 0$. Let $\mathcal{L}_\bullet$ be the sublinear system of $\mathcal{L}$ given by the surfaces of $\mathcal{L}$ having a triple point at a general point $w \in W$. Then the linear system $\mathcal{L}_\bullet$ has dimension $r - 10$ and one of the following conditions occurs:

(A) a general element $S_\bullet \in \mathcal{L}_\bullet$ is an irreducible surface which has irregular desingularization $\widetilde{S}$ with $q(\widetilde{S}_\bullet) = 1$, $p_g(\widetilde{S}_\bullet) = 0$ and $p_a(\widetilde{S}_\bullet) = -1$;

(B) the surfaces $S_\bullet \in \mathcal{L}_\bullet$ are reducible in the union $S_\bullet = F_\bullet \cup M_\bullet$ of two rational surfaces passing through the point $w$, where the surface $M_\bullet$ changes by varying $S_\bullet$, the surface $F_\bullet$ is fixed and $F_\bullet \cap M_\bullet$ is a rational curve;

(C) the surfaces $S_\bullet \in \mathcal{L}_\bullet$ have the same genera as a general $S \in \mathcal{L}$.

Let us suppose that case (A) of Castelnuovo's conjecture occurs. Let us consider the blow-ups necessary to obtain a birational morphism $bl : \widetilde{W} \to W$ such that the strict transform $\widetilde{S}_\bullet$ of $S_\bullet$ is a smooth irreducible surface and such that it moves in an

$r$-dimensional base point free linear system, given by the strict transform $\widetilde{\mathcal{L}}_\bullet$ of $\mathcal{L}_\bullet$. If $r \geq 13$, then $r_\bullet := \dim \mathcal{L}_\bullet = r - 10 \geq 3$ and we can apply Proposition 10.5 to the pair $(\widetilde{W}, \widetilde{\mathcal{L}}_\bullet)$. Thus the intersection of two general surfaces of $\widetilde{\mathcal{L}}_\bullet$ is the union of some elements of a congruence of curves of $\widetilde{W}$. These curves are fibres of a pencil of genus $b$ on a general surface $\widetilde{S}_\bullet \in \widetilde{\mathcal{L}}_\bullet$, where $0 \leq b \leq q(\widetilde{S}_\bullet) = 1$. In particular, if $\widetilde{\phi}_\bullet : \widetilde{W} \to \mathbb{P}^{r_\bullet}$ is the morphism defined by $\widetilde{\mathcal{L}}_\bullet$, we have that $b := p_g(\Gamma)$ where $\Gamma := \widetilde{\phi}_\bullet(S_\bullet)$. Furthermore $\Delta := \widetilde{\phi}_\bullet(\widetilde{W})$ is a rational surface of $\mathbb{P}^{r_\bullet}$ with general hyperplane section $\Gamma$ (see Remark 10.6).

**Remark 10.7.** If case (A) of Castelnuovo's conjecture occurs, if $r \geq 13$ and if $\Gamma$ is an elliptic curve, then $\Delta \subset \mathbb{P}^{r_\bullet}$ is a Del Pezzo surface (see [3, VI, Exercise (1)]). In this case $\Delta \subset \mathbb{P}^{r_\bullet}$ is represented on the projective plane $\mathbb{P}^2$ by a linear system $\mathcal{D}$ of elliptic curves with $\dim \mathcal{D} \leq 9$. Since the linear system $\mathcal{L}_\bullet$ is in birational correspondence with the linear system $\mathcal{D}$, we have $\dim \mathcal{D} = \dim \mathcal{L}_\bullet = r - 10 \leq 9$, which implies $r \leq 19$.

## 10.3 Castelnuovo's conjecture for singular threefolds

We can adapt Castelnuovo's conjecture, its consequences and preliminary results to singular threefolds. Let us see which ones and how.

Let $W$ be an irreducible threefold with *isolated singularities* and let $\mathcal{L}$ be an $r$-dimensional linear system on $W$, where $r \geq 2$, such that the general element $S \in \mathcal{L}$ is a smooth irreducible surface disjoint from the singular points of $W$. Let us take a resolution $f : \widehat{W} \to W$ of the singularities of $W$. Since $f$ is an isomorphism outside the singular points of $W$, we have that the surface $f^{-1}(S)$ is isomorphic to $S$. Furthermore $f^{-1}(S)$ moves in the linear system $\widehat{\mathcal{L}} := f^*\mathcal{L}$, which still has $\dim \widehat{\mathcal{L}} = r$. So we have a smooth irreducible threefold $\widehat{W}$ endowed with an $r$-dimensional linear system $\widehat{\mathcal{L}}$, where $r \geq 2$, such that the general element $\widehat{S} \in \widehat{\mathcal{L}}$ is a smooth irreducible surface. If in addition $W$ is rational and $p_g(S) = p_a(S) = 0$, then $\widehat{W}$ is rational too and $p_g(\widehat{S}) = p_a(\widehat{S}) = 0$. Let $w$ be a general point of $W$: since we may assume that $w$ is not a singular point of $W$, then $\widehat{w} := f^{-1}(w)$ is still a point of $\widehat{W}$. Furthermore if $\mathcal{L}_\bullet$ is the sublinear system of $\mathcal{L}$ given by the surfaces of $\mathcal{L}$ having a triple point at $w \in W$, then $\widehat{\mathcal{L}}_\bullet := f^*\mathcal{L}_\bullet$ is the sublinear system of $\widehat{\mathcal{L}}$ given by the surfaces of $\widehat{\mathcal{L}}$ having a triple point at $\widehat{w} \in \widehat{W}$. Thus we can adapt Castelnuovo's conjecture to a rational irreducible threefold $W$ with isolated singularities endowed with an $r$-dimensional linear system $\mathcal{L}$ whose general element is a smooth irreducible surface disjoint from the singular points of $W$, since we can birationally work with the pair $(\widehat{W}, \widehat{\mathcal{L}})$ defined as above.

In § 10.4 we will apply the ideas of Castelnuovo to the classical Enriques-Fano threefold $(W_F^{13}, \mathcal{L})$, found by Fano in [23, §8]. We recall that $W_F^{13}$ is a rational threefold with eight singular points and $\mathcal{L}$ is a linear system on $W$ whose general element is an Enriques surface, which is a smooth surface with zero geometric genus and zero arithmetic genus and which is disjoint to the singular points of $W$, since it is a Cartier divisor on $W$. We will see that case (A) of Castelnuovo's conjecture occurs for $(W_F^{13}, \mathcal{L})$ (see Theorem 10.25). From this it will follow that case (A) of Castelnuovo's conjecture also occurs for the P-EF 3-fold $W_P^{17}$ (see Corollary 10.26). We observe that it actually makes sense to ask ourselves about the link between Castelnuovo's arguments and the P-EF 3-folds, since the Remark 10.6 also holds for a unirational variety.

Finally, for completeness, let us state the following results.

**Theorem 10.8.** Let $W$ be an irreducible threefold with isolated singularities and let $\mathcal{L}$ be an $r$-dimensional linear system on $W$, where $r \geq 2$, such that the general element is a smooth irreducible surface disjoint from the singular points of $W$. If the elements of $\mathcal{L}$ are big and nef divisors, then a desingularization of $W$ has same irregularity of a general surface $S \in \mathcal{L}$.

*Proof.* Let us apply Proposition 10.1 to the pair $(\widehat{W}, \widehat{\mathcal{L}})$, constructed as above. □

**Theorem 10.9.** Let $W$ be an irreducible threefold with isolated singularities and let $\mathcal{L}$ be an $r$-dimensional linear system on $W$, where $r \geq 3$, such that the general element is a smooth irreducible surface disjoint from the singular points of $W$. If $W$ has regular desingularization and if the intersection of two general surfaces of $\mathcal{L}$ (outside the base locus) is an irreducible curve, then a general element $S \in \mathcal{L}$ is a regular surface.

*Proof.* Let us apply Proposition 10.3 to the pair $(\widehat{W}, \widehat{\mathcal{L}})$, constructed as above. □

**Theorem 10.10.** Let $W$ be an irreducible threefold with isolated singularities and let $\mathcal{L}_\bullet$ be an $r_\bullet$-dimensional linear system on $W$, where $r_\bullet \geq 3$, such that the general element is an irregular smooth irreducible surface disjoint from the singular points of $W$. If $W$ has regular desingularization, then two general elements $S_\bullet$ and $S'_\bullet$ of $\mathcal{L}_\bullet$ intersect each other (outside the base locus) along reducible curves. In particular on a fixed $S_\bullet$, the components of these curves are fibres of a pencil of genus $b$ with $0 \leq b \leq q(S_\bullet)$. Furthermore, by varying the surface $S_\bullet$, these component curves give a congruence of curves of $W$.

*Proof.* Let us apply Proposition 10.5 to the pair $(\widehat{W}, \widehat{\mathcal{L}_\bullet})$, constructed as above. □

## 10.4 Castelnuovo's conjecture for the classical Enriques-Fano threefold

Let us consider the classical Enriques-Fano threefold $(W = W_F^{13}, \mathcal{L})$. We want to study the sublinear system $\mathcal{L}_\bullet \subset \mathcal{L}$ of the hyperplane sections of $W$ with triple point at a general point $w \in W$.

We recall that $W$ is the image of $\mathbb{P}^3$ via the birational map $\nu_{\mathcal{S}} : \mathbb{P}^3 \dashrightarrow W \subset \mathbb{P}^{13}$, defined by the linear system $\mathcal{S}$ of the sextic surfaces singular along the edges of a tetrahedron $T$ (we will use the notations of § 5.2). By definition, for each surface $S \in \mathcal{L}$ there is a unique sextic surface $\Sigma \in \mathcal{S}$ such that $S = \nu_{\mathcal{S}}(\Sigma)$. Hence, if we take a surface $S_\bullet \in \mathcal{L}_\bullet \subset \mathcal{L}$, there must exist a unique sextic surface $\Sigma_\bullet \in \mathcal{S}$ such that $S_\bullet = \nu_{\mathcal{S}}(\Sigma_\bullet)$. This surface $\Sigma_\bullet$ is a particular surface of $\mathcal{S}$, which has triple point at the point $p \in \mathbb{P}^3$ such that $w = \nu_{\mathcal{S}}(p)$. We can so represent the linear system $\mathcal{L}_\bullet$ on $W$ via the sublinear system $\mathcal{S}_\bullet \subset \mathcal{S}$ on $\mathbb{P}^3$ given by the sextic surfaces of $\mathbb{P}^3$ double along the six edges of the tetrahedron $T$ and triple at the point $p \in \mathbb{P}^3$, such that $\nu_\mathcal{S}(p) = w$. Since $w$ is a general point of $W$, we may consider $p$ as a general point of $\mathbb{P}^3$. A priori we have that $r_\bullet := \dim \mathcal{S}_\bullet \geq \dim \mathcal{S} - 10 = 13 - 10 = 3$ and the linear system $\mathcal{S}_\bullet$ defines a rational map $\nu_\bullet : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{r_\bullet}$.

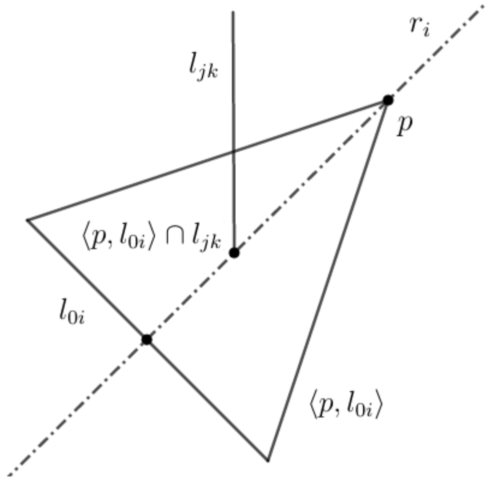


*Figure 16: Construction of the line $r_i$ where $i, j, k \in \{1, 2, 3\}$ with $j < k$ and $j, k \neq i$.*

Let us take the plane $\langle p, l_{0i}\rangle$ generated by the point $p$ and the edge $l_{0i}$, for a fixed index $1 \leq i \leq 3$. If $1 \leq j < k \leq 3$ with $j, k \neq i$, then the edges $l_{0i}$ and $l_{jk}$ are disjoint lines of $\mathbb{P}^3$; so the plane $\langle p, l_{0i}\rangle$ and the line $l_{jk}$ intersect at a point, outside $l_{0i}$. Let $r_i$ be the line joining this point and the point $p$, i.e. $r_i := \langle p, \langle p, l_{0i}\rangle \cap l_{jk}\rangle$ as in Figure 16.

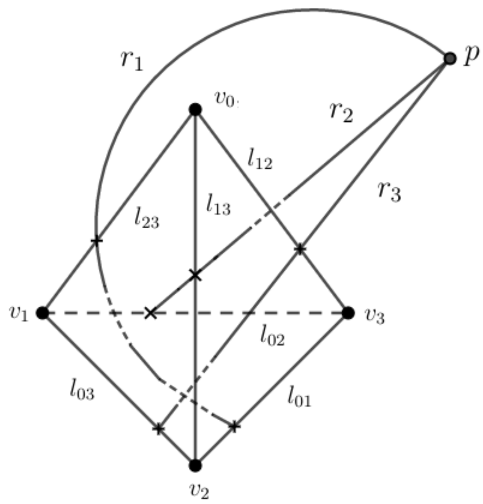


*Figure 17: The position of the lines $r_1, r_2, r_3$ with respect to the tetrahedron $T$.*

**Proposition 10.11.** Let $\mathcal{S}_\bullet$ be the linear system on $\mathbb{P}^3$ given by the sextic surfaces of $\mathbb{P}^3$ double along the six edges of the tetrahedron $T$ and triple at the general point $p \in \mathbb{P}^3$. The three lines $r_1$, $r_2$, $r_3$ are contained in the base locus of $\mathcal{S}_\bullet$.

*Proof.* Assume the contrary. Let us take a surface $\Sigma_\bullet \in \mathcal{S}_\bullet$ and let us fix $1 \leq i, j, k \leq 3$ with $j < k$ and $j, k \neq i$. By Bezout's Theorem, $\Sigma_\bullet \cap r_i$ is given by 6 points. Since $r_i$ is a line of the plane $\langle p, l_{0i} \rangle$, then it intersects the line $l_{0i}$ at a point. Thus $r_i$ is a line joining the triple point $p$ of $\Sigma_\bullet$ and two particular double points of $\Sigma_\bullet$, each one lying in one of the two opposite disjoint edges $l_{0i}$ and $l_{jk}$ (see Figure 17). Then $\Sigma_\bullet \cap r_i$ contains at least $3+2+2 = 7$ points, counted with multiplicity. This is a contradiction, so $r_i \subset \Sigma_\bullet$. □

Let us denote by $\mathcal{A}$ the two-dimensional linear system of the planes of $\mathbb{P}^3$ passing through the point $p$. In a general plane $\alpha \in \mathcal{A}$ we can construct a cubic plane curve $\gamma_\alpha$ with node at $p$ and passing through the six points given by the intersection of $\alpha$ with the six edges of the tetrahedron $T$. Let us denote these six points by $A_{ij} := \alpha \cap l_{ij}$ for $0 \leq i < j \leq 3$.

**Lemma 10.12.** In a general plane $\alpha \subset \mathbb{P}^3$ passing through the point $p$, there is a unique cubic plane curve $\gamma_\alpha$, defined as above.

*Proof.* Let $\mathfrak{g}$ be the linear system of the cubic plane curves on $\alpha$ passing through the six points $\{A_{ij} | 0 \leq i < j \leq 3\}$ and having a node at $p$. If the six fixed points had been general, we would have imposed $\frac{2 \cdot 3}{2} + \sum_{i=1}^{6} 1 = 9$ independent conditions. In our case the points $\{A_{ij} | 0 \leq i < j \leq 3\}$ are not in general position: indeed they are the vertices of a complete quadrilateral whose edges are the intersection of the plane $\alpha$ with the four faces of the tetrahedron $T$. Hence $\dim \mathfrak{g} \geq \binom{3+2}{2} - 9 - 1 = 0$. We want to show that the equality holds. In order to do it, we take the blow-up $bl : \widetilde{\alpha} \to \alpha$ of the plane $\alpha$ at the points $\{A_{ij} | 0 \leq i < j \leq 3\} \cup p$, by denoting the exceptional divisors by $e_{ij} = bl^{-1}(A_{ij})$ and $e_p = bl^{-1}(p)$, for $1 \leq i < j \leq 3$. If we denote by $\ell$ the strict transform of a general line of $\alpha$, then the strict transform of a general $\gamma_\alpha \in \mathfrak{g}$ is $\widetilde{\gamma}_\alpha \sim 3\ell - 2e_p - \sum_{0 \leq i < j \leq 3} e_{ij}$. By the generality of the point $p \in \mathbb{P}^3$, we may assume that the five points $p$, $A_{02}$, $A_{13}$, $A_{03}$, $A_{12}$ are in general position, since no three of them are collinear. So we can consider the unique irreducible conic $\delta$ passing through $p$, $A_{02}$, $A_{13}$, $A_{03}$, $A_{12}$, as in Figure 18, which has strict transform $\widetilde{\delta} \sim 2\ell - e_p - e_{02} - e_{13} - e_{03} - e_{12}$. Since $\widetilde{\gamma}_\alpha \cdot \widetilde{\delta} = 0$, then we have the following exact sequence

$$0 \to \mathcal{O}_{\widetilde{\alpha}}(l - e_p - e_{01} - e_{23}) \to \mathcal{O}_{\widetilde{\alpha}}(\widetilde{\gamma}_\alpha) \to \mathcal{O}_{\widetilde{\delta}} \to 0.$$

Obviously $h^0(\widetilde{\alpha}, \mathcal{O}_{\widetilde{\alpha}}(l - e_p - e_{01} - e_{23})) = 0$, since the three points $p, A_{01}, A_{23}$ are not collinear, by the generality of the point $p$ again. Hence $h^0(\widetilde{\alpha}, \mathcal{O}_{\widetilde{\alpha}}(\widetilde{\gamma}_\alpha)) \leq h^0(\mathcal{O}_{\widetilde{\delta}}) = 1$ and $\dim \mathfrak{g} = h^0(\widetilde{\alpha}, \mathcal{O}_{\widetilde{\alpha}}(\widetilde{\gamma}_\alpha)) - 1 = 0$. □

**Lemma 10.13.** Let $\mathcal{S}_\bullet$ be the linear system on $\mathbb{P}^3$ given by the sextic surfaces of $\mathbb{P}^3$ double along the six edges of the tetrahedron $T$ and triple at the general point $p \in \mathbb{P}^3$. The rational map $\nu_\bullet : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{r_\bullet}$ defined by $\mathcal{S}_\bullet$ contracts the cubic plane curves $\gamma_\alpha$, constructed as above.

*Proof.* By Bezout's Theorem, a general element $\Sigma_\bullet \in \mathcal{S}_\bullet$ intersects a cubic plane curve $\gamma_\alpha$ in $6 \cdot 3 = 18$ points. Furthermore $\Sigma_\bullet$ and $\gamma_\alpha$ have in common, in the base locus of $\mathcal{S}_\bullet$, the point $p$ (which is a triple point for $\Sigma_\bullet$ and a node for $\gamma_\alpha$) and the six points $\{A_{ij} | 0 \leq i < j \leq 3\}$ (which are nodes for $\Sigma_\bullet$ and simple points for $\gamma_\alpha$). Hence, outside

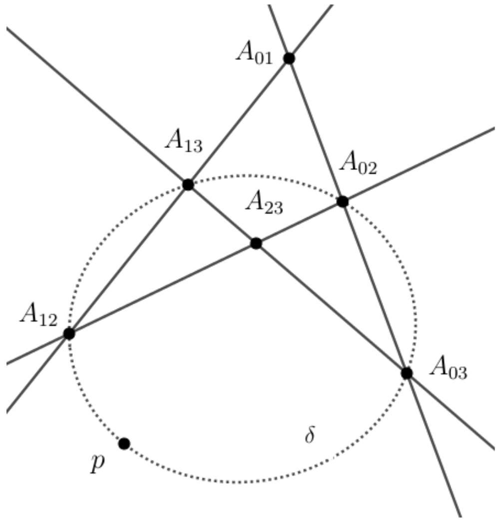


*Figure 18:* *The complete quadrilateral on $\alpha$ with vertices at the points $\{A_{ij}|0 \leq i < j \leq 3\}$ and the conic $\delta$ uniquely determined by the points $p, A_{02}, A_{13}, A_{03}, A_{12}$.*

the base locus, we have that $\Sigma_\bullet \cap \gamma_\alpha$ is given by $6 \cdot 3 - \sum_{i=1}^{6} 2 \cdot 1 - 3 \cdot 2 = 0$ points. So $\gamma_\alpha$ is contracted to a point by $\nu_\bullet : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{r_\bullet}$. □

**Remark 10.14.** Thanks to a computational analysis via Macaulay2 one can see that the general fibre of the rational map $\nu_\bullet : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{r_\bullet}$ defined by $\mathcal{S}_\bullet$ is a cubic plane curve $\gamma_\alpha$ (see Code B.11 of Appendix).

**Proposition 10.15.** The cubic plane curves $\gamma_\alpha$, defined as above, give a congruence $\mathcal{V}$ of curves of $\mathbb{P}^3$.

*Proof.* The set of the cubic plane curves $\gamma_\alpha$ is a 2-dimensional family $\mathcal{V}$ (see Lemma 10.12). In particular $\mathcal{V}$ is birationally parametrized by the same projective plane $\mathbb{P}^2$ parametrizing the planes passing through $p$. It remains to show that, given a general point $p' \in \mathbb{P}^3$, there is a unique curve of $\mathcal{V}$ passing through it. By Lemma 10.13 and Remark 10.14 we have that the curves of $\mathcal{V}$ are the general fibres of the rational map $\nu_\bullet : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{r_\bullet}$ defined by $\mathcal{S}_\bullet$. Hence $\nu_\bullet^{-1}(\nu_\bullet(p'))$ is the unique curve of $\mathcal{V}$ passing through $p'$. □

**Corollary 10.16.** Let $\mathcal{S}_\bullet$ be the linear system on $\mathbb{P}^3$ given by the sextic surfaces of $\mathbb{P}^3$ double along the six edges of the tetrahedron $T$ and triple at the general point $p \in \mathbb{P}^3$. The image of $\mathbb{P}^3$ via the rational map $\nu_\bullet : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{r_\bullet}$ defined by $\mathcal{S}_\bullet$ is a surface $\Delta \subset \mathbb{P}^{r_\bullet}$.

*Proof.* Let $\Delta$ be the image of $\mathbb{P}^3$ via $\nu_\bullet$. By Lemma 10.13 and Remark 10.14, the general fibre of $\nu_\bullet$ is a cubic plane curve, so we have $\dim \Delta = 3 - 1 = 2$. □

Let us now pay attention to a particular surface of $\mathbb{P}^3$. Let us consider the linear system $\mathfrak{c}$ on $\mathbb{P}^2$ given by the cubic plane curves passing through the six vertices of a complete quadrilateral. The image of $\mathbb{P}^2$ via the rational map defined by $\mathfrak{c}$ is a special Del Pezzo surface of degree 3 and it is called *Cayley cubic surface* (see [17, §9.2.2]). This surface has four singular points whose tangent cone is a quadric cone: we will refer to these singularities as *nodes*. The four nodes of the Cayley cubic surface are given by the image of the four edges of the fixed complete quadrilateral.

**Theorem 10.17.** Let $\mathcal{S}_\bullet$ be the linear system on $\mathbb{P}^3$ given by the sextic surfaces of $\mathbb{P}^3$ double along the six edges of the tetrahedron $T$ and triple at the general point $p \in \mathbb{P}^3$. The image of $\mathbb{P}^3$ via the rational map $\nu_\bullet : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{r_\bullet}$ defined by $\mathcal{S}_\bullet$ is a Cayley cubic surface $\Delta \subset \mathbb{P}^3$. Thus $r_\bullet = 3$.

*Proof.* Let us take a general element $\alpha \in \mathcal{A}$, i.e. a general plane passing through $p$. If we restrict the linear system $\mathcal{S}_\bullet$ to this plane, we obtain the linear system $\mathfrak{s}$ on $\alpha$ of the sextic plane curves with triple point at $p$ and nodes at the six points $\{A_{ij} | 0 \leq i < j \leq 3\}$. The plane $\alpha$ and a general fibre of $\nu_\bullet$ intersect, outside the base locus of $\mathcal{S}_\bullet$, at a single point: indeed the general fibre of $\nu_\bullet$ is a cubic plane curve $\gamma_{\alpha'}$ contained in a plane $\alpha' \in \mathcal{A}$, where $\alpha' \neq \alpha$; so we have that $\alpha$ intersects $\gamma_{\alpha'}$, outside the base locus of $\mathcal{S}_\bullet$, at $1 \cdot 3 - 1 \cdot 2 = 1$ point. Then the linear system $\mathfrak{s}$ defines the rational map $\nu_\bullet|_\alpha : \alpha \cong \mathbb{P}^2 \dashrightarrow \mathbb{P}^{r_\bullet}$ which is generically $1:1$. In the following we will see that, by applying three quadratic transformations, we obtain, from $\mathfrak{s}$, the linear system $\mathfrak{c}$ of the cubic plane curves passing through the six vertices of a complete quadrilateral. Thus the image of $\alpha$ via $\nu_\bullet|_\alpha$ is the image of $\mathbb{P}^2$ via the rational map defined by $\mathfrak{c}$, that is a Cayley cubic surface. By Corollary 10.16, this is the image $\Delta$ of $\mathbb{P}^3$ via $\nu_\bullet : \mathbb{P}^3 \dashrightarrow \mathbb{P}^{r_\bullet}$. Hence $r_\bullet = 3$.

Let us recall that the four faces of the tetrahedron $T$ intersect the plane $\alpha$ along four lines: the line $\langle A_{01}, A_{02}, A_{03} \rangle$ passing through $A_{01}$, $A_{02}$, $A_{03}$, the line $\langle A_{01}, A_{12}, A_{13} \rangle$ passing through $A_{01}$, $A_{12}$, $A_{13}$, the line $\langle A_{02}, A_{12}, A_{23} \rangle$ passing through $A_{02}$, $A_{12}$, $A_{23}$ and the line $\langle A_{03}, A_{13}, A_{23} \rangle$ passing through $A_{03}$, $A_{13}$, $A_{23}$. These four lines are the edges of a complete quadrilateral $Q_A$ with six vertices at the points $\{A_{ij} | 0 \leq i < j \leq 3\}$ (see Figure 19). Hence $\mathfrak{s}$ is the linear system of the sextic plane curves triple at $p$ e double at the six vertices of $Q_A$. Let us consider the quadratic trasformation $q_{p,A_{12},A_{03}} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$

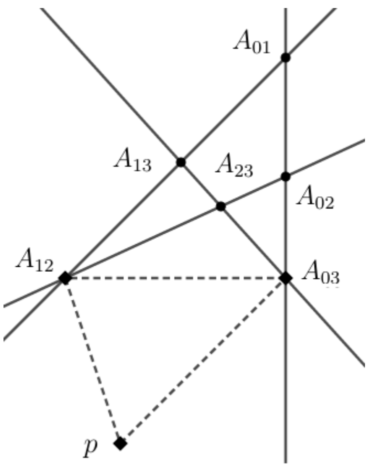


*Figure 19: The complete quadrilateral $Q_A$ on $\alpha$ with vertices at the points $\{A_{ij} | 0 \leq i < j \leq 3\}$ and the lines between the three base points of the quadratic transformation $q_{p,A_{12},A_{03}}$.*

given by the linear system of the conics passing through the three points $p, A_{12}, A_{03}$. Let $B_{23}$, $B_{13}$, $B_{01}$, $B_{02}$ be the images of the points $A_{23}$, $A_{13}$, $A_{01}$, $A_{02}$. We have that each of the lines $\langle p, A_{12} \rangle$, $\langle p, A_{03} \rangle$, and $\langle A_{12}, A_{03} \rangle$ is contracted by $q_{p,A_{12},A_{03}}$ to a point, denoted respectively by $B_{03}$, $B_{12}$ and $p'$. Furthermore the four edges of the complete quadrilateral $Q_A$ are sent to the four edges of a new complete quadrilateral $Q_B$ with six vertices at the points $\{B_{ij}|0 \leq i < j \leq 3\}$: in particular we have that

$$q_{p,A_{12},A_{03}}(\langle A_{01}, A_{02}, A_{03}\rangle) = \langle B_{01}, B_{02}, B_{03}\rangle\,,\ q_{p,A_{12},A_{03}}(\langle A_{01}, A_{12}, A_{13}\rangle) = \langle B_{01}, B_{12}, B_{13}\rangle\,,$$

$$q_{p,A_{12},A_{03}}(\langle A_{02}, A_{12}, A_{23}\rangle) = \langle B_{02}, B_{12}, B_{23}\rangle\,,\ q_{p,A_{12},A_{03}}(\langle A_{03}, A_{13}, A_{23}\rangle) = \langle B_{03}, B_{13}, B_{23}\rangle\,.$$

Then the linear system $\mathfrak{s}$ of the sextic plane curves triple at the point $p$ and double at the six points $\{A_{ij}|0 \leq i < j \leq 3\}$ is transformed in the linear system $\mathfrak{q}_5$ of the quintic plane curves double at $p'$, $B_{23}$, $B_{13}$, $B_{01}$, $B_{02}$ and passing through $B_{12}$ and $B_{03}$. Let us consider the quadratic trasformation $q_{p',B_{23},B_{01}} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by the linear system of the conics passing through the three points $p'$, $B_{23}$, $B_{01}$. Let $C_{13}$, $C_{12}$, $C_{02}$, $C_{03}$ be the images of the points $B_{13}$, $B_{12}$, $B_{02}$, $B_{03}$. We have that each of the lines $\langle p', B_{23}\rangle$, $\langle p', B_{01}\rangle$, and $\langle B_{23}, B_{01}\rangle$ is contracted by $q_{p',B_{23},B_{01}}$ to a point, denoted respectively by $C_{01}$, $C_{23}$ and $p''$. Furthermore the four edges of the complete quadrilateral $Q_B$ are sent to the four edges of a new complete quadrilateral $Q_C$ with six vertices at the points $\{C_{ij}|0 \leq i < j \leq 3\}$, in the following way:

$$q_{p',B_{23},B_{01}}(\langle B_{01}, B_{02}, B_{03}\rangle) = \langle C_{01}, C_{02}, C_{03}\rangle\,,\ q_{p',B_{23},B_{01}}(\langle B_{01}, B_{12}, B_{13}\rangle) = \langle C_{01}, C_{12}, C_{13}\rangle\,,$$

$$q_{p',B_{23},B_{01}}(\langle B_{02}, B_{12}, B_{23}\rangle) = \langle C_{02}, C_{12}, C_{23}\rangle\,,\ q_{p',B_{23},B_{01}}(\langle B_{03}, B_{13}, B_{23}\rangle) = \langle C_{03}, C_{13}, C_{23}\rangle\,.$$

Then the linear system $\mathfrak{q}_5$ of the quintic plane curves double at $p'$, $B_{23}$, $B_{13}$, $B_{01}$, $B_{02}$ and passing through $B_{12}$ and $B_{03}$ is transformed in the linear system $\mathfrak{q}_4$ of the quartic plane curves double at $C_{13}$ and $C_{02}$ and passing through $p''$, $C_{23}$, $C_{12}$, $C_{01}$, $C_{03}$. Let us consider the quadratic trasformation $q_{p'',C_{13},C_{02}} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by the linear system of the conics passing through the three points $p''$, $C_{13}$, $C_{02}$. Let $D_{23}$, $D_{12}$ $D_{01}$, $D_{03}$ be the images of the points $C_{23}$, $C_{12}$, $C_{01}$, $C_{03}$. We have that each of the lines $\langle p'', C_{13}\rangle$, $\langle p'', C_{02}\rangle$, and $\langle C_{13}, C_{02}\rangle$ are contracted by $q_{p'',C_{13},C_{02}}$ to a point, denoted respectively with $D_{02}$, $D_{13}$ and $p'''$. Furthermore the four edges of the complete quadrilateral $Q_C$ are sent to the four edges of a new complete quadrilateral $Q_D$ with six vertices the points $\{D_{ij}|0 \leq i < j \leq 3\}$:

$$q_{p'',C_{13},C_{02}}(\langle C_{01}, C_{02}, C_{03}\rangle) = \langle D_{01}, D_{02}, D_{03}\rangle\,,\ q_{p'',C_{13},C_{02}}(\langle C_{01}, C_{12}, C_{13}\rangle) = \langle D_{01}, D_{12}, D_{13}\rangle\,,$$

$$q_{p'',C_{13},C_{02}}(\langle C_{02}, C_{12}, C_{23}\rangle) = \langle D_{02}, D_{12}, D_{23}\rangle\,,\ q_{p'',C_{13},C_{02}}(\langle C_{03}, C_{13}, C_{23}\rangle) = \langle D_{03}, D_{13}, D_{23}\rangle\,.$$

Then the linear system $\mathfrak{q}_4$ of the quartic plane curves double at $C_{13}$ and $C_{02}$ and passing through $p'''$, $C_{23}$, $C_{12}$, $C_{01}$, $C_{03}$ is transformed in the linear system $\mathfrak{c}$ of the cubic plane curves passing through $\{D_{ij}|0 \leq i < j \leq 3\}$, which are the six vertices of a complete quadrilateral $Q_D$. □

**Corollary 10.18.** Let $\mathcal{S}_\bullet$ be the linear system on $\mathbb{P}^3$ given by the sextic surfaces of $\mathbb{P}^3$ double along the six edges of the tetrahedron $T$ and triple at the general point $p \in \mathbb{P}^3$. The only base curves of $\mathcal{S}_\bullet$ are the six edges of $T$ and the three lines $r_1$, $r_2$, $r_3$.

*Proof.* Let $\Delta$ be the image of $\mathbb{P}^3$ via the rational map $\nu_\bullet$. Two of its general hyperplane sections intersect each other at $\deg \Delta = 3$ points (see Theorem 10.17). Let us consider the preimages of these two curves: they are two elements $\Sigma_\bullet$ and $\Sigma'_\bullet$ of $\mathcal{S}_\bullet$, intersecting, outside the base locus of $\mathcal{S}_\bullet$, along a nonic curve. Indeed the intersection of $\Sigma_\bullet$ and $\Sigma'_\bullet$, outside the base locus of $\mathcal{S}_\bullet$, is given by the union of $\deg \Delta = 3$ fibres of $\nu_\bullet$, which are cubic plane curves (see Lemma 10.13 and Remark 10.14). The base locus of $\mathcal{S}_\bullet$ contains the six edges of $T$ and the three lines $r_1, r_2, r_3$ (see Proposition 10.11). If another curve existed in the base locus of $\mathcal{S}_\bullet$, then $\Sigma_\bullet$ would intersect $\Sigma'_\bullet$, outside it, along a curve of degree less than 9, and so $\deg \Delta < 3$, which is a contradiction. □

By using the notations of the proof of Theorem 10.17, we have the following facts.

**Proposition 10.19.** Let $\mathcal{S}_\bullet$ be the linear system on $\mathbb{P}^3$ given by the sextic surfaces of $\mathbb{P}^3$ double along the six edges of the tetrahedron $T$ and triple at the general point $p \in \mathbb{P}^3$. Let $\Delta \subset \mathbb{P}^3$ be the Cayley cubic surface given by the image of the rational map $\nu_\bullet : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by $\mathcal{S}_\bullet$. Then the four nodes of $\Delta$ are given by the image via $\nu_\bullet$ of the four faces of the tetrahedron $T$.

*Proof.* The faces of $T$ intersect a general plane $\alpha \in \mathcal{A}$ along the four edges of the complete quadrilateral $Q_A$. The edges of $Q_A$ are sent by $\mathfrak{s}$ to the edges of $Q_B$, which are mapped by $\mathfrak{q}_5$ to the edges of $Q_C$, which are transformed by $\mathfrak{q}_4$ in the edges of $Q_D$, which are finally sent by $\mathfrak{c}$ to the four singular points of $\Delta$. □

Let us consider the lines $s_i := \langle p, v_i \rangle$ joining the point $p \in \mathbb{P}^3$ and the vertex $v_i$ of the tetrahedron $T$, for $0 \leq i \leq 3$.

**Corollary 10.20.** Let $\mathcal{S}_\bullet$ be the linear system on $\mathbb{P}^3$ given by the sextic surfaces of $\mathbb{P}^3$ double along the six edges of the tetrahedron $T$ and triple at the general point $p \in \mathbb{P}^3$. Let $\Delta \subset \mathbb{P}^3$ be the Cayley cubic surface given by the image of the rational map $\nu_\bullet : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ defined by $\mathcal{S}_\bullet$. The four lines $s_0$, $s_1$, $s_2$, $s_3$ are sent via $\nu_\bullet$ to the four nodes of $\Delta$.

*Proof.* By Bezout's Theorem, a general sextic surface $\Sigma_\bullet \in \mathcal{S}_\bullet$ intersects each of the four lines at 6 points. We also observe that $\Sigma_\bullet$ and each of these lines have in common, in the base locus of $\mathcal{S}_\bullet$, the point $p$ and a vertex of $T$, which are triple points for $\Sigma_\bullet$. Hence, outside the base locus, we have that $\Sigma_\bullet \cap s_i$ is given by $6 - 3 - 3 = 0$ points, for all $0 \leq i \leq 3$. So the four lines $s_0$, $s_1$, $s_2$, $s_3$ are contracted by $\nu_\bullet$ to four points. Let us fix now $0 \leq i \leq 3$. We have that $s_i$ intersects at a point the face of $T$ opposite to the vertex $v_i$. Hence the point to which the line $s_i$ is sent by $\nu_\bullet$ is the same point to which the opposite face to $v_i$ is sent by $\nu_\bullet$, that is one of the four nodes of $\Delta$ by Proposition 10.19. □

Now we want to study the surfaces of the linear system $\mathcal{S}_\bullet$.

**Remark 10.21.** Let us recall some facts about the surfaces of the linear system $\mathcal{S}$, by using notations of § 5.2. If we blow-up $\mathbb{P}^3$ first at the vertices of $T$ and then along the (strict transforms of the) edges of $T$, we obtain a smooth threefold $Y''$. Let $\Sigma''$ be the strict transform of a general element $\Sigma \in \mathcal{S}$: it is a smooth surface, since it is the blow-up of a surface $\Sigma \in \mathcal{S}$ with ordinary singularities along its singular curves (see [27, pp.620-621]). Let us take the following exact sequence

$$0 \to \mathcal{O}_{Y''}(K_{Y''}) \to \mathcal{O}_{Y''}(K_{Y''} + \Sigma'') \to \mathcal{O}_{\Sigma''}(K_{\Sigma''}) \to 0,$$

where $K_{Y''} + \Sigma'' \sim 2H - \sum_{i=0}^{3} \widetilde{E}_i - \sum_{0 \leq i < j \leq 3} F_{ij}$ (see [27, p.187]). We have that $h^{i=0,1,2}(Y'', \mathcal{O}_{Y''}(K_{Y''})) = 0$ and $h^3(Y'', \mathcal{O}_{Y''}(K_{Y''})) = 1$ by Serre Duality, since $Y''$ is a rational smooth threefold by construction; furthermore we have that $h^0(\Sigma'', \mathcal{O}_{\Sigma''}(K_{\Sigma''})) = p_g(\Sigma'') = 0$, $h^1(\Sigma'', \mathcal{O}_{\Sigma''}(K_{\Sigma''})) = h^1(\Sigma'', \mathcal{O}_{\Sigma''}) = q(\Sigma'') = 0$ and $h^2(\Sigma'', \mathcal{O}_{\Sigma''}(K_{\Sigma''})) = h^0(\Sigma'', \mathcal{O}_{\Sigma''}) = 1$ by Serre Duality and by Theorem 5.15. So we obtain $h^0(Y'', \mathcal{O}_{Y''}(K_Y'' + \Sigma'')) = h^0(\Sigma'', \mathcal{O}_{\Sigma''}(K_{\Sigma''})) = 0$, i.e. there are no quadric surfaces of $\mathbb{P}^3$ containing the edges of $T$. We also have that $h^1(Y'', \mathcal{O}_{Y''}(K_Y'' + \Sigma'')) = h^1(\Sigma'', \mathcal{O}_{\Sigma''}(K_{\Sigma''})) = 0$.

In our case, first we blow-up $\mathbb{P}^3$ at the vertices of $T$, at the point $p$ and at the six points $r_i \cap l_{0i}$, $r_i \cap l_{jk}$, for $i, j, k \in \{1, 2, 3\}$ with $j < k$ and $j, k \neq i$. In this way we obtain a smooth threefold $X'$ and a birational morphism $bl' : X' \to \mathbb{P}^3$ with exceptional divisors

$$E_h = bl'^{-1}(v_h), \quad E_p = bl'^{-1}(p), \quad E_i' = bl'^{-1}(r_i \cap l_{0i}), \quad E_i'' = bl'^{-1}(r_i \cap l_{jk}).$$

where $0 \leq h \leq 3$. Let us denote by $\widetilde{l}_{0i}$, $\widetilde{l}_{jk}$ and $\widetilde{r}_i$, respectively, the strict transforms of the lines $l_{0i}$, $l_{jk}$ and $r_i$. Then we blow-up $X'$ along these objects. We obtain a smooth threefold $X''$ and a birational morphism $bl'' : X'' \to X'$, with exceptional divisors

$$F_{0i} = bl''^{-1}(\widetilde{l}_{0i}), \quad F_{jk} = bl''^{-1}(\widetilde{l}_{jk}), \quad R_i = bl''^{-1}(\widetilde{r}_i).$$

Furthermore let us denote by $\widetilde{E}_h$, $\widetilde{E}_p$, $\widetilde{E}_i'$, $\widetilde{E}_i''$, respectively the strict transforms of $E_h$, $E_p$, $E_i'$, $E_i''$. We denote by $H$ the pullback of a general plane of $\mathbb{P}^3$ via the birational morphism $bl' \circ bl'' : X'' \to \mathbb{P}^3$. Then the strict transform $\Sigma_\bullet''$ of an element $\Sigma_\bullet \in \mathcal{S}_\bullet$, via the blow-ups $bl' \circ bl'' : X'' \to \mathbb{P}^3$, is

$$\Sigma_\bullet'' \sim 6H - 3\widetilde{E}_p - \sum_{i=0}^{3} 3\widetilde{E}_i - \sum_{i=1}^{3} 2\widetilde{E}_i' - \sum_{i=1}^{3} 2\widetilde{E}_i'' - \sum_{0 \leq i < j \leq 3} 2F_{ij} - \sum_{i=1}^{3} R_i.$$

**Remark 10.22.** The anticanonical divisor of $X''$ is linearly equivalent to the strict transform of a quartic surface of $\mathbb{P}^3$ with double points at the vertices of $T$ and at the point $p$ and containing the six edges of $T$ and the three lines $r_1, r_2, r_3$, i.e.

$$K_{X''} \sim -4H + 2\widetilde{E}_p + 2\sum_{i=0}^{3} \widetilde{E}_i + \sum_{i=1}^{3} 2\widetilde{E}_i' + \sum_{i=1}^{3} 2\widetilde{E}_i'' + \sum_{0 \leq i < j \leq 3} F_{ij} + \sum_{i=1}^{3} R_i$$

(see [27, p.187]). Then we have $K_{X''} + \Sigma_\bullet'' \sim 2H - \widetilde{E}_p - \sum_{i=0}^{3} \widetilde{E}_i - \sum_{0 \leq i < j \leq 3} F_{ij}$. Since there are no quadric surfaces of $\mathbb{P}^3$ containing the edges of $T$ (see Remark 10.21), there are also no quadric surfaces of $\mathbb{P}^3$ containing the edges of $T$ and the point $p$. So we obtain $h^0(X'', \mathcal{O}_{X''}(K_{X''} + \Sigma_\bullet'')) = 0$.

**Theorem 10.23.** Let $\mathcal{S}_\bullet$ be the linear system on $\mathbb{P}^3$ given by the sextic surfaces of $\mathbb{P}^3$ double along the six edges of the tetrahedron $T$ and triple at the general point $p \in \mathbb{P}^3$. The strict transform $\Sigma_\bullet''$ on $X''$ of a general element $\Sigma_\bullet \in \mathcal{S}_\bullet$, via the blow-ups above described, is a smooth surface with $p_g(\Sigma_\bullet'') = 0$, $q(\Sigma_\bullet'') = 1$ and $p_a(\Sigma_\bullet'') = -1$.

*Proof.* It is known that $bl' \circ bl'' : X'' \to \mathbb{P}^3$ solves the singularities of a general $\Sigma_\bullet \in \mathcal{S}_\bullet \subset \mathcal{S}$ at the vertices of the tetrahedron $T$ and along its edges. In order to obtain the smoothness of the strict transform $\Sigma''_\bullet$ on $X''$ of $\Sigma_\bullet$, it remains to show that $bl' \circ bl'' : X'' \to \mathbb{P}^3$ also solves the triple point $p$ of $\Sigma_\bullet$. By Bertini's Theorem, it is sufficient to prove that the linear system $|\Sigma''_\bullet|$ is base point free on $\widetilde{E}_p$. We recall that $\widetilde{E}_p$ is the blow-up of the plane $E_p \cong \mathbb{P}^2$ at the three points $E_p \cap \widetilde{r}_1$, $E_p \cap \widetilde{r}_2$, $E_p \cap \widetilde{r}_3$. We also recall that $\Sigma'_\bullet \cap E_p = \mathbb{P}(TC_p\Sigma_\bullet)$, where $\Sigma'_\bullet := bl''(\Sigma''_\bullet)$ and where $TC_p\Sigma_\bullet$ denotes the tangent cone to $\Sigma_\bullet$ at $p$. Thanks to a computational analysis via Macaulay2, we find that $\mathbb{P}(TC_p\Sigma_\bullet)$ is a cubic plane curve passing through the points $E_p \cap \widetilde{r}_1$, $E_p \cap \widetilde{r}_2$, $E_p \cap \widetilde{r}_3$ (see Code B.11 of Appendix). In particular we have that $|\Sigma''_\bullet|$ cuts on $\widetilde{E}_p$ the strict transform on $\widetilde{E}_p$ of a linear system of cubic curves on $E_p$, whose base points are only the points $E_p \cap \widetilde{r}_1$, $E_p \cap \widetilde{r}_2$, $E_p \cap \widetilde{r}_3$ (see Code B.11 of Appendix). Thus $|\Sigma''_\bullet||_{\widetilde{E}_p}$ is base point free and so $\Sigma''_\bullet$ is smooth. By using the adjunction formula we have the exact seguence

$$0 \to \mathcal{O}_{X''}(K_{X''}) \to \mathcal{O}_{X''}(K_{X''} + \Sigma''_\bullet) \to \mathcal{O}_{\Sigma''_\bullet}(K_{\Sigma''_\bullet}) \to 0.$$

Since $X''$ is a smooth rational threefold, we have $h^{i=0,1,2}(X'', \mathcal{O}_{X''}(K_{X''})) = 0$. Then we obtain $p_g(\Sigma''_\bullet) = h^0(\Sigma''_\bullet, \mathcal{O}_{\Sigma''_\bullet}(K_{\Sigma''_\bullet})) = h^0(X'', \mathcal{O}_{X''}(K_{X''} + \Sigma''_\bullet)) = 0$ (see Remark 10.22). Furthermore we have that

$$q(\Sigma''_\bullet) = h^1(\Sigma''_\bullet, \mathcal{O}_{\Sigma''_\bullet}) = h^1(\Sigma''_\bullet, \mathcal{O}_{\Sigma''_\bullet}(K_{\Sigma''_\bullet})) = h^1(X'', \mathcal{O}_{X''}(K_{X''} + \Sigma''_\bullet)).$$

In order to verify that the last value is equal to 1, we observe that the strict transform on $X''$ of a quadric surface of $\mathbb{P}^3$ containing the edges of $T$ is linearly equivalent to $2H - \sum_{i=0}^{3} \widetilde{E}_i - \sum_{0 \leq i < j \leq 3} F_{ij}$. By Remark 10.22 we have the following exact sequence

$$0 \to \mathcal{O}_{X''}(K_{X''} + \Sigma''_\bullet) \to \mathcal{O}_{X''}(2H - \sum_{i=0}^{3} \widetilde{E}_i - \sum_{0 \leq i < j \leq 3} F_{ij}) \to \mathcal{O}_{E_p} \to 0.$$

Since $h^{i=0,1}(X'', \mathcal{O}_{X''}(2H - \sum_{i=0}^{3} \widetilde{E}_i - \sum_{0 \leq i < j \leq 3} F_{ij})) = 0$ (see Remark 10.21), then $h^1(X'', \mathcal{O}_{X''}(K_{X''} + \Sigma''_\bullet)) = h^0(E_p, \mathcal{O}_{E_p}) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 1$. Finally, by Riemann-Roch we have that $p_a(\Sigma''_\bullet) = p_g(\Sigma''_\bullet) - q(\Sigma''_\bullet) = -1$. □

Let us recall now some definitions. Let $R$ be a smooth surface and $\Gamma$ a smooth, irreducible curve. We say that $R$ is a *ruled surface over* $\Gamma$ if there is a surjective morphism $f : R \to \Gamma$ such that, for a general point $x \in \Gamma$, we have that $f^{-1}(x)$ is isomorphic to $\mathbb{P}^1$. It is equivalent to say that $R$ is birational to $\Gamma \times \mathbb{P}^1$ (see [3, Theorem III.4]). Furthermore we say that a smooth variety $Z$ is *uniruled* if it is covered by a family of rational curves. More precisely, $Z$ is an uniruled variety if there is a variety $K$ with $\dim K = \dim Z - 1$ and there is a dominant rational map $K \times \mathbb{P}^1 \dashrightarrow Z$. Every uniruled variety $Z$ has Kodaira dimension $\kappa(Z) = -\infty$.

**Theorem 10.24.** Let $\mathcal{S}_\bullet$ be the linear system on $\mathbb{P}^3$ given by the sextic surfaces of $\mathbb{P}^3$ double along the six edges of the tetrahedron $T$ and triple at the general point $p \in \mathbb{P}^3$. The strict transform $\Sigma''_\bullet$ of a general element $\Sigma_\bullet \in \mathcal{S}_\bullet$, via the blow-ups above described, is an elliptic ruled surface.

*Proof.* Let us take a general $\Sigma_\bullet \in \mathcal{S}_\bullet$ and its image $\Gamma := \nu_\bullet(\Sigma_\bullet)$, which is a general hyperplane section of the Cayley cubic surface $\Delta \subset \mathbb{P}^3$. Since $\Delta$ has only isolated singularities, then $\Gamma$ is a smooth elliptic cubic plane curve. Furthermore, $\Sigma_\bullet$ is union of $\infty^1$ rational cubic plane curves, fibres of $\nu_\bullet$, given by the preimages of the $\infty^1$ points of $\Gamma$ (see Lemma 10.13 and Remark 10.14). So $(\nu_\bullet \circ bl'' \circ bl') : \Sigma''_\bullet \to \Gamma$ is an uniruled surface. Since $\kappa(\Sigma''_\bullet) = -\infty$, we have that $\Sigma''_\bullet$ is an irrational elliptic ruled surface by Enriques-Kodaira classification and by Theorem 10.23. □

By construction, for a general surface $S_\bullet \in \mathcal{L}_\bullet$ there exists a unique surface $\Sigma_\bullet \in \mathcal{S}_\bullet$ such that $S_\bullet = \nu_\bullet(\Sigma_\bullet)$. So if we denote by $\phi_\bullet : W \dashrightarrow \mathbb{P}^3$ the rational map defined by the linear system $\mathcal{L}_\bullet$, we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{P}^3 & & \\ {\scriptstyle \nu}\,\downarrow & \searrow^{\nu_\bullet} & \\ W & \overset{\phi_\bullet}{\dashrightarrow} & \Delta \subset \mathbb{P}^3 \\ \downarrow {\scriptstyle \phi_{\mathcal{L}}} & & \\ \mathbb{P}^{13} & & \end{array}$$

and we obtain the following result (see Theorems 10.17, 10.24).

**Theorem 10.25.** Let $(W_F^{13}, \mathcal{L})$ be the classical Enriques-Fano threefold. Let $\mathcal{L}_\bullet \subset \mathcal{L}$ be the sublinear system of the hyperplane sections having a triple point at a general point $w \in W_F^{13}$. Then

(i) a general $S_\bullet \in \mathcal{L}_\bullet$ is birational to an elliptic ruled surface;

(ii) the image of $W_F^{13}$ via the rational map defined by $\mathcal{L}_\bullet$ is a Cayley cubic surface.

We have thus proved that case (A) of Castelnuovo's conjecture occurs for the classical Enriques-Fano threefold and that the consequences stated in Remark 10.7 are verified.

## 10.5 Consequences for the P-EF 3-folds

It is known that all Enriques surfaces appear as the desingularization of some sextic surface of $\mathbb{P}^3$ double along the six edges of a tetrahedron and triple at the four vertices (see [16, p.275]). By using notations of previous sections, we can say that all Enriques surfaces are birational to a surface $\Sigma \in \mathcal{S}$ and so to a hyperplane section of the classical Enriques-Fano threefold $W_F^{13}$. If we consider an Enriques-Fano threefold $(W, \mathcal{L})$ of genus $13 \leq p \leq 17$, we can say that a general hyperplane section of $W$ is birational to a hyperplane section of the classical Enriques-Fano threefold $W_F^{13}$. In particular, a general hyperplane section of $W$ with triple point at a general point $w \in W$ is birational to a hyperplane section of the classical Enriques-Fano threefold $W_F^{13}$ with triple point at a point on it.

Since the general hyperplane section of the P-EF 3-fold $W_P^{17}$ is a general Enriques surface (see proof of [10, Proposition 4.7]), we obtain the following result by Theorem 10.25.

**Corollary 10.26.** Let $(W_P^{17}, \mathcal{L})$ be the P-EF 3-fold of genus 17. Let $\mathcal{L}_\bullet$ be the linear system of hyperplane sections of $W_P^{17}$ with a triple point at a general point $w \in W_P^{17}$. Then a general element $S_\bullet \in \mathcal{L}_\bullet$ is birational to an elliptic ruled surface.

It would be interesting to verify if the linear system $\mathcal{L}_\bullet$ on $W_P^{17}$ has dimension $7 = 17 - 10$ and if its image is still a Del Pezzo surface. Finally it would be interesting to understand what happens on the P-EF 3-fold $W_P^{13}$ of genus 13.

# A Appendix: configurations of the singularities of some known EF 3-folds

Let us suppose that each singular point $P_i$ of an Enriques-Fano threedold $W \subset \mathbb{P}^p$ is associated with $0 \leq m_i \leq n-1$ of the other singular points, for $1 \leq i \leq n$ (see Definition 4.4). If the singular points are similar (see Assumption CM3 in § 4.1), then we have that $m_i$ is constant, i.e. $m_i = m$ for all $1 \leq i \leq n$. We can graphically represent the way in which the singular points $P_1, \dots, P_n$ of an Enriques-Fano threefold are associated: if two singular points $P_i$ and $P_j$ are associated, we draw a segment joining them, otherwise not. We will refer to the graph obtained by the union of all these segments as the *configuration* of the singular points $P_1, \dots, P_n$. Let us see two example: if we have $n = 4$ singular points $P_1, P_2, P_3, P_4$, such that $P_1$ is associated with $P_2, P_3, P_4$, and $P_3$ is associated with $P_4$ then we have a configuration as in the left of Figure 20; if we have $n = 3$ singular points $P_1, P_2, P_3$ mutually associated we have a configuration as in the right of Figure 20.

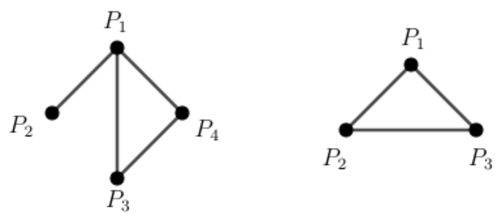


*Figure 20: Two examples of configurations: on the left with $(m_1, m_2, m_3, m_4) = (3, 1, 2, 2)$ and on the right with $m_i = m = 2$ for $1 \leq i \leq 3$.*

In this Appendix we represent the configurations of the singular points of the known Enriques-Fano threefolds $(W, \mathcal{L})$ of genus $p$ which are embedded in $\mathbb{P}^p$ via the map defined by $\mathcal{L}$. By using the notations of § 3.2 to indicate the known Enriques-Fano threefolds, we have the following configurations:

(i) Figure 21 for the BS-EF 3fold $W^6_{BS}$ and the F-EF 3fold $W^6_F$ (see [23, §3], Theorem 5.96, Example 6.2);

(ii) Figure 22 for the BS-EF 3fold $W^7_{BS}$ and the F-EF 3fold $W^7_F$ (see [23, §4], Theorem 5.61, Remark 6.4);

(iii) Figure 23 for the BS-EF 3fold $W^8_{BS}$ (see Remark 6.6);

(iv) Figure 24 for the BS-EF 3fold $W^9_{BS}$ and the F-EF 3fold $W^9_F$ (see [23, §7], Theorem 5.41, Remark 6.10);

(v) Figure 25 for the BS-EF 3fold $W^{10}_{BS}$ (see Remark 6.13);

(vi) Figure 26 for the BS-EF 3fold $W^{13}_{BS}$ and the F-EF 3fold $W^{13}_F$ (see [23, §8], Theorem 5.17, Remark 6.16);

(vii) Figure 27 for the P-EF 3folds $W^{13}_P$ and $W^{17}_P$ (see Remark 8.5 and Remark 8.12);

(viii) Figure 28 for the KLM-EF 3fold $W^9_{KLM}$ (see Remark 7.7).

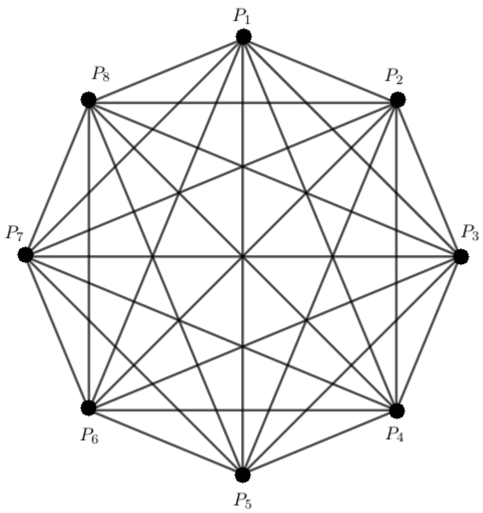


*Figure 21:* *Configuration of the eight quadruple points of the Enriques-Fano threefolds $W^6_F \subset \mathbb{P}^6$ and $W^6_{BS} \xhookrightarrow{\phi_{\mathcal{L}}} \mathbb{P}^6$, with $m = 7$.*

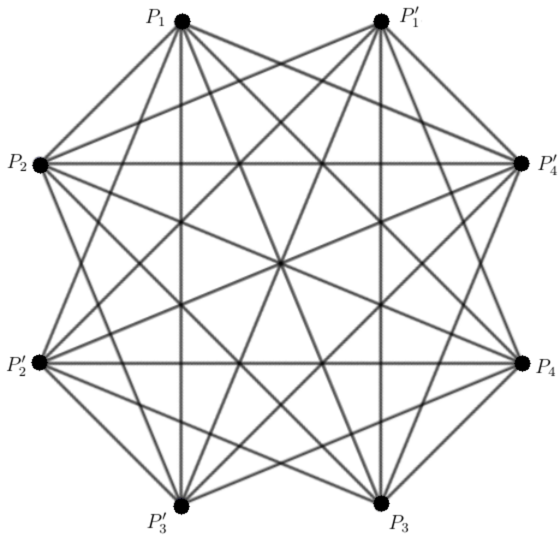


*Figure 22:* *Configuration of the eight quadruple points of the Enriques-Fano threefolds $W^7_F \subset \mathbb{P}^7$ and $W^7_{BS} \xhookrightarrow{\phi_{\mathcal{L}}} \mathbb{P}^7$, with $m = 6$.*

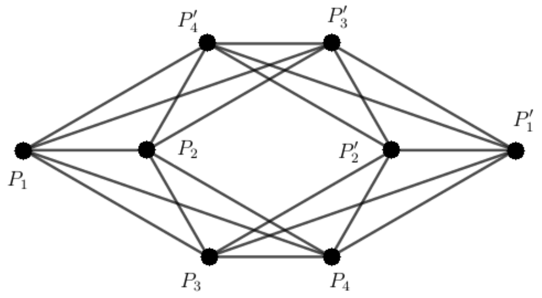


*Figure 23: Configuration of the eight quadruple points of the Enriques-Fano threefold $W^8_{BS} \xhookrightarrow{\phi_{\mathcal{L}}} \mathbb{P}^8$, with $m = 5$.*

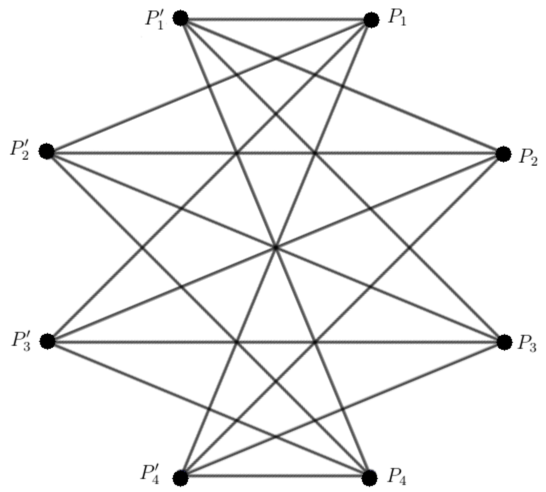


*Figure 24: Configuration of the eight quadruple points of the Enriques-Fano threefold $W^9_F = \phi_{\mathcal{L}}(W^9_{BS}) \subset \mathbb{P}^9$, with $m = 4$.*

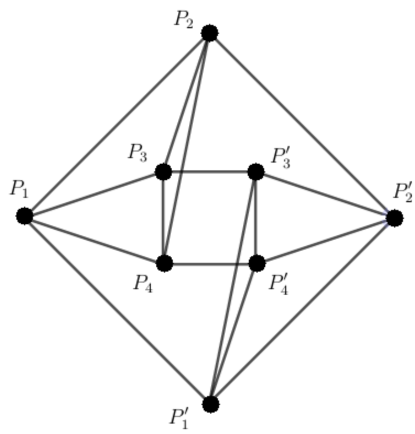


*Figure 25: Configuration of the eight quadruple points of the Enriques-Fano threefold $W^{10}_{BS} \xhookrightarrow{\phi_{\mathcal{L}}} \mathbb{P}^{10}$, with $m = 4$.*

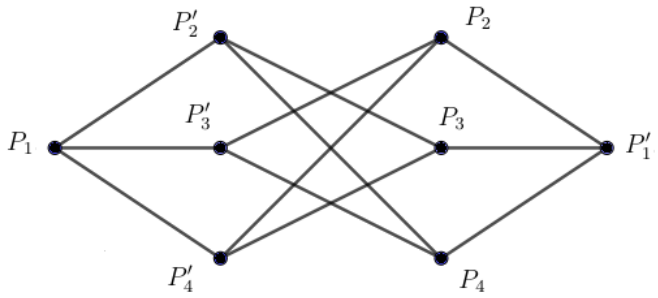


*Figure 26: Configuration of the eight quadruple points of the Enriques-Fano threefold $W_F^{13} = \phi_{\mathcal{L}}(W_{BS}^{13}) \subset \mathbb{P}^{13}$, with $m = 3$.*

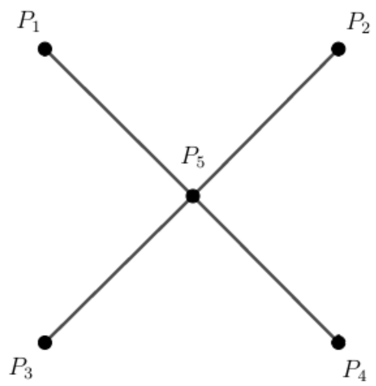


*Figure 27: Configuration of the five singular points of the Enriques-Fano threefolds $W_P^{13} \subset \mathbb{P}^{13}$ and $W_P^{17} \subset \mathbb{P}^{17}$, with $(m_1, m_2, m_3, m_4, m_5) = (1, 1, 1, 1, 4)$.*

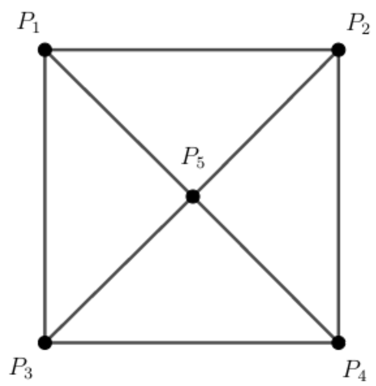


*Figure 28: Configuration of the five singular points of the Enriques-Fano threefold $W_{KLM}^9 \subset \mathbb{P}^9$, with $(m_1, m_2, m_3, m_4, m_5) = (3, 3, 3, 3, 4)$, obtained by the eight singular points of the classical Enriques-Fano threefold $W_F^{13} \subset \mathbb{P}^{13}$.*

# B Appendix: Macaulay2 codes

For the computational analysis via Macaulay2 we will work over a finite field (we will choose $\mathbb{F}_n := \mathbb{Z}/n\mathbb{Z}$ with $n = 10000019$). We will essentially use the package *Cremona* of Staglianò (see [49]) and in particular the following functions, commands and methods:

- *toMap*, to construct the rational map defined by a linear system;
- *rationalMap*, to construct rational maps between projective varieties;
- *image*, to compute the image of a rational map;
- *degree*, to compute the degree of a rational map;
- *isBirational*, to verify the birationality of a rational map;
- *inverseMap*, to compute the inverse of a birational map;
- *ideal*, to compute the base locus of a rational map.

For more information visit the website

http://www2.macaulay2.com/Macaulay2/doc/Macaulay2-1.12/share/doc/Macaulay2/Cremona/html/

We will also use the function *tangentCone*, to compute the tangent cone to an affine variety at the origin, and the following standard functions: *associatedPrimes*, to compute the irreducible components of a variety; *jacobian*, to compute the Jacobian matrix of the generators of an ideal; *minors*, to compute the ideal generated by the minors of a certain order of a given matrix.

In the following we will collect the input codes used in Macaulay2 for this thesis.

**Code B.1.** Computational analysis of $W^6_{BS}$ (see § 6.2).

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP3xPP3 = ZZ/10000019[x_0,x_1,x_2,x_3]**ZZ/10000019[y_0,y_1,y_2,y_3];
i3 : X = ideal{ x_0*y_0-7*x_1*y_1+4*x_2*y_2+2*x_3*y_3,
     x_0*y_0-6*x_1*y_1+2*x_2*y_2+3*x_3*y_3, x_0*y_0-x_1*y_1-7*x_2*y_2+7*x_3*y_3};
i4 : PP9 = ZZ/10000019[Z_0..Z_9];
i5 : phi = rationalMap map(PP3xPP3, PP9, matrix{{x_0*y_0, x_1*y_1, x_2*y_2, x_3*y_3, x_0*y_1+x_1*y_0,
     x_0*y_2+x_2*y_0, x_0*y_3+x_3*y_0, x_1*y_2+x_2*y_1, x_1*y_3+x_3*y_1, x_2*y_3+x_3*y_2}});
i6 : (dim(image phi) -1, degree(image phi)) == (6,10)
i7 : image phi == ideal{-2*Z_1*Z_5*Z_6+Z_4*Z_6*Z_7+Z_4*Z_5*Z_8-2*Z_0*Z_7*Z_8+4*Z_0*Z_1*Z_9-Z_4^2*Z_9,
     -2*Z_2*Z_4*Z_6+Z_5*Z_6*Z_7+4*Z_0*Z_2*Z_8-Z_5^2*Z_8+Z_4*Z_5*Z_9-2*Z_0*Z_7*Z_9,
     -4*Z_1*Z_2*Z_6+Z_6*Z_7^2+2*Z_2*Z_4*Z_8-Z_5*Z_7*Z_8+2*Z_1*Z_5*Z_9-Z_4*Z_7*Z_9,
     -2*Z_3*Z_4*Z_5+4*Z_0*Z_3*Z_7-Z_6^2*Z_7+Z_5*Z_6*Z_8+Z_4*Z_6*Z_9-2*Z_0*Z_8*Z_9,
     -4*Z_1*Z_3*Z_5+2*Z_3*Z_4*Z_7-Z_6*Z_7*Z_8+Z_5*Z_8^2+2*Z_1*Z_6*Z_9-Z_4*Z_8*Z_9,
     -4*Z_2*Z_3*Z_4+2*Z_3*Z_5*Z_7+2*Z_2*Z_6*Z_8-Z_6*Z_7*Z_9-Z_5*Z_8*Z_9+Z_4*Z_9^2,
     -4*Z_1*Z_2*Z_3+Z_3*Z_7^2+Z_2*Z_8^2-Z_7*Z_8*Z_9+Z_1*Z_9^2,
     -4*Z_0*Z_2*Z_3+Z_3*Z_5^2+Z_2*Z_6^2-Z_5*Z_6*Z_9+Z_0*Z_9^2,
     -4*Z_0*Z_1*Z_3+Z_3*Z_4^2+Z_1*Z_6^2-Z_4*Z_6*Z_8+Z_0*Z_8^2,
```

```
     -4*Z_0*Z_1*Z_2+Z_2*Z_4^2+Z_1*Z_5^2-Z_4*Z_5*Z_7+Z_0*Z_7^2}
i8 : phiX = ideal{Z_2-Z_3,Z_1-Z_3,Z_0-Z_3,
     -2*Z_1*Z_5*Z_6+Z_4*Z_6*Z_7+Z_4*Z_5*Z_8-2*Z_0*Z_7*Z_8+4*Z_0*Z_1*Z_9-Z_4^2*Z_9,
     -2*Z_2*Z_4*Z_6+Z_5*Z_6*Z_7+4*Z_0*Z_2*Z_8-Z_5^2*Z_8+Z_4*Z_5*Z_9-2*Z_0*Z_7*Z_9,
     -4*Z_1*Z_2*Z_6+Z_6*Z_7^2+2*Z_2*Z_4*Z_8-Z_5*Z_7*Z_8+2*Z_1*Z_5*Z_9-Z_4*Z_7*Z_9,
     -2*Z_3*Z_4*Z_5+4*Z_0*Z_3*Z_7-Z_6^2*Z_7+Z_5*Z_6*Z_8+Z_4*Z_6*Z_9-2*Z_0*Z_8*Z_9,
     -4*Z_1*Z_3*Z_5+2*Z_3*Z_4*Z_7-Z_6*Z_7*Z_8+Z_5*Z_8^2+2*Z_1*Z_6*Z_9-Z_4*Z_8*Z_9,
     -4*Z_2*Z_3*Z_4+2*Z_3*Z_5*Z_7+2*Z_2*Z_6*Z_8-Z_6*Z_7*Z_9-Z_5*Z_8*Z_9+Z_4*Z_9^2,
     -4*Z_1*Z_2*Z_3+Z_3*Z_7^2+Z_2*Z_8^2-Z_7*Z_8*Z_9+Z_1*Z_9^2,
     -4*Z_0*Z_2*Z_3+Z_3*Z_5^2+Z_2*Z_6^2-Z_5*Z_6*Z_9+Z_0*Z_9^2,
     -4*Z_0*Z_1*Z_3+Z_3*Z_4^2+Z_1*Z_6^2-Z_4*Z_6*Z_8+Z_0*Z_8^2,
     -4*Z_0*Z_1*Z_2+Z_2*Z_4^2+Z_1*Z_5^2-Z_4*Z_5*Z_7+Z_0*Z_7^2};
i9 : (dim oo -1, degree oo, oo == phi(X) ) == (3, 10, true)
i10 : H6 = ideal{Z_2-Z_3,Z_1-Z_3,Z_0-Z_3};
i11 : PP6 = ZZ/10000019[w_0..w_6];
i12 : inclusion = rationalMap map(PP6,PP9,matrix{{w_0,w_0,w_0,w_0,w_1,w_2,w_3,w_4,w_5,w_6}});
i13 : image oo == H6
i14 : pigreca = phi*(rationalMap map(PP9,PP6, sub(matrix inverseMap(inclusion||H6), PP9) ))
i15 : pigreca(X) == inclusion^*(phiX)
i16 : WB6 = ideal{-2*w_0*w_2*w_3+w_1*w_3*w_4+w_1*w_2*w_5-2*w_0*w_4*w_5+4*w_0^2*w_6-w_1^2*w_6,
      -2*w_0*w_1*w_3+w_2*w_3*w_4+4*w_0^2*w_5-w_2^2*w_5+w_1*w_2*w_6-2*w_0*w_4*w_6,
      -4*w_0^2*w_3+w_3*w_4^2+2*w_0*w_1*w_5-w_2*w_4*w_5+2*w_0*w_2*w_6-w_1*w_4*w_6,
      -2*w_0*w_1*w_2+4*w_0^2*w_4-w_3^2*w_4+w_2*w_3*w_5+w_1*w_3*w_6-2*w_0*w_5*w_6,
      -4*w_0^2*w_2+2*w_0*w_1*w_4-w_3*w_4*w_5+w_2*w_5^2+2*w_0*w_3*w_6-w_1*w_5*w_6,
      -4*w_0^2*w_1+2*w_0*w_2*w_4+2*w_0*w_3*w_5-w_3*w_4*w_6-w_2*w_5*w_6+w_1*w_6^2,
      -4*w_0^3+w_0*w_4^2+w_0*w_5^2-w_4*w_5*w_6+w_0*w_6^2, -4*w_0^3+w_0*w_2^2+w_0*w_3^2-w_2*w_3*w_6+w_0*w_6^2,
      -4*w_0^3+w_0*w_1^2+w_0*w_3^2-w_1*w_3*w_5+w_0*w_5^2, -4*w_0^3+w_0*w_1^2+w_0*w_2^2-w_1*w_2*w_4+w_0*w_4^2};
i17 : WB6 == pigreca(X)
i18 : (dim ooo -1, degree ooo) == (3, 10)
i19 : P1 = ideal{w_1-2*w_0,w_2-2*w_0,w_3-2*w_0,w_4-2*w_0,w_5-2*w_0,w_6-2*w_0};
i20 : P2 = ideal{w_1+2*w_0,w_2+2*w_0,w_3+2*w_0,w_4-2*w_0,w_5-2*w_0,w_6-2*w_0};
i21 : P3 = ideal{w_1+2*w_0,w_2-2*w_0,w_3-2*w_0,w_4+2*w_0,w_5+2*w_0,w_6-2*w_0};
i22 : P4 = ideal{w_1-2*w_0,w_2+2*w_0,w_3+2*w_0,w_4+2*w_0,w_5+2*w_0,w_6-2*w_0};
i23 : P5 = ideal{w_1-2*w_0,w_2+2*w_0,w_3-2*w_0,w_4+2*w_0,w_5-2*w_0,w_6+2*w_0};
i24 : P6 = ideal{w_1+2*w_0,w_2-2*w_0,w_3+2*w_0,w_4+2*w_0,w_5-2*w_0,w_6+2*w_0};
i25 : P7 = ideal{w_1+2*w_0,w_2+2*w_0,w_3-2*w_0,w_4-2*w_0,w_5+2*w_0,w_6+2*w_0};
i26 : P8 = ideal{w_1-2*w_0,w_2-2*w_0,w_3+2*w_0,w_4-2*w_0,w_5+2*w_0,w_6+2*w_0};
i27 : -- let us see if the lines lij joining the points Pi and Pj
      -- are contained in the threefold WB6
      l12 = ideal{(toMap(saturate(P1*P2),1,1)).matrix};
i28 : (l12 + WB6 == l12) == true
i29 : l13 = ideal{(toMap(saturate(P1*P3),1,1)).matrix};
i30 : (l13 + WB6 == l13) == true
i31 : l14 = ideal{(toMap(saturate(P1*P4),1,1)).matrix};
i32 : (l14 + WB6 == l14) == true
i33 : l15 = ideal{(toMap(saturate(P1*P5),1,1)).matrix};
i34 : (l15 + WB6 == l15) == true
i35 : l16 = ideal{(toMap(saturate(P1*P6),1,1)).matrix};
i36 : (l16 + WB6 == l16) == true
i37 : l17 = ideal{(toMap(saturate(P1*P7),1,1)).matrix};
i38 : (l17 + WB6 == l17) == true
i39 : l18 = ideal{(toMap(saturate(P1*P8),1,1)).matrix};
i40 : (l18 + WB6 == l18) == true
i41 : -- etc...
      -- let us now change the coordinates of PP6
      -- in order to have P1 = [1:0...0]
      PP6'=ZZ/10000019[z_0..z_6];
i42 : W' = sub(WB6, {(gens PP6)_0 => (gens PP6')_0,
      (gens PP6)_1 => (gens PP6')_1 + 2*(gens PP6')_0, (gens PP6)_2 => (gens PP6')_2 + 2*(gens PP6')_0,
      (gens PP6)_3 => (gens PP6')_3 + 2*(gens PP6')_0, (gens PP6)_4 => (gens PP6')_4 + 2*(gens PP6')_0,
      (gens PP6)_5 => (gens PP6')_5 + 2*(gens PP6')_0, (gens PP6)_6 => (gens PP6')_6 + 2*(gens PP6')_0})
i43 : W'U0 = sub(oo, {(gens PP6')_0  => 1})
i44 : ConeP1 = sub(tangentCone oo, {(gens PP6')_0 => (gens PP6)_0,
      (gens PP6')_1 => (gens PP6)_1 - 2*(gens PP6)_0, (gens PP6')_2 => (gens PP6)_2 - 2*(gens PP6)_0,
      (gens PP6')_3 => (gens PP6)_3 - 2*(gens PP6)_0, (gens PP6')_4 => (gens PP6)_4 - 2*(gens PP6)_0,
      (gens PP6')_5 => (gens PP6)_5 - 2*(gens PP6)_0, (gens PP6')_6 => (gens PP6)_6 - 2*(gens PP6)_0})
i45 : degree oo == 4
i46 : -- similarly for P2,P3,P4,P5,P6,P7,P8
      -- we observe now that WB6 is not contained in a quadric hypersurface of PP6
```

```
      rationalMap toMap(WB6,2,1)
i47 : -- let us also see that a general hyperplane section S is not
      -- contained in a quadric hypersurface of PP5, where
      -- S = ideal{random(1,PP6)}+WB6
      -- for example:
      S = ideal{w_0-w_1+72*w_2-13*w_3+4*w_4+8*w_5+35*w_6}+WB6
i48: PP5 = ZZ/10000019[t_0..t_5]
i49 : inc = rationalMap map(PP5,PP6,matrix{{t_0-72*t_1+13*t_2-4*t_3-8*t_4-35*t_5,t_0,t_1,t_2,t_3,t_4,t_5}})
i50 : image oo == ideal{S_0}
i51 : inc^*S
i52 : (dim oo -1, degree oo) == (2, 10)
i53 : toMap(ooo,2,1)
```

**Code B.2.** Computational analysis of $W^7_{BS}$ (see § 6.3).

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP1xPP1xPP1xPP1 = ZZ/10000019[x_0,x_1]**ZZ/10000019[y_0,y_1]**ZZ/10000019[z_0,z_1]**ZZ/10000019[t_0,t_1];
i3 : a0001=1;
i4 : a0010=1;
i5 : a0100=1;
i6 : a1000=1;
i7 : a1110=1;
i8 : a1101=1;
i9 : a1011=1;
i10 : a0111=1;
i11 : X = ideal{a0001*x_0*y_0*z_0*t_1+a0010*x_0*y_0*z_1*t_0+a0100*x_0*y_1*z_0*t_0+a1000*x_1*y_0*z_0*t_0+
      a1110*x_1*y_1*z_1*t_0+a1101*x_1*y_1*z_0*t_1+a1011*x_1*y_0*z_1*t_1+a0111*x_0*y_1*z_1*t_1};
i12 : phi = rationalMap map(PP1xPP1xPP1xPP1, ZZ/10000019[w_0..w_7], matrix(PP1xPP1xPP1xPP1,{{x_1*y_1*z_1*t_1,
      x_1*y_0*z_0*t_1, x_0*y_0*z_1*t_1, x_1*y_0*z_1*t_0, x_0*y_0*z_0*t_0,
      x_0*y_1*z_1*t_0, x_1*y_1*z_0*t_0, x_0*y_1*z_0*t_1}}));
i13 : WB7 = phi(X);
i14 : (dim oo -1, degree oo) == (3,12)
i15 : PP7 = ring WB7;
i16 : P1 = ideal{w_1, w_2, w_3, w_4, w_5, w_6, w_7};
i17 : P2 = ideal{w_0, w_2, w_3, w_4, w_5, w_6, w_7};
i18 : P3 = ideal{w_0, w_1, w_3, w_4, w_5, w_6, w_7};
i19 : P4 = ideal{w_0, w_1, w_2, w_4, w_5, w_6, w_7};
i20 : P1' = ideal{w_0, w_1, w_2, w_3, w_5, w_6, w_7};
i21 : P2' = ideal{w_0, w_1, w_2, w_3, w_4, w_6, w_7};
i22 : P3' = ideal{w_0, w_1, w_2, w_3, w_4, w_5, w_7};
i23 : P4' = ideal{w_0, w_1, w_2, w_3, w_4, w_5, w_6};
i24 : -- let us see if the lines lij joining the points Pi and Pj
      -- are contained in the threefold WB7
      l12 = ideal{(toMap(saturate(P1*P2),1,1)).matrix};
i25 : (l12 + WB7 == l12) == true
i26 : l13 = ideal{(toMap(saturate(P1*P3),1,1)).matrix};
i27 : (l13 + WB7 == l13) == true
i28 : l14 = ideal{(toMap(saturate(P1*P4),1,1)).matrix};
i29 : (l14 + WB7 == l14) == true
i30 : l11' = ideal{(toMap(saturate(P1*P1'),1,1)).matrix};
i31 : (l11' + WB7 == l11') == false
i32 : l12' = ideal{(toMap(saturate(P1*P2'),1,1)).matrix};
i33 : (l12' + WB7 == l12') == true
i34 : l13' = ideal{(toMap(saturate(P1*P3'),1,1)).matrix};
i35 : (l13' + WB7 == l13') == true
i36 : l14' = ideal{(toMap(saturate(P1*P4'),1,1)).matrix};
i37 : (l14' + WB7 == l14') == true
i38 : -- etc...
      sub(WB7, {(gens PP7)_0=>1});
i39 : ConeP1 = tangentCone oo
i40 : degree oo == 4
i41 : sub(WB7, {(gens PP7)_1=>1});
i42 : ConeP2 = tangentCone oo
i43 : degree oo == 4
i44 : -- etc.. similarly for P3,P4,P5,P1',P2',P3',P4'
```

**Code B.3.** Computational analysis of $W^8_{BS}$ (see § 6.4).

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP4 = ZZ/10000019[x_0..x_4];
i3 : Q = ideal{x_0^2-x_1^2 -x_2^2+x_3^2};
i4 : R = ideal{2*x_0^2-x_1^2-3*x_2^2+2*x_3^2};
i5 : fixedconic1 = ideal{x_2,x_3,x_4^2-R_0};
i6 : fixedconic2 = ideal{x_0,x_1,x_4^2+R_0};
i7 : four = associatedPrimes (fixedconic1+Q)
i8 : p1 = four#0;
i9 : p2 = four#1;
i10 : p1' = four#2;
i11 : p2' = four#3;
i12 : four' = associatedPrimes (fixedconic2+Q)
i13 : p3 = four'#0;
i14 : p4 = four'#1;
i15 : p3' = four'#2;
i16 : p4' = four'#3;
i17 : PP9 = ZZ/10000019[z_0..z_9];
i18 : phi = rationalMap map(PP4,PP9,matrix{{x_4^2*x_0+x_0*R_0,x_4^2*x_1+x_1*R_0,x_4^2*x_2-x_2*R_0,
      x_4^2*x_3-x_3*R_0,x_4*x_0^2,x_4*x_1^2,x_4*x_2^2,x_4*x_3^2,x_4*x_0*x_1,x_4*x_2*x_3}});
i19 : phiY = phi(Q);
i20 : H8 = ideal{phiY_0}
i21 : PP8 = ZZ/10000019[w_0,w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8];
i22 : inclusion = rationalMap map(PP8,PP9, matrix(PP8,{{w_0,w_1,w_2,w_3,w_4+w_5-w_6,w_4,w_5,w_6,w_7,w_8}}));
i23 : H8 == image inclusion
i24 : WB8 = inclusion^* phiY;
i25 : (dim oo -1, degree oo) == (3,14)
i26 : P1 = inclusion^* phi(p1)
i27 : P2 = inclusion^* phi(p2)
i28 : P3 = inclusion^* phi(p3)
i29 : P4 = inclusion^* phi(p4)
i30 : P1' = inclusion^* phi(p1')
i31 : P2' = inclusion^* phi(p2')
i32 : P3' = inclusion^* phi(p3')
i33 : P4' = inclusion^* phi(p4')
i34 : -- let us see if the lines lij joining the points Pi and Pj
      -- are contained in the threefold WB8
      l12 = ideal{(toMap(saturate(P1*P2),1,1)).matrix};
i35 : (l12 + WB8 == l12) == true
i36 : l13 = ideal{(toMap(saturate(P1*P3),1,1)).matrix};
i37 : (l13 + WB8 == l13) == true
i38 : l14 = ideal{(toMap(saturate(P1*P4),1,1)).matrix};
i39 : (l14 + WB8 == l14) == true
i40 : l11' = ideal{(toMap(saturate(P1*P1'),1,1)).matrix};
i41 : (l11' + WB8 == l11') == false
i42 : l12' = ideal{(toMap(saturate(P1*P2'),1,1)).matrix};
i43 : (l12' + WB8 == l12') == false
i44 : l13' = ideal{(toMap(saturate(P1*P3'),1,1)).matrix};
i45 : (l13' + WB8 == l13') == true
i46 : l14' = ideal{(toMap(saturate(P1*P4'),1,1)).matrix};
i47 : (l14' + WB8 == l14') == true
i48 : -- etc...
      proj1 = rationalMap toMap(P1,1,1);
i49 : proj1' = rationalMap toMap(proj1(P1'),1,1);
i50 : proj2 = rationalMap toMap(proj1'(proj1(P2)),1,1);
i51 : proj3 = rationalMap toMap(proj2(proj1'(proj1(P3))),1,1);
i52 : proj3' = rationalMap toMap(proj3(proj2(proj1'(proj1(P3')))),1,1);
i53 : proj = proj1*proj1'*proj2*proj3*proj3'
i54 : isBirational(proj | WB8)
i55 : PP3 = target proj;
i56 : septies = rationalMap map( PP3, PP8, matrix(inverseMap(proj|WB8)) )
i57 : image oo == WB8
i58 : baseL = associatedPrimes ideal septies
i59 : e0= baseL#0;
i60 : l1= baseL#1;
```

```
i61 : l2= baseL#2;
i62 : s1= baseL#3;
i63 : s2= baseL#4;
i64 : l2'= baseL#5;
i65 : l1'= baseL#6;
i66 : l0= baseL#7;
i67 : r1= baseL#8;
i68 : t1= baseL#9;
i69 : r2= baseL#10;
i70 : t2= baseL#11;
i71 : C= baseL#12;
i72 : v = saturate(l1+l2+l0);
i73 : q1 = saturate(l1+r1+s1+e0+l2')
i74 : q2 = saturate(l2+r2+s2+e0+l1')
i75 : ar = saturate(r1+r2+l0)
i76 : as = saturate(s1+s2+l0)
i77 : at = saturate(t1+t2+l0)
i78 : a1 = saturate(l1+t1)
i79 : a2 = saturate(l2+t2)
i80 : b1 = saturate(r1+t1+C)
i81 : b2 = saturate(r2+t2+C)
i82 : c1 = saturate(s1+t1)
i83 : c2 = saturate(s2+t2)
i84 : q1' = saturate(l1'+t1)
i85 : q2' = saturate(l2'+t2)
i86 : -- general septic surface of the linear system :
      N = septies^* ideal{random(1,PP8)};
i87 : (dim oo -1, degree oo) == (2, 7)
i88 : -- N is double along l0,l1,l2,l1',l2',C
      (minors(1,jacobian(N))+ l1 == l1) == true
i89 : (minors(1,jacobian(N))+ l2 == l2) == true
i90 : (minors(1,jacobian(N))+ l2' == l2') == true
i91 : (minors(1,jacobian(N))+ l1' == l1') == true
i92 : (minors(1,jacobian(N))+ l0 == l0) == true
i93 : (minors(1,jacobian(N))+ C == C) == true
i94 : -- N is triple at v
      (minors(1,jacobian(jacobian(N)))+minors(1,jacobian(N))+ v == v) == true
i95 : -- N is quadruple at q1 and q2
      (minors(1,jacobian(jacobian(jacobian(N))))+minors(1,jacobian(jacobian(N)))+
      minors(1,jacobian(N))+ q1 == q1) == true
i96 : (minors(1,jacobian(jacobian(jacobian(N))))+minors(1,jacobian(jacobian(N)))+
       minors(1,jacobian(N))+ q2 == q2) == true
```

**Code B.4.** Computational analysis of $W^9_{BS}$ (see § 6.5).

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP5 = ZZ/10000019[x_0, x_1, x_2, y_3, y_4, y_5];
i3 : s1 = x_0^2-3*x_1^2+2*x_2^2;
i4 : s2 = 3*x_0^2-8*x_1^2+5*x_2^2;
i5 : r1 = 3*y_3^2-8*y_4^2+5*y_5^2;
i6 : r2 = y_3^2-3*y_4^2+2*y_5^2;
i7 : X = ideal{s1+r1, s2+r2};
i8 : (dim oo -1, degree oo) == (3,4)
i9 : PP11 = ZZ/10000019[Z_0..Z_11];
i10 : phi = rationalMap map(PP5, PP11, matrix(PP5,{{x_0^2, x_1^2, x_2^2, x_0*x_1,
      x_0*x_2, x_1*x_2, y_3^2, y_4^2, y_5^2, y_3*y_4, y_3*y_5, y_4*y_5}}));
i11 : phi(X)
i12 : (dim oo -1, degree oo) == (3,16)
i13 : H9 = ideal{ooo_0, ooo_1}
i14 : phi(X) + H9 == phi(X)
i15 : PP9 = ZZ/10000019[w_0..w_9];
i16 : inclusion = rationalMap map(PP9,PP11, matrix(PP9,{{w_0+21*w_4-55*w_5+34*w_6,
       w_0+8*w_4-21*w_5+13*w_6, w_0, w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9 }}));
i17 : image oo == H9
i18 : WB9 = inclusion^* (phi(X));
i19 : (dim oo -1, degree oo) == (3, 16)
```

```
i20 : rationalMap map(PP11,PP9, sub(matrix inverseMap(inclusion||H9), PP11))
i21 : pigreca = phi* oo
i23 : fixedPlanex = associatedPrimes (X+ideal{x_0,x_1,x_2});
i24 : fixedPlaney = associatedPrimes (X+ideal{y_3,y_4,y_5});
i25 : P1 = inclusion^* phi(fixedPlaney#0);
i26 : P4 = inclusion^* phi(fixedPlaney#1);
i27 : P2 = inclusion^* phi(fixedPlaney#2);
i28 : P3 = inclusion^* phi(fixedPlaney#3);
i29 : P1' = inclusion^* phi(fixedPlanex#0);
i30 : P4' = inclusion^* phi(fixedPlanex#1);
i31 : P2' = inclusion^* phi(fixedPlanex#2);
i32 : P3' = inclusion^* phi(fixedPlanex#3);
i33 : -- let us see if the lines lij joining the points Pi and Pj
      -- are contained in the threefold WB9
      l12 = ideal{(toMap(saturate(P1*P2),1,1)).matrix};
i34 : (l12 + WB9 == l12) == false
i35 : l13 = ideal{(toMap(saturate(P1*P3),1,1)).matrix};
i36 : (l13 + WB9 == l13) == false
i37 : l14 = ideal{(toMap(saturate(P1*P4),1,1)).matrix};
i38 : (l14 + WB9 == l14) == false
i39 : l11' = ideal{(toMap(saturate(P1*P1'),1,1)).matrix};
i40 : (l11' + WB9 == l11') == true
i41 : l12' = ideal{(toMap(saturate(P1*P2'),1,1)).matrix};
i42 : (l12' + WB9 == l12') == true
i43 : l13' = ideal{(toMap(saturate(P1*P3'),1,1)).matrix};
i44 : (l13' + WB9 == l13') == true
i45 : l14' = ideal{(toMap(saturate(P1*P4'),1,1)).matrix};
i46 : (l14' + WB9 == l14') == true
i47 : -- etc..
      proj1 = rationalMap toMap(P2,1,1);
i48 : proj2 = rationalMap toMap(proj1(P3),1,1);
i49 : proj3 = rationalMap toMap(proj2(proj1(P4)),1,1);
i50 : proj4 = rationalMap toMap(proj3(proj2(proj1(P2'))),1,1);
i51 : proj5 = rationalMap toMap(proj4(proj3(proj2(proj1(P3')))),1,1);
i52 : proj6 = rationalMap toMap(proj5(proj4(proj3(proj2(proj1(P4'))))),1,1);
i53 : proj = proj1*proj2*proj3*proj4*proj5*proj6;
i54 : proj(WB9)
i55 : PP3 = ring oo;
i56 : isBirational( proj|WB9 )
i57 : septics = rationalMap map( PP3, PP9, matrix(inverseMap( proj|WB9 )));
i58 : time image oo == WB9
i59 : comp = associatedPrimes(ideal septics)
i60 : l3' = comp#0;
i61 : l2' = comp#1;
i62 : r21 = comp#2;
i63 : r11 = comp#3;
i64 : r31 = comp#4;
i65 : l1' = comp#5;
i66 : r23 = comp#6;
i67 : r13 = comp#7;
i68 : r33 = comp#8;
i69 : r22 = comp#9;
i70 : r12 = comp#10;
i71 : r32 = comp#11;
i72 : l1 = comp#12;
i73 : l2 = comp#13;
i74 : l3 = comp#14;
i75 : -- trihedron T' :
       f1' = ideal{(gens PP3)_3};
i76 : f2' = ideal{(gens PP3)_1+(gens PP3)_3};
i77 : f3' = ideal{(gens PP3)_2+(gens PP3)_3};
i78 : f1'+f2' == l3'
i79 : f1'+f3' == l2'
i80 : f2'+f3' == l1'
i81 : v'= saturate(f1'+f2'+f3')
i82 : -- trihedron T :
       f1 = ideal{(gens PP3)_0-55*(gens PP3)_1+34*(gens PP3)_2};
i83 : f2 = ideal{(gens PP3)_0  - 21*(gens PP3)_1  +13*(gens PP3)_2};
i84 : f3 = ideal{(gens PP3)_0};
```

```
i85 : f1+f2 == l3
i86 : f1+f3 == l2
i87 : f2+f3 == l1
i88 : v = saturate(l1+l2+l3)
i89 : r11 == f1+f1'
i90 : r12 == f1+f2'
i91 : r13 == f1+f3'
i92 : r21 == f2+f1'
i93 : r22 == f2+f2'
i94 : r23 == f2+f3'
i95 : r31 == f3+f1'
i96 : r32 == f3+f2'
i97 : r33 == f3+f3'
i98 : -- general septic surface of the linear system :
       K = septics^* ideal{random(1,PP9)};
i99 : (dim oo -1, degree oo) == (2,7)
i100 : -- K has double point along l1,l2,l3,l1',l2',l3' :
       (minors(1,jacobian(K))+l1 == l1) == true
i101 : (minors(1,jacobian(K))+l2 == l2) == true
i102 : (minors(1,jacobian(K))+l3 == l3) == true
i103 : (minors(1,jacobian(K))+l1' == l1') == true
i104 : (minors(1,jacobian(K))+l2' == l2') == true
i105 : (minors(1,jacobian(K))+l3' == l3') == true
i106 : -- K has triple point at v and v' :
       (minors(1,jacobian(jacobian(K)))+minors(1,jacobian(K))+v == v) == true
i107 : (minors(1,jacobian(jacobian(K)))+minors(1,jacobian(K))+v' == v') == true
i108 : -- remark
       septics(f1) == P2
i109 : septics(f1') == P2'
i110 : septics(f2) == P3
i111 : septics(f2') == P3'
i112 : septics(f3) == P4
i113 : septics(f3') == P4'
```

**Code B.5.** Computational analysis of $W^{10}_{BS}$ (see § 6.6).

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP2=ZZ/10000019[u_0,u_1,u_2];
i3 : PP6 = ZZ/10000019[x_0,x_1,x_2,x_3,x_4,x_5,x_6];
i4 : cubics3points = rationalMap map(PP2, PP6 , matrix{{u_1^2*u_2,
     u_1*u_2^2, u_0^2*u_2,u_0*u_2^2, u_0^2*u_1,u_0*u_1^2, u_0*u_1*u_2}});
i5 : S6 = image cubics3points
i6 : PP1 = ZZ/10000019[y_0,y_1];
i7 : PP1xPP6= PP1 ** PP6;
i8 : pr2 = rationalMap(PP1xPP6,PP6, matrix{{x_0,x_1,x_2,x_3,x_4,x_5,x_6}});
i9 : PP10 = ZZ/10000019[w_0..w_10];
i10 : phi = rationalMap map(PP1xPP6,PP10, matrix{{y_0^2*x_6,y_0^2*x_0+y_0^2*x_2,
      y_0^2*x_1+y_0^2*x_4,y_0^2*x_3+y_0^2*x_5,y_1^2*x_6,y_1^2*x_0+y_1^2*x_2,
      y_1^2*x_1+y_1^2*x_4,y_1^2*x_3+y_1^2*x_5,y_0*y_1*x_0-y_0*y_1*x_2,
      y_1*y_0*x_1-y_1*y_0*x_4,y_1*y_0*x_3-y_1*y_0*x_5}});
i11 : PP1xS6 = pr2^* S6;
i12 : WB10 = phi(PP1xS6);
i13 : (dim WB10 -1, degree WB10) == (3,18)
i14 : ideal{WB10_0,WB10_1,2*WB10_2,WB10_3,WB10_4,2*WB10_5,WB10_6,WB10_7,WB10_8,
      2*WB10_9,WB10_10,WB10_11,WB10_12,2*WB10_13,WB10_14,2*WB10_15,2*WB10_16,
      4*WB10_17,4*WB10_18,4*WB10_19}
i15 : oo == WB10
i16 : P1 = ideal{w_0,w_1,w_2,w_3,w_5-2*w_4,w_6-2*w_4,w_7-2*w_4,w_8,w_9,w_10};
i17 : P2 = ideal{w_0,w_1,w_2,w_3,w_5-2*w_4,w_6+2*w_4,w_7+2*w_4,w_8,w_9,w_10};
i18 : P3 = ideal{w_0,w_1,w_2,w_3,w_5+2*w_4,w_6-2*w_4,w_7+2*w_4,w_8,w_9,w_10};
i19 : P4 = ideal{w_0,w_1,w_2,w_3,w_5+2*w_4,w_6+2*w_4,w_7-2*w_4,w_8,w_9,w_10};
i20 : P1' = ideal{w_1-2*w_0,w_2-2*w_0,w_3-2*w_0,w_4,w_5,w_6,w_7,w_8,w_9,w_10};
i21 : P2' = ideal{w_1-2*w_0,w_2+2*w_0,w_3+2*w_0,w_4,w_5,w_6,w_7,w_8,w_9,w_10};
i22 : P3' = ideal{w_1+2*w_0,w_2-2*w_0,w_3+2*w_0,w_4,w_5,w_6,w_7,w_8,w_9,w_10};
i23 : P4' = ideal{w_1+2*w_0,w_2+2*w_0,w_3-2*w_0,w_4,w_5,w_6,w_7,w_8,w_9,w_10};
i24 : -- let us see if the lines lij joining the points Pi and Pj
```

```
      -- are contained in the threefold WB10
      l12 = ideal{(toMap(saturate(P1*P2),1,1)).matrix};
i25 : (l12 + WB10 == l12) == true
i26 : l13 = ideal{(toMap(saturate(P1*P3),1,1)).matrix};
i27 : (l13 + WB10 == l13) == true
i28 : l14 = ideal{(toMap(saturate(P1*P4),1,1)).matrix};
i29 : (l14 + WB10 == l14) == true
i30 : l11' = ideal{(toMap(saturate(P1*P1'),1,1)).matrix};
i31 : (l11' + WB10 == l11') == true
i32 : l12' = ideal{(toMap(saturate(P1*P2'),1,1)).matrix};
i33 : (l12' + WB10 == l12') == false
i34 : l13' = ideal{(toMap(saturate(P1*P3'),1,1)).matrix};
i35 : (l13' + WB10 == l13') == false
i36 : l14' = ideal{(toMap(saturate(P1*P4'),1,1)).matrix};
i37 : (l14' + WB10 == l14') == false
i38 : -- etc...
      proj1 = rationalMap toMap(P1,1,1);
i39 : proj2 = rationalMap toMap(proj1(P2),1,1);
i40 : proj3 = rationalMap toMap(proj2(proj1(P3)),1,1);
i41 : proj4 = rationalMap toMap(proj3(proj2(proj1(P4))),1,1);
i42 : proj1' = rationalMap toMap(proj4(proj3(proj2(proj1(P1')))),1,1);
i43 : proj2' = rationalMap toMap(proj1'(proj4(proj3(proj2(proj1(P2'))))),1,1);
i44 : proj3' = rationalMap toMap(proj2'(proj1'(proj4(proj3(proj2(proj1(P3')))))),1,1);
i45 : proj = proj1*proj2*proj3*proj4*proj1'*proj2'*proj3'
i46 : isBirational(proj | WB10)
i47 : PP3 = target proj;
i48 : sexties = rationalMap map( PP3, PP10, matrix(inverseMap(proj|WB10)) )
i49 : image oo == WB10
i50 : baseL = associatedPrimes ideal sexties
i51 : l23 = baseL#0
i52 : r1 = baseL#1
i53 : l12 = baseL#2
i54 : r3 = baseL#3
i55 : l13 = baseL#4
i56 : r2 = baseL#5
i57 : l02 = baseL#6
i58 : l03 = baseL#7
i59 : l01 = baseL#8
i60 : v1 = baseL#9
i61 : v2 = baseL#10
i62 : v3 = baseL#11
i63 : f0 =ideal{(gens PP3)_0};
i64 : f1 =ideal{(gens PP3)_1+(gens PP3)_2+(gens PP3)_3};
i65 : f2=ideal{(gens PP3)_1-(gens PP3)_2+(gens PP3)_3};
i66 : f3 =ideal{(gens PP3)_1+(gens PP3)_2-(gens PP3)_3};
i67 : plane = ideal{(gens PP3)_1-(gens PP3)_2-(gens PP3)_3};
i68 : l12 == f1+f2
i69 : l13 == f1+f3
i70 : l23 == f2+f3
i71 : l01 == f0+f1
i72 : l02 == f0+f2
i73 : l03 == f0+f3
i74 : r1 == plane+f1
i75 : r2 == plane+f2
i76 : r3 == plane+f3
i77 : v0 = f1+f2+f3+plane
i78 : v1 == f0+f2+f3
i79 : v2 == f0+f1+f3
i80 : v3 == f0+f1+f2
i81 : q1 = saturate(l01+r1)
i82 : q2 = saturate(l02+r2)
i83 : q3 = saturate(l03+r3)
i84 : -- general element of the linear system defining sexties :
       M = sexties^* ideal{random(1,PP10)};
i85 : (dim oo -1, degree oo)
i86 : -- M has double points along r1,r2,r3 :
       (minors(1,jacobian(M))+r1 == r1) == true
i87 : (minors(1,jacobian(M))+r2 == r2) == true
i88 : (minors(1,jacobian(M))+r3 == r3) == true
```

```
i89 : -- M has triple points at v1,v2,v3  :
       (minors(1,jacobian(jacobian(M)))+minors(1,jacobian(M))+ v1 == v1) == true
i90 : (minors(1,jacobian(jacobian(M)))+minors(1,jacobian(M))+ v2 == v2) == true
i91 : (minors(1,jacobian(jacobian(M)))+minors(1,jacobian(M))+ v3 == v3) == true
i92 : -- v0 is a quadruple point of M :
       (minors(1,jacobian(jacobian(jacobian(M))))+minors(1,jacobian(jacobian(M)))+
        minors(1,jacobian(M))+ v0 == v0) == true
```

**Code B.6.** Computational analysis of $W_{BS}^{13}$ (see § 6.7).

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP1x = ZZ/10000019[x_0,x_1];
i3 : PP1y = ZZ/10000019[y_0,y_1];
i4 : PP1z = ZZ/10000019[z_0,z_1];
i5 : X = PP1x ** PP1y ** PP1z;
i6 : use X;
i7 : pigreca = rationalMap map(X, ZZ/10000019[w_0..w_13], matrix{{x_0^2*y_0^2*z_0^2,
     x_0^2*y_0^2*z_1^2, x_0^2*y_0*y_1*z_0*z_1, x_0^2*y_1^2*z_0^2, x_0^2*y_1^2*z_1^2,
     x_0*x_1*y_0^2*z_0*z_1, x_0*x_1*y_0*y_1*z_0^2, x_0*x_1*y_0*y_1*z_1^2,
     x_0*x_1*y_1^2*z_0*z_1, x_1^2*y_0^2*z_0^2, x_1^2*y_0^2*z_1^2, x_1^2*y_0*y_1*z_0*z_1,
     x_1^2*y_1^2*z_0^2, x_1^2*y_1^2*z_1^2}});
i8 : WB13 = image pigreca;
i9 : (dim oo -1, degree oo) == (3, 24)
i10 : PP13 = ring WB13;
i11 : P1 = pigreca(ideal{x_1,y_0,z_0});
i12 : P2 = pigreca(ideal{x_1,y_1,z_1});
i13 : P3 = pigreca(ideal{x_0,y_1,z_0});
i14 : P4 = pigreca(ideal{x_0,y_0,z_1});
i15 : P1' = pigreca(ideal{x_0,y_1,z_1});
i16 : P2' = pigreca(ideal{x_0,y_0,z_0});
i17 : P3' = pigreca(ideal{x_1,y_0,z_1});
i18 : P4' = pigreca(ideal{x_1,y_1,z_0});
i19 : -- let us see if the lines lij joining the points Pi and Pj
      -- are contained in the threefold WB13
      l12 = ideal{(toMap(saturate(P1*P2),1,1)).matrix};
i20 : (l12 + WB13 == l12) == false
i21 : l13 = ideal{(toMap(saturate(P1*P3),1,1)).matrix};
i22 : (l13 + WB13 == l13) == false
i23 : l14 = ideal{(toMap(saturate(P1*P4),1,1)).matrix};
i24 : (l14 + WB13 == l14) == false
i25 : l11' = ideal{(toMap(saturate(P1*P1'),1,1)).matrix};
i26 : (l11' + WB13 == l11') == false
i27 : l12' = ideal{(toMap(saturate(P1*P2'),1,1)).matrix};
i28 : (l12' + WB13 == l12') == true
i29 : l13' = ideal{(toMap(saturate(P1*P3'),1,1)).matrix};
i30 : (l13' + WB13 == l13') == true
i31 : l14' = ideal{(toMap(saturate(P1*P4'),1,1)).matrix};
i32 : (l14' + WB13 == l14') == true
i33 : -- etc..
      proj1 = rationalMap toMap(P1,1,1);
i34 : proj2 = rationalMap toMap(proj1(P2),1,1);
i35 : proj3 = rationalMap toMap(proj2(proj1(P3)),1,1);
i36 : proj4 = rationalMap toMap(proj3(proj2(proj1(P4))),1,1);
i37 : proj5 = rationalMap toMap(proj4(proj3(proj2(proj1(P1')))),1,1);
i38 : proj6 = rationalMap toMap(proj5(proj4(proj3(proj2(proj1(P2'))))),1,1);
i40 : proj7 = rationalMap toMap(proj6(proj5(proj4(proj3(proj2(proj1(P3')))))),1,1);
i41 : proj8 = rationalMap toMap(proj7(proj6(proj5(proj4(proj3(proj2(proj1(P4'))))))),1,1);
i42 : proj = proj1*proj2*proj3*proj4*proj5*proj6*proj7*proj8;
i43 : T4 = proj(WB13)
i44 : (dim oo -1, degree oo) == (3, 4)
i45 : isBirational((proj|WB13)||T4)
i46 : PP5 = ring T4;
i47 : PP3 = ZZ/10000019[t_0..t_3];
i48 : quadricsThroughVertices = rationalMap map(PP3, PP5, matrix{{(gens PP3)_0*(gens PP3)_1,
      (gens PP3)_1*(gens PP3)_2 ,(gens PP3)_1*(gens PP3)_3 ,(gens PP3)_0*(gens PP3)_2,
      (gens PP3)_0*(gens PP3)_3, (gens PP3)_2*(gens PP3)_3}});
```

```
i49 : image oo == T4
i50 : isBirational(quadricsThroughVertices||T4)
i51 : mapP5toP3 = rationalMap map( PP5, PP3, sub(matrix(inverseMap(quadricsThroughVertices||T4)), PP5))
i52 : mapWB13toP3 = (proj*mapP5toP3) | WB13;
i53 : isBirational mapWB13toP3
i54 : sexties = rationalMap map( PP3, ring WB13, matrix(inverseMap(mapWB13toP3)))
i55 : image oo == WB13
```

**Code B.7.** Computational analysis of $W_F^7$ (see § 5.4). We will use Remark 5.44, Theorem 6.17 and Remark 6.16.

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP3 = ZZ/10000019[s_0..s_3];
i3 : -- edges of the trivial tetrahedon :
     l12 = ideal{(gens PP3)_1, (gens PP3)_2};
i4 : l13 = ideal{(gens PP3)_1, (gens PP3)_3};
i5 : l23 = ideal{(gens PP3)_2, (gens PP3)_3};
i6 : l01 = ideal{(gens PP3)_0, (gens PP3)_1};
i7 : l02 = ideal{(gens PP3)_0, (gens PP3)_2};
i8 : l03 = ideal{(gens PP3)_0, (gens PP3)_3};
i9 : PP13 = ZZ/10000019[w_0..w_13];
i10 : sextiesSigma = rationalMap map(PP3,PP13, matrix{{s_0*s_1^3*s_2*s_3, s_0^2*s_1^2*s_2^2,
      s_0^2*s_1^2*s_2*s_3, s_0^2*s_1^2*s_3^2, s_0^3*s_1*s_2*s_3, s_0*s_1^2*s_2^2*s_3,
      s_0*s_1^2*s_2*s_3^2,  s_0^2*s_1*s_2^2*s_3, s_0^2*s_1*s_2*s_3^2, s_1^2*s_2^2*s_3^2,
      s_0*s_1*s_2^3*s_3, s_0*s_1*s_2^2*s_3^2, s_0*s_1*s_2*s_3^3, s_0^2*s_2^2*s_3^2}});
i11 : -- classical F-EF 3-fold
      WF13 = image sextiesSigma;
i12 : (dim WF13 -1, degree WF13) == (3, 24)
i13 : -- singular points of WF13
      -- (which is equal to the BS-EF 3-fold WB13) :
      q1 = ideal{w_0,w_1,w_2,w_3,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_12,w_13};
i14 : q2 = ideal{w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_12,w_13};
i15 : q3 = ideal{w_0,w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_11,w_12,w_13};
i16 : q4 = ideal{w_0,w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_13};
i17 : q1' = ideal{w_0,w_1,w_2,w_3,w_5,w_4,w_6,w_7,w_8,w_10,w_11,w_12,w_13};
i18 : q2' = ideal{w_0,w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_12};
i19 : q3' = ideal{w_0,w_1,w_2,w_4,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_12,w_13};
i20 : q4' = ideal{w_0,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_12,w_13};
i21 : -- let us take a general plane of PP3
      -- and the intersection points with the edges lij
      -- plane = ideal{random(1,PP3)}
      -- example:
      plane = ideal{s_0+s_1+s_2+s_3};
i22 : PP2 = ZZ/10000019[x_0,x_1,x_2];
i23 : inclusion = rationalMap map(PP2,PP3, matrix{{-(gens PP2)_0-(gens PP2)_1-(gens PP2)_2,
      (gens PP2)_0, (gens PP2)_1, (gens PP2)_2}})
i24 : image oo == plane
i25 : p01 = saturate(plane+l01) -- [0:0:-1:1]
i26 : p02 = saturate(plane+l02) -- [0:-1:0:1]
i27 : p03 = saturate(plane+l03) -- [0:-1:1:0]
i28 : p12 = saturate(plane+l12) -- [-1:0:0:1]
i29 : p13 = saturate(plane+l13) -- [-1:0:1:0]
i30 : p23 = saturate(plane+l23) -- [-1:1:0:0]
i31 : a01=inclusion^*p01; -- [0:-1:1]
i32 : a02=inclusion^*p02; -- [-1:0:1]
i33 : a03=inclusion^*p03; -- [-1:1:0]
i34 : a12=inclusion^*p12; -- [0:0:1]
i35 : a13=inclusion^*p13; -- [0:1:0]
i36 : a23=inclusion^*p23; -- [1:0:0]
i37 : -- in the above plane, let us take
      -- a general cubic curve through the six points pij
      -- rationalMap toMap(saturate(a01*a02*a03*a12*a13*a23),3,1);
      -- cubicThrough6points = oo^*(ideal{random(1,ring(image oo))})
      -- example:
      cubicThrough6points = ideal{(gens PP2)_1^2*(gens PP2)_2+(gens PP2)_1*(gens PP2)_2^2+
      (gens PP2)_0^2*(gens PP2)_2+(gens PP2)_0*(gens PP2)_2^2+(gens PP2)_0^2*(gens PP2)_1+
```

```
      (gens PP2)_0*(gens PP2)_1^2+(gens PP2)_0*(gens PP2)_1*(gens PP2)_2}
--              2        2    2               2       2      2
-- o37 = ideal(x x  + x x  + x x  + x x x  + x x  + x x  + x x )
--              0 1    0 1    0 2    0 1 2    1 2    0 2    1 2
i38 : (dim oo -1, degree oo, genus oo)==(1,3,1)
i39 : delta = inclusion(cubicThrough6points)
--                                 2       2    2               2       2      2
-- o39 = ideal (s  + s  + s  + s , s s  + s s  + s s  + s s s  + s s  + s s  + s s )
--               0    1    2    3   1 2    1 2    1 3    1 2 3    2 3    1 3    2 3
i40 : (dim oo -1, degree oo, genus oo)==(1,3,1)
i41 : (delta+l01==p01) == true
i42 : (delta+l02==p02) == true
i43 : (delta+l03==p03) == true
i44 : (delta+l12==p12) == true
i45 : (delta+l13==p13) == true
i46 : (delta+l23==p23) == true
i47 : nudelta=sextiesSigma(delta)
i48 : (dim oo -1, degree oo, genus oo)==(1,6,1)
i49 : (nudelta+q1 == q1) == false
i50 : (nudelta+q2 == q2) == false
i51 : (nudelta+q3 == q3) == false
i52 : (nudelta+q4 == q4) == false
i53 : (nudelta+q1' == q1') == false
i54 : (nudelta+q2' == q2') == false
i55 : (nudelta+q3' == q3') == false
i56 : (nudelta+q4' == q4') == false
i57 : spannudelta=ideal{nudelta_0,nudelta_1,nudelta_2,nudelta_3,nudelta_4,
      nudelta_5,nudelta_6,nudelta_7}
i58 : (dim oo -1, degree oo)==(5,1)
i59 : -- let us construct the F-EF 3-fold WF7
      -- as projection og WF13 from spannudelta
      proj = rationalMap toMap(spannudelta,1,1)
i60 : WF7 = proj(WF13)
i61 : (dim oo -1, degree oo)==(3,12)
i62 : -- let us see the configuration of
      -- the singular points of WF7:
      P1 = proj(q1);   -- [ 0: 0: 0: 0: 1:-1: 1: 1]
i63 : P2 = proj(q2);   -- [ 0: 0: 0: 0: 0: 0: 0: 1]
i64 : P3 = proj(q3);   -- [ 1: 0: 0:-2: 0: 0: 4: 2]
i65 : P4 = proj(q4);   -- [ 1: 0:-2: 0: 4: 0: 0: 2]
i66 : P1' = proj(q1'); -- [ 1: 0: 0: 0: 0: 0: 0: 0]
i67 : P2' = proj(q2'); -- [ 1: 2: 0:-2: 2:-2: 4: 4]
i68 : P3' = proj(q3'); -- [ 0: 0: 0: 0: 1: 0: 0: 0]
i69 : P4' = proj(q4'); -- [ 0: 0: 0: 0: 0: 0: 1: 0]
i70 : line12 = ideal{(toMap(saturate(P1*P2),1,1)).matrix};
i71 : (line12 + WF7 == line12) == true
i72 : line13 = ideal{(toMap(saturate(P1*P3),1,1)).matrix};
i73 : (line13 + WF7 == line13) == true
i74 : line14 = ideal{(toMap(saturate(P1*P4),1,1)).matrix};
i75 : (line14 + WF7 == line14) == true
i76 : line23 = ideal{(toMap(saturate(P2*P3),1,1)).matrix};
i77 : (line23 + WF7 == line23) == true
i78 : line24 = ideal{(toMap(saturate(P2*P4),1,1)).matrix};
i79 : (line24 + WF7 == line24) == true
i80 : line34 = ideal{(toMap(saturate(P3*P4),1,1)).matrix};
i81 : (line34 + WF7 == line34) == true
i82 : line11' = ideal{(toMap(saturate(P1*P1'),1,1)).matrix};
i83 : (line11' + WF7 == line11') == false
i84 : line22' = ideal{(toMap(saturate(P2*P2'),1,1)).matrix};
i85 : (line22' + WF7 == line22') == false
i86 : line33' = ideal{(toMap(saturate(P3*P3'),1,1)).matrix};
i87 : (line33' + WF7 == line33') == false
i88 : line44' = ideal{(toMap(saturate(P4*P4'),1,1)).matrix};
i89 : (line44' + WF7 == line44') == false
i90 : -- let us take the rational map
      -- defined by the linear system
      -- of the sextics of PP3 double along the edges lij
      -- and containing the curve delta
      sextiesX = sextiesSigma*proj
```

```
i91 : WF7 == image oo
i92 : -- base locus of sextiesX
      baseX = associatedPrimes(ideal sextiesX)
i93 : baseX#0 == delta
i94 : baseX#1 == l13
i95 : baseX#2 == l12
i96 : baseX#3 == l01
i97 : baseX#4 == l23
i98 : baseX#5 == l02
i99 : baseX#6 == l03
i100 : PP7 = ring WF7
i101 : -- let us take a general element X of the
       -- linear system of the sextics double
       -- along the edges of a trivial tetrahedron
       -- and containing the cubic curve delta:
       -- X = sextiesX^*(ideal{random(1,PP7)})
       -- for example let us take:
       matrix sextiesX
i102 : X = ideal{2*oo_(0,0)+oo_(0,4)+oo_(0,6)-oo_(0,7)}
--              2 2 2    3            3             3      2 2 2     2 2 2        2 2     2 2 2           3
-- o102 = ideal(s s s  - s s s s  - s s s s  - 2s s s s  + s s s  + 2s s s  + s s s s  + 2s s s  - 2s s s s )
--              0 1 2    0 1 2 3    0 1 2 3     0 1 2 3    0 1 3     0 2 3    0 1 2 3     1 2 3     0 1 2 3
i103 : (dim oo -1, degree oo) == (2,6)
i104 : -- remark: its image via sextiesX is contains no
       -- singular points of WF7
       S = sextiesX(X);
i105 : (S+P1 == P1) == false
i106 : (S+P2 == P2) == false
i107 : (S+P3 == P3) == false
i108 : (S+P4 == P4) == false
i109 : (S+P1' == P1') == false
i110 : (S+P2' == P2') == false
i111 : (S+P3' == P3') == false
i112 : (S+P4' == P4') == false
i113 : -- as for a general Enriques sextic,
       -- the tangent cone to X at a vertex of the tetrahedron
       -- is the union of the three faces containing that vertex
       PP3' = ZZ/10000019[x_0..x_3];
i114 : Conev0 = tangentCone(sub(X, {(gens PP3)_0 => 1 }))
i115 : degree oo == 3
i116 : Conev1 = tangentCone(sub(X, {(gens PP3)_1 => 1 }))
i117 : degree oo == 3
i118 : Conev2 = tangentCone(sub(X, {(gens PP3)_2 => 1 }))
i119 : degree oo == 3
i120 : Conev3 = tangentCone(sub(X, {(gens PP3)_3 => 1 }))
i121 : degree oo == 3
i122 : -- the tangent cone to X at a point pij
       -- it the union of two planes containing lij:
       -- let us take a change of coordinates
       -- in order to see p01 as the point [0:0:0:1]
       sub(X, {(gens PP3)_0 => (gens PP3')_0, (gens PP3)_1 => (gens PP3')_1,
       (gens PP3)_2 => (gens PP3')_2-(gens PP3')_3, (gens PP3)_3 => (gens PP3')_3 });
i123 : sub(oo, {(gens PP3')_3  => 1})
i124 : tangentCone oo
i125 : Conep01 = sub(oo, {(gens PP3')_0 => (gens PP3)_0, (gens PP3')_1 => (gens PP3)_1,
       (gens PP3')_2 => (gens PP3)_2+(gens PP3)_3, (gens PP3')_3 => (gens PP3)_3 })
i126 : degree oo == 2
i127 : -- let us take a change of coordinates
       -- in order to see p02 as the point [0:0:0:1]
       sub(X, {(gens PP3)_0 => (gens PP3')_0, (gens PP3)_1 => (gens PP3')_1-(gens PP3')_3,
       (gens PP3)_2 => (gens PP3')_2, (gens PP3)_3 => (gens PP3')_3 });
i128 : sub(oo, {(gens PP3')_3  => 1})
i129 : tangentCone oo
i130 : Conep02 = sub(oo, {(gens PP3')_0 => (gens PP3)_0, (gens PP3')_1 => (gens PP3)_1+(gens PP3)_3,
       (gens PP3')_2 => (gens PP3)_2, (gens PP3')_3 => (gens PP3)_3 })
i131 : degree oo == 2
i132 : -- let us take a change of coordinates
       -- in order to see p03 as the point [0:0:1:0]
       sub(X, {(gens PP3)_0 => (gens PP3')_0, (gens PP3)_1 => (gens PP3')_1-(gens PP3')_2,
```

```
       (gens PP3)_2 => (gens PP3')_2, (gens PP3)_3 => (gens PP3')_3 });
i133 : sub(oo, {(gens PP3')_2  => 1})
i134 : tangentCone oo
i135 : Conep03 = sub(oo, {(gens PP3')_0 => (gens PP3)_0, (gens PP3')_1 => (gens PP3)_1+(gens PP3)_2,
       (gens PP3')_2 => (gens PP3)_2, (gens PP3')_3 => (gens PP3)_3 })
i136 : degree oo == 2
i137 : -- let us take a change of coordinates
       -- in order to see p12 as the point [0:0:0:1]
       sub(X, {(gens PP3)_0 => (gens PP3')_0-(gens PP3')_3, (gens PP3)_1 => (gens PP3')_1,
       (gens PP3)_2 => (gens PP3')_2, (gens PP3)_3 => (gens PP3')_3 });
i138 : sub(oo, {(gens PP3')_3  => 1})
i139 : tangentCone oo
i140 : Conep12 = sub(oo, {(gens PP3')_0 => (gens PP3)_0+(gens PP3)_3, (gens PP3')_1 => (gens PP3)_1,
       (gens PP3')_2 => (gens PP3)_2, (gens PP3')_3 => (gens PP3)_3 })
i141 : degree oo == 2
i142 : -- let us take a change of coordinates
       -- in order to see p13 as the point [0:0:1:0]
       sub(X, {(gens PP3)_0 => (gens PP3')_0-(gens PP3')_2, (gens PP3)_1 => (gens PP3')_1,
       (gens PP3)_2 => (gens PP3')_2, (gens PP3)_3 => (gens PP3')_3 });
i143 : sub(oo, {(gens PP3')_2  => 1})
i144 : tangentCone oo
i145 : Conep13 = sub(oo, {(gens PP3')_0 => (gens PP3)_0+(gens PP3)_2, (gens PP3')_1 => (gens PP3)_1,
       (gens PP3')_2 => (gens PP3)_2, (gens PP3')_3 => (gens PP3)_3 })
i146 : degree oo == 2
i147 : -- let us take a change of coordinates
       -- in order to see p23 as the point [0:1:0:0]
       sub(X, {(gens PP3)_0 => (gens PP3')_0-(gens PP3')_1, (gens PP3)_1 => (gens PP3')_1,
       (gens PP3)_2 => (gens PP3')_2, (gens PP3)_3 => (gens PP3')_3 });
i148 : sub(oo, {(gens PP3')_1  => 1})
i149 : tangentCone oo
i150 : Conep23 = sub(oo, {(gens PP3')_0 => (gens PP3)_0+(gens PP3)_1, (gens PP3')_1 => (gens PP3)_1,
       (gens PP3')_2 => (gens PP3)_2, (gens PP3')_3 => (gens PP3)_3 })
i151 : degree oo == 2
i152 : -- let us see that the tangent cone to X
       -- at a point of lij is a couple of planes
       -- containing lij

       -- let us take a point [x:0:0:y] of l12
       -- since we have already studied
       -- the points v0=[1:0:0:0] and v3=[1:0:0:0]
       -- we can assume x and y not equal to zero
       -- so let us consider the point [a:0:0:1]
       -- with a not equal to zero
       -- let us take a change of coordinates
       -- in order to see [a:0:0:1] as the point [0:0:0:1]
       A = ZZ/10000019[a];
i153 : R = A[s_0..s_3];
i154 : R' = A[r_0..r_3];
i155 : newX = sub(X,R)
i156 : sub(newX, {(gens R)_0 => (gens R')_0+(gens A)_0*(gens R')_3, (gens R)_1 => (gens R')_1,
       (gens R)_2 => (gens R')_2, (gens R)_3 => (gens R')_3 })
i157 : sub(oo,(gens R')_3=>1)
i158 : sub(tangentCone oo, {(gens R')_0 => (gens R)_0-(gens A)_0*(gens R)_3, (gens R')_1 => (gens R)_1,
       (gens R')_2 => (gens R)_2, (gens R')_3 => (gens R)_3 })
i159 : -- we obtain
       --          2 2       3               2 2
       --   ideal(a s  + (- a  - 2a)s s  + 2a s )
       --            1               1 2      2
       -- so the tangent cone to X at the point [a:0:0:1]
       -- is the union of two planes containing l12

       -- similarly for the points of l13,l23,l01,l02,l03

       -- let us study the singular locus of X
       -- in order to verify if X has other kinds of singularity
       JX = jacobian(X)
i160 : singX = minors(1,JX)+X
i161 : compSingX = associatedPrimes singX
i162 : compSingX#0 == l01
```

```
i163 : compSingX#1 == l02
i164 : compSingX#2 == l12
i165 : compSingX#3 == l23
i166 : compSingX#4 == l03
i167 : compSingX#5 == l13
i168 : -- compSingX#6 is the point [0:0:2:1]
       -- compSingX#7 is the point [0:0:1:2]
       x' = compSingX#8 -- is a point of l01
i169 : x'' = compSingX#9 -- is the point of l01
i170 : -- remark (in order to understand x' and x''):
       sub(X, QQ[s_0..s_3])
i171 : minors(1,jacobian(oo))+oo
i172 : compSingX' = associatedPrimes oo
i173 : (associatedPrimes(sub(compSingX'#6,PP3)))#0 == x'
i174 : (associatedPrimes(sub(compSingX'#6,PP3)))#1 == x''
       -- where compSingX'#6 is:
       --                2           2
       --  ideal (s , s , 2s  + 3s s  + 2s )
       --          1   0    2     2 3     3
i175 :
       -- furthermore, if r = sqrt(2) we have that
       -- compSingX#10 are the points [r:0:0:1] and [-r:0:0:1]
       -- compSingX#11 are the points [r:0:1:0] and [-r:0:1:0]
       -- compSingX#12 are the points [0:r:0:1] and [0:-r:0:1]
       -- compSingX#13 is the point [1:1:0:0]
       compSingX#14 == p23
i176 : -- compSingX#15 are the points [0:r:1:0] and [0:r:1:0]
       -- hence X has just the singularities described above
```

**Code B.8.** Computational analysis of $W^9_{KLM}$ (see § 7.3).

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP3 = ZZ/10000019[s_0..s_3];
i3 : PP13 = ZZ/10000019[w_0..w_13];
i4 : sexties = rationalMap map(PP3,PP13, matrix{{s_0*s_1^3*s_2*s_3, s_0^2*s_1^2*s_2^2,
     s_0^2*s_1^2*s_2*s_3, s_0^2*s_1^2*s_3^2, s_0^3*s_1*s_2*s_3, s_0*s_1^2*s_2^2*s_3,
     s_0*s_1^2*s_2*s_3^2, s_0^2*s_1*s_2^2*s_3, s_0^2*s_1*s_2*s_3^2, s_1^2*s_2^2*s_3^2,
     s_0*s_1*s_2^3*s_3, s_0*s_1*s_2^2*s_3^2, s_0*s_1*s_2*s_3^3, s_0^2*s_2^2*s_3^2}});
i5 : WF13 = image sexties;
i6 : (dim WF13 -1, degree WF13) == (3, 24)
i7 : P1 = ideal{w_0,w_1,w_2,w_3,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_12,w_13};
i8 : P2 = ideal{w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_12,w_13};
i9 : P3 = ideal{w_0,w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_11,w_12,w_13};
i10 : P4 = ideal{w_0,w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_13};
i11 : P1' = ideal{w_0,w_1,w_2,w_3,w_5,w_4,w_6,w_7,w_8,w_10,w_11,w_12,w_13};
i12 : P2' = ideal{w_0,w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_12};
i13 : P3' = ideal{w_0,w_1,w_2,w_4,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_12,w_13};
i14 : P4' = ideal{w_0,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_12,w_13};
i15 : J = jacobian((map sexties).matrix);
i16 : JJ = jacobian(J);
i17 : JJl23 = sub(JJ,{(gens PP3)_2=> 0, (gens PP3)_3 =>0})
i18 : SPANnuF23 = ideal{w_5,w_6,w_7,w_8,w_9,w_10,w_11,w_12,w_13};
i19 : -- H12 = ideal{random(1,PP13)};
      -- for example
      H12 = ideal{w_0+11*w_1+2*w_2+3*w_3+5*w_4+4*w_5+6*w_6-7*w_7-8*w_8-9*w_9+
      10*w_10-11*w_11+12*w_12+13*w_13};
i20 : S = H12+WF13;
i21 : E3 = saturate(S+SPANnuF23)
i22 : (dim oo -1, degree oo, genus oo) == (1, 4, 1)
i23 : SPANE3 = ideal{E3_0,E3_1,E3_2,E3_3,E3_4,E3_5,E3_6,E3_7,E3_8,E3_9};
i24 : PP9 = ZZ/10000019[z_0..z_9];
i25 : projE3 = rationalMap map(PP13,PP9, matrix{{SPANE3_9,SPANE3_8,SPANE3_7,SPANE3_6,
      SPANE3_5,SPANE3_4,SPANE3_3,SPANE3_2,SPANE3_1,SPANE3_0}})
i26 : KLM = projE3(WF13)
i27 : (dim oo -1, degree oo) == (3, 16)
i28 : isBirational((projE3|WF13)||KLM)
```

```
i29 : projE3(P1')==projE3(P4')
i30 : projE3(P4')==projE3(P2)
i31 : projE3(P2)==projE3(P3')
i32 : p1 = projE3(P1')
i33 : p2 = projE3(P2')
i34 : p3 = projE3(P3)
i35 : p4 = projE3(P4)
i36 : p5 = projE3(P1)
i37 : line15 = ideal{(toMap(saturate(p1*p5),1,1)).matrix};
i38 : (line15 + KLM == line15) == true
i39 : line25 = ideal{(toMap(saturate(p2*p5),1,1)).matrix};
i40 : (line25 + KLM == line25) == true
i41 : line35 = ideal{(toMap(saturate(p3*p5),1,1)).matrix};
i42 : (line35 + KLM == line35) == true
i43 : line45 = ideal{(toMap(saturate(p4*p5),1,1)).matrix};
i44 : (line45 + KLM == line45) == true
i45 : line12 = ideal{(toMap(saturate(p1*p2),1,1)).matrix};
i46 : (line12 + KLM == line12) == false
i47 : line13 = ideal{(toMap(saturate(p1*p3),1,1)).matrix};
i48 : (line13 + KLM == line13) == true
i49 : line14 = ideal{(toMap(saturate(p1*p4),1,1)).matrix};
i50 : (line14 + KLM == line14) == true
i51 : line23 = ideal{(toMap(saturate(p2*p3),1,1)).matrix};
i52 : (line23 + KLM == line23) == true
i53 : line24 = ideal{(toMap(saturate(p2*p4),1,1)).matrix};
i54 : (line24 + KLM == line24) == true
i55 : line34 = ideal{(toMap(saturate(p3*p4),1,1)).matrix};
i56 : (line34 + KLM == line34) == false
i57 : sub(KLM, {(gens PP9)_5=>1});
i58 : Conep1 = tangentCone oo
i59 : degree oo == 4
i60 : sub(KLM, {(gens PP9)_9=>1});
i61 : Conep2 = tangentCone oo
i62 : degree oo == 4
i63 : sub(KLM, {(gens PP9)_6=>1});
i64 : Conep3 = tangentCone oo
i65 : degree oo == 4
i66 : sub(KLM, {(gens PP9)_8=>1});
i67 : Conep4 = tangentCone oo
i68 : degree oo == 4
i69 : sub(KLM, {(gens PP9)_0=>1});
i70 : Conep5 = tangentCone oo
i71 : degree oo == 6
i72 : M6 = Conep5+ideal{(gens PP9)_0}
i73 : irredCompM6 = associatedPrimes M6;
i74 : plane1 = irredCompM6#0
i75 : plane2 = irredCompM6#1
i76 : plane2' = irredCompM6#2
i77 : plane1' = irredCompM6#3
i78 : Q = irredCompM6#4
i79 : line1 = Q+plane1;
i80 : line1' = Q+plane1';
i81 : line2 = Q+plane2;
i82 : line2' = Q+plane2';
i83 : dim(line1+line1')-1 == -1
i84 : dim(line2+line2')-1 == -1
i85 : q12 = saturate(line1+line2)
i86 : q12' = saturate(line1+line2')
i87 : q1'2 = saturate(line1'+line2)
i88 : q1'2' = saturate(line1'+line2')
```

**Code B.9.** Computational analysis of $W_P^{13}$ (see § 8.2).

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : needsPackage "Points";
i3 : PP2=ZZ/10000019[u_0,u_1,u_2];
```

```
i4 : a1 = ideal{u_1,u_2};
i5 : a2 = ideal{u_0,u_2};
i6 : a3 = ideal{u_0,u_1};
i7 : cubics3points = rationalMap toMap(saturate(a1*a2*a3),3,1);
i8 : DelPezzo6ic = image cubics3points;
i9 : (dim DelPezzo6ic -1, degree DelPezzo6ic)
i10 : PP6 = ring DelPezzo6ic;
i11 : PP7 = ZZ/10000019[x_0,x_1,x_2,x_3,x_4,x_5,x_6,y];
i12 : inclusion = rationalMap map(PP6,PP7, matrix{{(gens PP6)_0,(gens PP6)_1,
      (gens PP6)_2,(gens PP6)_3,(gens PP6)_4,(gens PP6)_5,(gens PP6)_6,0}});
i13 : S6 = inclusion(DelPezzo6ic);
i14 : v = ideal{x_0,x_1,x_2,x_3,x_4,x_5,x_6};
i15 : numgens S6 == 10
i16 : V = ideal{S6_1,S6_2,S6_3,S6_4,S6_5,S6_6,S6_7,S6_8,S6_9};
i17 : (dim V -1, degree V) == (3, 6)
i18 : tau = rationalMap map(PP7,PP7, matrix{{x_2,x_4,x_0,x_5,x_1,x_3,x_6,-y}});
i19 : F1 = ideal{(gens PP7)_0+(gens PP7)_2, (gens PP7)_1+(gens PP7)_4,
      (gens PP7)_3+(gens PP7)_5, (gens PP7)_6};
i20 : tau(F1) == F1
i21 : F2 = ideal{(gens PP7)_0-(gens PP7)_2, (gens PP7)_1-(gens PP7)_4,
      (gens PP7)_3-(gens PP7)_5, y};
i22 : tau(F2) == F2
i23 : F2intV = associatedPrimes saturate(F2+V);
i24 : v1 = F2intV#0;
i25 : v2 = F2intV#3;
i26 : v3 = F2intV#2;
i27 : v4 = F2intV#1;
i28 : v1 == points matrix(PP7, {{1},{1},{1},{1},{1},{1},{1},{0}})
i29 : v2 == points matrix(PP7, {{1},{-1},{1},{-1},{-1},{-1},{1},{0}})
i30 : v3 == points matrix(PP7, {{-1},{1},{-1},{-1},{1},{-1},{1},{0}})
i31 : v4 == points matrix(PP7, {{-1},{-1},{-1},{1},{-1},{1},{1},{0}})
i32 : PP13 = ZZ/10000019[z_0..z_13];
i33 : pigreco = rationalMap map(PP7,PP13, matrix{{x_6^2, x_0^2+x_2^2, x_1^2+x_4^2, x_3^2+x_5^2,
      (x_0+x_2)*x_6, (x_1+x_4)*x_6, (x_3+x_5)*x_6, x_0*x_1+x_2*x_4, x_2*x_3+x_0*x_5, x_1*x_3+x_4*x_5,
      (x_0-x_2)*y, (x_1-x_4)*y, (x_3-x_5)*y, y^2}});
i34 : PP19 = ZZ/10000019[Z_0..Z_19]
i35 : phi = rationalMap map(PP7,PP19,matrix{{x_6^2, x_0^2+x_2^2, x_1^2+x_4^2,
      x_3^2+x_5^2, (x_0+x_2)*x_6, (x_1+x_4)*x_6, (x_3+x_5)*x_6, x_0*x_1+x_2*x_4,
      x_2*x_3+x_0*x_5, x_1*x_3+x_4*x_5, (x_0-x_2)*y, (x_1-x_4)*y, (x_3-x_5)*y, y^2,
      2*x_0*x_2, 2*x_1*x_4, 2*x_3*x_5, x_4*x_3+x_1*x_5, x_0*x_3+x_2*x_5, x_1*x_2+x_0*x_4}});
i36 : phi(V)
i37 : phiV = sub(phi(V), {Z_14 => 2*Z_0,Z_15 => 2*Z_0,Z_16 => 2*Z_0, Z_19 => Z_6,
      Z_18 => Z_5, Z_17 => Z_4})
i38 : PP13' = ZZ/10000019[Z_0..Z_13];
i39 : ideal(submatrix(gens (sub(ooo, PP13')), {6..47}))
i40 : WP13 = sub(oo, { (gens PP13')_0 => (gens PP13)_0, (gens PP13')_1 => (gens PP13)_1,
      (gens PP13')_2 => (gens PP13)_2, (gens PP13')_3 => (gens PP13)_3, (gens PP13')_4 => (gens PP13)_4,
      (gens PP13')_5 => (gens PP13)_5, (gens PP13')_6 => (gens PP13)_6, (gens PP13')_7 => (gens PP13)_7,
      (gens PP13')_8 => (gens PP13)_8, (gens PP13')_9 => (gens PP13)_9, (gens PP13')_10 => (gens PP13)_10,
      (gens PP13')_11 => (gens PP13)_11, (gens PP13')_12 => (gens PP13)_12, (gens PP13')_13 => (gens PP13)_13 })
i41 : (dim oo -1, degree oo) == (3, 24)
i42 : WP13 == pigreco(V)
i43 : P1 = ideal{z_1 -2*z_0,z_2 -2*z_0,z_3 -2*z_0,z_4 -2*z_0,z_5 -2*z_0,
       z_6 -2*z_0,z_7 -2*z_0,z_8 -2*z_0,z_9 -2*z_0,z_10,z_11,z_12,z_13};
i44 : P2 = ideal{z_1 -2*z_0,z_2 -2*z_0,z_3 -2*z_0,z_4 -2*z_0,z_5 +2*z_0,
      z_6 +2*z_0,z_7 +2*z_0,z_8 +2*z_0,z_9 -2*z_0,z_10,z_11,z_12,z_13};
i45 : P3 = ideal{z_1 -2*z_0,z_2 -2*z_0,z_3 -2*z_0,z_4 +2*z_0,z_5 -2*z_0,
      z_6 +2*z_0,z_7 +2*z_0,z_8 -2*z_0,z_9 +2*z_0,z_10,z_11,z_12,z_13};
i46 : P4 = ideal{z_1 -2*z_0,z_2 -2*z_0,z_3 -2*z_0,z_4 +2*z_0,z_5 +2*z_0,
      z_6 -2*z_0,z_7 -2*z_0,z_8 +2*z_0,z_9 +2*z_0,z_10,z_11,z_12,z_13};
i47 : P1 == pigreco(v1)
i48 : P2 == pigreco(v2)
i49 : P3 == pigreco(v3)
i50 : P4 == pigreco(v4)
i51 : P5 = pigreco(v);
i52 : l12 = ideal{(toMap(saturate(P1*P2),1,1)).matrix};
i53 : (l12 + WP13 == l12 ) == false
i54 : l13 = ideal{(toMap(saturate(P1*P3),1,1)).matrix};
i55 : (l13 + WP13 == l13) == false
```

```
i56 : l14 = ideal{(toMap(saturate(P1*P4),1,1)).matrix};
i57 : (l14 + WP13 == l14 ) == false
i58 : l15 = ideal{(toMap(saturate(P1*P5),1,1)).matrix};
i59 : (l15 + WP13 == l15) == true
i60 : l23 = ideal{(toMap(saturate(P2*P3),1,1)).matrix};
i61 : (l23 + WP13 == l23) == false
i62 : l24 = ideal{(toMap(saturate(P2*P4),1,1)).matrix};
i63 : (l24 + WP13 == l24) == false
i64 : l25 = ideal{(toMap(saturate(P2*P5),1,1)).matrix};
i65 : (l25 + WP13 == l25) == true
i66 : l34 = ideal{(toMap(saturate(P3*P4),1,1)).matrix};
i67 : (l34 + WP13 == l34) == false
i68 : l35 = ideal{(toMap(saturate(P3*P5),1,1)).matrix};
i69 : (l35 + WP13 == l35) == true
i70 : l45 = ideal{(toMap(saturate(P4*P5),1,1)).matrix};
i71 : (l45 + WP13 == l45) == true
i72 : W' = sub(WP13, {(gens PP13)_0 => (gens PP13')_0, (gens PP13)_1 => (gens PP13')_1 + 2*(gens PP13')_0,
      (gens PP13)_2 => (gens PP13')_2 + 2*(gens PP13')_0, (gens PP13)_3 => (gens PP13')_3 + 2*(gens PP13')_0,
      (gens PP13)_4 => (gens PP13')_4 + 2*(gens PP13')_0, (gens PP13)_5 => (gens PP13')_5 + 2*(gens PP13')_0,
      (gens PP13)_6 => (gens PP13')_6 + 2*(gens PP13')_0, (gens PP13)_7 => (gens PP13')_7 + 2*(gens PP13')_0,
      (gens PP13)_8 => (gens PP13')_8 + 2*(gens PP13')_0, (gens PP13)_9 => (gens PP13')_9 + 2*(gens PP13')_0,
      (gens PP13)_10 => (gens PP13')_10, (gens PP13)_11 => (gens PP13')_11,
      (gens PP13)_12 => (gens PP13')_12, (gens PP13)_13 => (gens PP13')_13});
i73 : W'U0 = sub(oo, {(gens PP13')_0  => 1});
i74 : ConeP1 = sub(tangentCone oo, {(gens PP13')_0 => (gens PP13)_0,
      (gens PP13')_1 => (gens PP13)_1 - 2*(gens PP13)_0, (gens PP13')_2 => (gens PP13)_2 - 2*(gens PP13)_0,
      (gens PP13')_3 => (gens PP13)_3 - 2*(gens PP13)_0, (gens PP13')_4 => (gens PP13)_4 - 2*(gens PP13)_0,
      (gens PP13')_5 => (gens PP13)_5 - 2*(gens PP13)_0, (gens PP13')_6 => (gens PP13)_6 - 2*(gens PP13)_0,
      (gens PP13')_7 => (gens PP13)_7 - 2*(gens PP13)_0, (gens PP13')_8 => (gens PP13)_8 - 2*(gens PP13)_0,
      (gens PP13')_9 => (gens PP13)_9 - 2*(gens PP13)_0,
      (gens PP13')_10 => (gens PP13)_10, (gens PP13')_11 => (gens PP13)_11,
      (gens PP13')_12 => (gens PP13)_12, (gens PP13')_13 => (gens PP13)_13 });
i75 : degree oo == 4
i76 : TCOW'U0 = ideal{-9*Z_1+8*Z_7+8*Z_8-4*Z_9, -9*Z_2+8*Z_7-4*Z_8+8*Z_9,
     -9*Z_3-4*Z_7+8*Z_8+8*Z_9, -9*Z_4+2*Z_7+2*Z_8-Z_9, -9*Z_5+2*Z_7-Z_8+2*Z_9,
     -9*Z_6-Z_7+2*Z_8+2*Z_9, Z_10-Z_11+Z_12, 9*Z_12^2-(-4*Z_7+8*Z_8+8*Z_9)*Z_13,
     9*Z_11^2-(8*Z_7-4*Z_8+8*Z_9)*Z_13, 9*Z_11*Z_12+(2*Z_7+2*Z_8-10*Z_9)*Z_13,
     (2*Z_7-10*Z_8+2*Z_9)*Z_11+(-10*Z_7+2*Z_8+2*Z_9)*Z_12,
     (6*Z_7-6*Z_8-18*Z_9)*Z_11+(6*Z_7-6*Z_8+18*Z_9)*Z_12,
     Z_7^2- 2*Z_7*Z_8+Z_8^2-2*Z_7*Z_9-2*Z_8*Z_9+Z_9^2}
i77 : oo == tangentCone W'U0
i78 : sub(ooo, {(gens PP13')_0 => (gens PP13)_0,
      (gens PP13')_1 => (gens PP13)_1 - 2*(gens PP13)_0, (gens PP13')_2 => (gens PP13)_2 - 2*(gens PP13)_0,
      (gens PP13')_3 => (gens PP13)_3 - 2*(gens PP13)_0, (gens PP13')_4 => (gens PP13)_4 - 2*(gens PP13)_0,
      (gens PP13')_5 => (gens PP13)_5 - 2*(gens PP13)_0, (gens PP13')_6 => (gens PP13)_6 - 2*(gens PP13)_0,
      (gens PP13')_7 => (gens PP13)_7 - 2*(gens PP13)_0, (gens PP13')_8 => (gens PP13)_8 - 2*(gens PP13)_0,
      (gens PP13')_9 => (gens PP13)_9 - 2*(gens PP13)_0,
      (gens PP13')_10 => (gens PP13)_10, (gens PP13')_11 => (gens PP13)_11,
      (gens PP13')_12 => (gens PP13)_12, (gens PP13')_13 => (gens PP13)_13 })
i79 : oo == ConeP1
i80 : -- similarly with P2,P3,P4
     sub(WP13, {(gens PP13)_13=>1});
i81 : ConeP5 = tangentCone oo;
i82 : degree oo == 5
i83 : TCOW'U13 = ideal{ z_6-z_7, z_5-z_8, z_4-z_9, z_2-z_3, z_1-z_3, 2*z_0-z_3,
     z_9*z_10-z_8*z_11+z_7*z_12, z_8*z_10-z_9*z_11+z_3*z_12,
     z_7*z_10-z_3*z_11+z_9*z_12, z_3*z_10-z_7*z_11+z_8*z_12,
     z_8^2-z_9^2, z_7*z_8-z_3*z_9, z_3*z_8-z_7*z_9,
     z_7^2-z_9^2, z_3*z_7-z_8*z_9, z_3^2-z_9^2 }
i84 : ConeP5 == oo
i85 : M5 = ConeP5+ideal{(gens PP13)_13}
i86 : (dim oo -1, degree oo) == (2, 5)
i87 : irredCompM5 = associatedPrimes M5;
i88 : plane0=irredCompM5#0
i89 : plane1=irredCompM5#1
i90 : plane2=irredCompM5#2
i91 : plane3=irredCompM5#3
i92 : plane4=irredCompM5#4
i93 : (dim(plane0+plane1)-1, degree (plane0+plane1)) == (1,1)
```

```
i94 : (dim(plane0+plane2)-1, degree (plane0+plane2)) == (1,1)
i95 : (dim(plane0+plane3)-1, degree (plane0+plane3)) == (1,1)
i96 : (dim(plane0+plane4)-1, degree (plane0+plane4)) == (1,1)
i97 : (dim(plane1+plane2)-1, degree (plane1+plane2)) == (0,1)
i98 : (dim(plane1+plane3)-1, degree (plane1+plane3)) == (0,1)
i99 : (dim(plane1+plane4)-1, degree (plane1+plane4)) == (0,1)
i100 : (dim(plane2+plane3)-1, degree (plane2+plane3)) == (0,1)
i101 : (dim(plane2+plane4)-1, degree (plane2+plane4)) == (0,1)
i102 : (dim(plane3+plane4)-1, degree (plane3+plane4)) == (0,1)
```

**Code B.10.** Computational analysis of $W_P^{17}$ (see § 8.3).

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP1 = ZZ/10000019[u_0,u_1];
i3 : PP1'= ZZ/10000019[v_0,v_1];
i4 : P1P1 = PP1 ** PP1';
i5 : PP9 = ZZ/10000019[y_{0,0},y_{0,1},y_{0,2},y_{1,0},y_{1,1},
     y_{1,2},y_{2,0},y_{2,1},y_{2,2},x];
i6 : antiCanonicalEmbeddingP = rationalMap map(P1P1,PP9, matrix{{u_1^2*v_1^2,
     u_1^2*v_0*v_1,u_1^2*v_0^2,u_1*u_0*v_1^2,u_1*u_0*v_0*v_1,u_1*u_0*v_0^2,
     u_0^2*v_1^2,u_0^2*v_0*v_1,u_0^2*v_0^2,0}});
i7 : P = image oo;
i8 : (dim P -1, degree P) == (2, 8)
i9 : v = ideal{y_{0,0},y_{0,1},y_{0,2},y_{1,0},y_{1,1},y_{1,2},y_{2,0},y_{2,1},y_{2,2}};
i10 : numgens P == 21
i11 : V = ideal{P_1,P_2,P_3,P_4,P_5,P_6,P_7,P_8,P_9,P_10,
      P_11,P_12,P_13,P_14,P_15,P_16,P_17,P_18,P_19,P_20}
i12 : (dim V -1, degree V) == (3, 8)
i13 : v00 = ideal{y_{0,1},y_{0,2},y_{1,0},y_{1,1},y_{1,2},y_{2,0},y_{2,1},y_{2,2},x};
i14 : (dim oo -1, degree oo) == (0, 1)
i15 : v02 = ideal{y_{0,0},y_{0,1},y_{1,0},y_{1,1},y_{1,2},y_{2,0},y_{2,1},y_{2,2},x};
i16 : (dim oo -1, degree oo) == (0, 1)
i17 : v20 = ideal{y_{0,0},y_{0,1},y_{0,2},y_{1,0},y_{1,1},y_{1,2},y_{2,1},y_{2,2},x};
i18 : (dim oo -1, degree oo) == (0, 1)
i19 : v22 = ideal{y_{0,0},y_{0,1},y_{0,2},y_{1,0},y_{1,1},y_{1,2},y_{2,0},y_{2,1},x};
i20 : (dim oo -1, degree oo) == (0, 1)
i21 : PP29 = ZZ/10000019[Z_0..Z_29];
i22 : phi = rationalMap map(PP9, PP29, matrix(PP9, {{y_{1,1}^2, y_{0,0}^2, y_{0,2}^2, y_{2,0}^2,
      y_{2,2}^2, x^2, y_{0,1}^2, y_{1,0}^2, y_{1,2}^2, y_{2,1}^2, y_{0,1}*x, y_{1,0}*x, y_{1,2}*x,
      y_{2,1}*x, y_{0,0}*y_{1,1}, y_{0,2}*y_{1,1}, y_{2,0}*y_{1,1}, y_{2,2}*y_{1,1}, y_{0,1}*y_{1,0},
      y_{0,1}*y_{1,2}, y_{1,0}*y_{2,1}, y_{1,2}*y_{2,1}, y_{0,0}*y_{0,2}, y_{0,0}*y_{2,0}, y_{0,2}*y_{2,2},
      y_{2,0}*y_{2,2}, y_{0,1}*y_{2,1}, y_{0,0}*y_{2,2}, y_{0,2}*y_{2,0}, y_{1,0}*y_{1,2} }}));
i23 : phi(V)
i24 : H17 = ideal{Z_18 - Z_14, Z_19 - Z_15, Z_20 - Z_16, Z_21 - Z_17, Z_22 - Z_6, Z_23 - Z_7,
      Z_24 - Z_8, Z_25 - Z_9, Z_26 - Z_0, Z_27 - Z_0, Z_28 - Z_0, Z_29 - Z_0};
i25 : phi(V) + H17 == phi(V)
i26 : PP17=ZZ/10000019[z_0..z_17];
i27 : inclusion = rationalMap map(PP17, PP29, matrix(PP17, {{ z_0,z_1,z_2,z_3,z_4,z_5,z_6,z_7,z_8,z_9,
      z_10,z_11,z_12,z_13,z_14,z_15,z_16,z_17, z_14,z_15,z_16,z_17,z_6,z_7,z_8,z_9,z_0,z_0,z_0,z_0 }}));
i28 : image oo == H17
i29 : WP17 = inclusion^* (phi(V))
i30 : (dim oo -1, degree oo) == (3, 32)
i31 : pigreca = rationalMap map(PP9,PP17, matrix(PP9, {{y_{1,1}^2, y_{0,0}^2, y_{0,2}^2,
      y_{2,0}^2, y_{2,2}^2, x^2, y_{0,1}^2, y_{1,0}^2, y_{1,2}^2, y_{2,1}^2, y_{0,1}*x, y_{1,0}*x,
      y_{1,2}*x, y_{2,1}*x, y_{0,0}*y_{1,1}, y_{0,2}*y_{1,1}, y_{2,0}*y_{1,1}, y_{2,2}*y_{1,1} }}));
i32 : pigreca(V) == WP17
i33 : P1 = pigreca(v00)
i34 : P2 = pigreca(v02)
i35 : P3 = pigreca(v20)
i36 : P4 = pigreca(v22)
i37 : P5 = pigreca(v)
i38 : l12 = ideal{(toMap(saturate(P1*P2),1,1)).matrix};
i39 : (l12 + WP17 == l12) == false
i40 : l13 = ideal{(toMap(saturate(P1*P3),1,1)).matrix};
i41 : (l13 + WP17 == l13) == false
i42 : l14 = ideal{(toMap(saturate(P1*P4),1,1)).matrix};
```

```
i43 : (l14 + WP17 == l14) == false
i44 : l15 = ideal{(toMap(saturate(P1*P5),1,1)).matrix};
i45 : (l15 + WP17 == l15) == true
i46 : l23 = ideal{(toMap(saturate(P2*P3),1,1)).matrix};
i47 : (l23 + WP17 == l23) == false
i48 : l24 = ideal{(toMap(saturate(P2*P4),1,1)).matrix};
i49 : (l24 + WP17 == l24) == false
i50 : l25 = ideal{(toMap(saturate(P2*P5),1,1)).matrix};
i51 : (l25 + WP17 == l25) == true
i52 : l34 = ideal{(toMap(saturate(P3*P4),1,1)).matrix};
i53 : (l34 + WP17 == l34) == false
i54 : l35 = ideal{(toMap(saturate(P3*P5),1,1)).matrix};
i55 : (l35 + WP17 == l35) == true
i56 : l45 = ideal{(toMap(saturate(P4*P5),1,1)).matrix};
i57 : (l45 + WP17 == l45) == true
i58 : sub(WP17, {(gens PP17)_1=>1});
i59 : ConeP1 = tangentCone oo
i60 : degree oo == 4
i61 : sub(WP17, {(gens PP17)_2=>1});
i62 : ConeP2 = tangentCone oo
i63 : degree oo == 4
i64 : sub(WP17, {(gens PP17)_3=>1});
i65 : ConeP3 = tangentCone oo
i66 : degree oo == 4
i67 : sub(WP17, {(gens PP17)_4=>1});
i68 : ConeP4 = tangentCone oo
i69 : degree oo == 4
i70 : sub(WP17, {(gens PP17)_5=>1});
i71 : ConeP5 = tangentCone oo
i72 : degree oo == 6
i73 : M6 = ConeP5+ideal{(gens PP17)_5}
i74 : time irredCompM6 = associatedPrimes M6;
i75 : plane1 = irredCompM6#0
i76 : plane2 = irredCompM6#1
i77 : plane1' = irredCompM6#2
i78 : plane2' = irredCompM6#3
i79 : Q = irredCompM6#4
i80 : line1 = Q+plane1;
i81 : line1' = Q+plane1';
i82 : line2 = Q+plane2;
i83 : line2' = Q+plane2';
i84 : (dim(line1+line1')-1) == -1
i85 : (dim(line2+line2')-1) == -1
i86 : q12 = saturate(line1+line2)
i87 : q12' = saturate(line1+line2')
i88 : q1'2 = saturate(line1'+line2)
i89 : q1'2' = saturate(line1'+line2')
```

**Code B.11.** Let $\mathcal{S}_\bullet$ be the linear system on $\mathbb{P}^3$ given by the sextic surfaces of $\mathbb{P}^3$ double along the six edges of a tetrahedron $T$ and triple at a general point $p \in \mathbb{P}^3$. Let us use the notation of § 10.4 and in particular let us see the proof of Theorem 10.23. Let $\Sigma_\bullet$ be a general element of $\mathcal{S}_\bullet$ and let $\pi$ be a general plane of $\mathbb{P}^3$, that is a plane not containing the point $p$. Thanks to Macaulay2, one can find that the tangent cone to $\Sigma_\bullet$ at $p$ is a cone with vertex $p$ over a cubic plane curve on $\pi$ passing through the three points $\pi \cap r_1$, $\pi \cap r_3$ and $\pi \cap r_3$. In particular, by moving the surface $\Sigma_\bullet \in \mathcal{S}_\bullet$, these cubic cones cut on $\pi$ a linear system of cubic curves whose base locus is given exactly by the three points $\pi \cap r_1$, $\pi \cap r_3$ and $\pi \cap r_3$. Before providing the Macaulay2 code, let us explain the strategy to use:

(i) we consider the linear system $\mathcal{S}$ of the sextic surfaces of $\mathbb{P}^3_{[s_0:\cdots:s_3]}$ having double points along the six edges of the tetrahedron $T := \{s_0 s_1 s_2 s_3 = 0\}$, which has equation

$$l_0s_0s_1^3s_2s_3+l_1s_0^2s_1^2s_2^2+l_2s_0^2s_1^2s_2s_3+l_3s_0^2s_1^2s_3^2+l_4s_0^3s_1s_2s_3+$$

$$+l_5s_0s_1^2s_2^2s_3++l_6s_0s_1^2s_2s_3^2+l_7s_0^2s_1s_2^2s_3+l_8s_0^2s_1s_2s_3^2+$$

$$+l_9s_1^2s_2^2s_3^2+l_{10}s_0s_1s_2^3s_3+l_{11}s_0s_1s_2^2s_3^2+l_{12}s_0s_1s_2s_3^3+l_{13}s_0^2s_2^2s_3^2=0;$$

(ii) we choose a point $p\in\mathbb{P}^3$ sufficiently general such that, setting it as a triple point for the surfaces of $\mathcal{S}$, imposes 10 linearly independent conditions to the coefficients $l_0,\dots,l_{13}$: in our example we choose $p:=[1:1:1:-1]$;

(iii) we find the equation of $\mathcal{S}_\bullet$: in our example we have

$$l_{10}(s_0^3s_1s_2s_3-2s_0^2s_1s_2^2s_3+s_0s_1s_2^3s_3+s_0^2s_1^2s_3^2-2s_0s_1^2s_2s_3^2+s_1^2s_2^2s_3^2)+$$

$$+l_{11}(-s_0^3s_1s_2s_3+s_0^2s_1^2s_2s_3+s_0^2s_1s_2^2s_3-s_0s_1^2s_2^2s_3-s_0^2s_1s_2s_3^2+s_0s_1^2s_2s_3^2+s_0s_1s_2^2s_3^2-s_1^2s_2^2s_3^2)+$$

$$+l_{12}(s_0^2s_1^2s_2^2+s_0^3s_1s_2s_3+2s_0s_1^2s_2^2s_3+2s_0^2s_1s_2s_3^2+s_1^2s_2^2s_3^2+s_0s_1s_2s_3^3)+$$

$$+l_{13}(-s_0^3s_1s_2s_3+s_0s_1^3s_2s_3+2s_0^2s_1s_2^2s_3-2s_0s_1^2s_2^2s_3-2s_0^2s_1s_2s_3^2+2s_0s_1^2s_2s_3^2+s_0^2s_2^2s_3^2-s_1^2s_2^2s_3^2)=0.$$

We see that a general fibre of the rational map defined by $\mathcal{S}_\bullet$ is a cubic plane curve with node at $p$ and intersecting each edge of $T$ at a point. We also recall that the base locus of $\mathcal{S}_\bullet$ is given by the union of the six edges of $T$ and by three lines $r_1$, $r_2$, $r_3$ intersecting at $p$ (see Corollary 10.18);

(iv) we consider a change of coordinates of $\mathbb{P}^3$, with respect to which $p$ has coordinates $[0:0:0:1]$. By abuse of notation let us denote the new coordinates by $[s_0:\dots:s_3]$. Let $\Sigma_\bullet$ be a general element of $\mathcal{S}_\bullet$, obtained by fixing general values for $l_{10},\dots,l_{13}$. The point $p$ can be viewed as the origin of the open affine set $U_0:=\{s_3\neq 0\}$ and we can find the ideal of the tangent cone $TC_p(\Sigma_\bullet\cap U_0)$: in our example we obtain

$$(l_{10}-l_{11}+l_{12}-l_{13})s_0^3+(-l_{10}+l_{11}-l_{12}+l_{13})s_0^2s_1-l_{13}s_0s_1^2+l_{13}s_1^3+$$

$$-(l_{10}-l_{11}+l_{12}-l_{13})s_0^2s_2+(2l_{10}-l_{11})s_0s_1s_2-l_{13}s_1^2s_2-l_{10}s_0s_2^2-l_{10}s_1s_2^2+l_{10}s_2^3=0,$$

thus $TC_p\Sigma_\bullet$ is a cone with vertex $p$ over a cubic plane curve on the plane $\pi:=\{s_3=0\}$;

(v) by moving $\Sigma_\bullet\in\mathcal{S}_\bullet$, i.e. by varying the coefficients $l_{10},\dots,l_{13}$, the cubic cones $TC_p\Sigma_\bullet$ identify a linear system of cubic plane curves on $\pi$; we see that the base locus of this linear system is given by the union of the three points $r_1\cap\pi$, $r_2\cap\pi$, $r_3\cap\pi$: we verify this by studying the intersection of the four cubic curves given by $[l_{10}:\dots:l_{13}]\in\{[1:0:0:0],[0:1:0:0],[0:0:1:0],[0:0:0:1]\}$.

```
Macaulay2, version 1.11
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
               LLLBases, PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : needsPackage "Cremona";
i2 : PP3 = ZZ/10000019[s_0..s_3];
i3 : -- let us take a general point of PP3 with random coordinates:
     -- for i to 3 list random(-5,10)
     -- in our example we take p=[1: 1: 1: -1]
     p = ideal{s_0+s_3,s_1+s_3,s_2+s_3}
i4 : -- let us take the linear system of the sextic surfaces of PP3
     -- double along the six edges of the coordinate tetrahedron
     R = ZZ/10000019[l_0..l_13][s_0..s_3];
i5 : use R
i6 : Sigma = ideal{l_0*s_0*s_1^3*s_2*s_3+l_1*s_0^2*s_1^2*s_2^2+l_2*s_0^2*s_1^2*s_2*s_3+
     l_3*s_0^2*s_1^2*s_3^2+l_4*s_0^3*s_1*s_2*s_3+l_5*s_0*s_1^2*s_2^2*s_3+l_6*s_0*s_1^2*s_2*s_3^2+
     l_7*s_0^2*s_1*s_2^2*s_3+l_8*s_0^2*s_1*s_2*s_3^2+l_9*s_1^2*s_2^2*s_3^2+l_10*s_0*s_1*s_2^3*s_3+
     l_11*s_0*s_1*s_2^2*s_3^2+l_12*s_0*s_1*s_2*s_3^3+l_13*s_0^2*s_2^2*s_3^2};
i7 : -- for a fixed value of [l_0:..:l_13], we have that Sigma is a hypersurface of PP3
     -- let us find the values for [l_0:..:l_13] in order to have p as triple point for Sigma
     J = jacobian(Sigma);
i8 : JJ = jacobian(J);
i9 : triplelocus = minors(1,J)+minors(1,JJ)+Sigma;
i10 : substitute(triplelocus, {s_0=>1, s_1=>1, s_2=>1, s_3=>-1})
i11 : -- we have the following 10 independent conditions
      substitute(oo,{l_0 => l_13})
i12 : substitute(oo,{l_1 => l_12})
i13 : substitute(oo,{l_2 => l_11})
i14 : substitute(oo,{l_3 => l_10})
i15 : substitute(oo,{l_4 => l_10-l_11+l_12-l_13})
i16 : substitute(oo,{l_5 => -l_11 + 2*l_12 - 2*l_13})
i17 : substitute(oo,{l_6 => -2*l_10+l_11+2*l_13})
i18 : substitute(oo,{l_7 => -2*l_10+l_11+2*l_13})
i19 : substitute(oo,{l_8 => -l_11+2*l_12-2*l_13})
i20 : substitute(oo,{l_9 => l_10-l_11+l_12-l_13})
i21 : -- thus we let:
      substitute(Sigma,{l_0 => l_13})
i22 : substitute(oo,{l_1 => l_12})
i23 : substitute(oo,{l_2 => l_11})
i24 : substitute(oo,{l_3 => l_10})
i25 : substitute(oo,{l_4 => l_10-l_11+l_12-l_13})
i26 : substitute(oo,{l_5 => -l_11 + 2*l_12 - 2*l_13})
i27 : substitute(oo,{l_6 => -2*l_10+l_11+2*l_13})
i28 : substitute(oo,{l_7 => -2*l_10+l_11+2*l_13})
i29 : substitute(oo,{l_8 => -l_11+2*l_12-2*l_13})
i30 : substitute(oo,{l_9 => l_10-l_11+l_12-l_13})
i31 : -- the linear system of the sextic surfaces of PP3
      -- double along the edges of the coordinate tetrahedron
      -- and triple at the point p has the following equation,
      -- depending on the coefficients l_10,l_11,l_12,l_13
      SigmaTripleAtp = oo
i32 : -- let us find the rational map defined by SigmaTripleAtp
      generator1 = substitute( SigmaTripleAtp, {l_10 =>1, l_11=>0, l_12=>0, l_13=>0})
i33 : generator2 = substitute( SigmaTripleAtp, {l_10 =>0, l_11=>1, l_12=>0, l_13=>0})
i34 : generator3 = substitute( SigmaTripleAtp, {l_10 =>0, l_11=>0, l_12=>1, l_13=>0})
i35 : generator4 = substitute( SigmaTripleAtp, {l_10 =>0, l_11=>0, l_12=>0, l_13=>1})
i36 : PP3' = ZZ/10000019[x_0..x_3]
i37 : sexticsbullet = rationalMap map(PP3,PP3',matrix{{sub(generator1_0,PP3),
      sub(generator2_0,PP3),sub(generator3_0,PP3),sub(generator4_0,PP3)}});
i38 : CayleyCubic = image oo
i39 : (dim oo -1, degree oo) == (2, 3)
i40 : -- let us find the general fibre of sexticsbullet
      gamma = sexticsbullet^*(sexticsbullet(ideal{random(1,PP3),random(1,PP3),random(1,PP3)}))
i41 : (dim oo -1, degree oo) == (1, 3)
i42 : alpha = ideal{gamma_0}
i43 : (dim oo -1, degree oo) == (2, 1)
i44 : (dim(gamma+ideal{(gens PP3)_0,(gens PP3)_1})-1, degree(gamma+ideal{(gens PP3)_0,(gens PP3)_1}))==(0, 1)
i45 : (dim(gamma+ideal{(gens PP3)_0,(gens PP3)_2})-1, degree(gamma+ideal{(gens PP3)_0,(gens PP3)_2}))==(0, 1)
i46 : (dim(gamma+ideal{(gens PP3)_0,(gens PP3)_3})-1, degree(gamma+ideal{(gens PP3)_0,(gens PP3)_3}))==(0, 1)
i47 : (dim(gamma+ideal{(gens PP3)_1,(gens PP3)_2})-1, degree(gamma+ideal{(gens PP3)_1,(gens PP3)_2}))==(0, 1)
```

```
i48 : (dim(gamma+ideal{(gens PP3)_1,(gens PP3)_3})-1, degree(gamma+ideal{(gens PP3)_1,(gens PP3)_3}))==(0, 1)
i49 : (dim(gamma+ideal{(gens PP3)_2,(gens PP3)_3})-1, degree(gamma+ideal{(gens PP3)_2,(gens PP3)_3}))==(0, 1)
i50 : (alpha+p == p, gamma+p == p) == (true, true)
i51 : (p == saturate(gamma+minors(2,jacobian(gamma)))) == true
i52 : -- let us find the base locus of SigmaTripleAtp
      associatedPrimes(ideal sexticsbullet)
i53 : -- it is the union of the six edges of T, the point p
      -- and the following three lines r1, r2, r3 intersecting at p
      -- such that ri intersects the edges ideal{s_j,s_k}, ideal{s_0,s_i}
      -- for i,j,k distinct indices in {1,2,3}
      use PP3
i54 : r1 = ideal{s_2+s_3,s_0-s_1}
i55 : r2 = ideal{s_1+s_3,s_0-s_2}
i56 : r3 = ideal{s_1-s_2,s_0+s_3}
i57 : -- let us find the tangent cone at the point p
      -- to a general sextic surface of the linear system SigmaTripleAtp
      newR = ZZ/10000019[l_10,l_11,l_12,l_13][s_0..s_3];
i58 : -- let us consider the change of coordinates thanks to which
      -- the point p is the point [0:0:0:1]
      -- (by abuse of notation ,let [s_0..s_3] be the new coordinates)
      substitute(SigmaTripleAtp, newR)
i59 : sub(oo, {(gens newR)_0 =>(gens newR)_0-(gens newR)_3, (gens newR)_1=>(gens newR)_1-(gens newR)_3,
      (gens newR)_2=>(gens newR)_2-(gens newR)_3, (gens newR)_3=>(gens newR)_3});
i60 : sub(oo, {(gens newR)_3 => 1})
i61 : TCp = tangentCone oo
i62 : -- TCp is a cone of vertex p over a cubic plane curve on the plane ideal{s_3}.
      -- By moving the surfaces of the linear system, i.e. by varying the values l_10,l_11,l_12,l_13,
      -- we obtain a linear system of cubic plane curves on ideal{s_3} which has only three base points,
      -- given by the intersection with the three lines r1, r2, r3
      c0 =sub(ideal{sub(TCp,{l_10=>1, l_11=>0, l_12=>0, l_13=>0})},PP3)
i63 : c1 =sub(ideal{sub(TCp,{l_10=>0, l_11=>1, l_12=>0, l_13=>0})},PP3)
i64 : c2 =sub(ideal{sub(TCp,{l_10=>0, l_11=>0, l_12=>1, l_13=>0})},PP3)
i65 : c3 =sub(ideal{sub(TCp,{l_10=>0, l_11=>0, l_12=>0, l_13=>1})},PP3)
i66 : threepts = associatedPrimes(ideal{(gens PP3)_3}+c0+c1+c2+c3)
i67 : threepts#0 == ideal{(gens PP3)_3}+sub(r1, {(gens PP3)_0 =>(gens PP3)_0-(gens PP3)_3,
                     (gens PP3)_1=>(gens PP3)_1-(gens PP3)_3, (gens PP3)_2=>(gens PP3)_2-(gens PP3)_3,
                     (gens PP3)_3=>(gens PP3)_3})
i68 : threepts#1 == ideal{(gens PP3)_3}+sub(r2, {(gens PP3)_0 =>(gens PP3)_0-(gens PP3)_3,
                    (gens PP3)_1=>(gens PP3)_1-(gens PP3)_3, (gens PP3)_2=>(gens PP3)_2-(gens PP3)_3,
                    (gens PP3)_3=>(gens PP3)_3})
i69 : threepts#2 == ideal{(gens PP3)_3}+sub(r3, {(gens PP3)_0 =>(gens PP3)_0-(gens PP3)_3,
                     (gens PP3)_1=>(gens PP3)_1-(gens PP3)_3, (gens PP3)_2=>(gens PP3)_2-(gens PP3)_3,
                     (gens PP3)_3=>(gens PP3)_3})
```

# References

[1] L. Bayle, *Classification des variétés complexes projectives de dimension trois dont une section hyperplane générale est une surface d'Enriques.* J. Reine Angew. Math., 449, 9-63, (1994).

[2] W. Barth, C. Peters, A. Van de Ven, *Compact Complex Surfaces*, Springer-Verlag, (1984).

[3] A. Beauville, *Surfaces algébriques complexes.* Asterisque 54, (1978).

[4] G. Castelnuovo, *Memorie scelte*, N. Zanichelli, pp. X-588, (1937).

[5] G. Castelnuovo, F. Enriques, *Sur les int égrales simples de primière espèce d' une surface ou d'une variété algébrique à plusieurs dimensions.* Annales scientifiques de l'École Normale Supérieure, 3, 23, (1906).

[6] I. A. Cheltsov, *Singularities of 3-Dimensional Varieties Admitting an Ample Effective Divisor of Kodaira Dimension Zero.* Mathematical Notes, Vol. 59, No. 4, (1996).

[7] I.Cheltsov, *On the rationality of non-Gorenstein Q-Fano 3-folds with an integer Fano index.* Cont. Math. 207 (1997), 43-50.

[8] I. A. Cheltsov, *Rationality of an Enriques-Fano threefold of genus five.* Izv. RAN. Ser. Mat., 68:3 (2004), 181-194; Izv. Math., 68:3 (2004), 607-618.

[9] C. Ciliberto, T. Dedieu, C. Galati, A. L. Knutsen, *Irreducible unirational and uniruled components of moduli spaces of polarized Enriques surfaces*, arXiv:1809.10569 [math.AG], version 04 December 2019.

[10] C. Ciliberto, T. Dedieu, C. Galati, A. L. Knutsen, *Moduli of curves on Enriques surfaces*, Advances in Mathematics 365 (2020).

[11] C. Ciliberto, E. Sernesi, *Families of varieties and the Hilbert scheme*, Lectures on Riemann surfaces (Trieste 1987) 428-499, Teaneck, World Sci. Publishing, (1989).

[12] A. Conte, *On threefolds whose hyperplane sections are enriques surfaces.* In: Conte A. (eds) Algebraic Threefolds. Lecture Notes in Mathematics, vol 947. Springer, Berlin, Heidelberg, (1982).

[13] A. Conte, *Two examples of algebraic threefolds whose hyperplane sections are Enriques surfaces.* In: Ciliberto C., Ghione F., Orecchia F. (eds) Algebraic Geometry - Open Problems. Lecture Notes in Mathematics, vol 997. Springer, Berlin, Heidelberg, (1983).

[14] A. Conte, J. P. Murre, *Algebraic varieties of dimension three whose hyperplane sections are Enriques surfaces.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 12(1):43-80, (1985).

[15] A. Conte, J. P. Murre, *On the definition and on the nature of the singularities of Fano threefolds.* Conference on algebraic varieties of small dimension (Turin, 1985). Rend. Sem. Mat. Univ. Politec. Torino 1986, Special Issue, 51-67 (1987).

[16] F. R. Cossec, I. V. Dolgachev, *Enriques surfaces. I* Progress in Mathematics, 76, Birkhäuser Boston, Inc., Boston, Ma, (1989).

[17] I.V. Dolgachev, *Classical Algebraic Geometry. A Modern View*, Cambridge University Press, Cambridge, (2012).

[18] I.V. Dolgachev, *A brief introduction to Enriques surfaces. Development of Moduli Theory - Kyoto 2013*, 1-32, Mathematical Society of Japan, Tokyo, Japan, (2016).

[19] J.-M. Drézet. *Luna's slice theorem and applications.* Jaroslaw A. Wisniewski. Algebraic group actions and quotients, Hindawi Publishing Corporation, pp.39-90, (2004), 977-5945-12-7. hal-00742479.

[20] D.H.J. Epema, *Surfaces with canonical hyperplane sections*, PhD thesis, Leiden, (1983).

[21] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry.* Springer, (1995).

[22] D. Eisenbud, A. Van de Ven, *On the normal bundles of smooth rational space curves.* Math. Ann.256, 453-463, (1981).

[23] G. Fano, *Sulle varietà algebriche a tre dimensioni le cui sezioni iperpiane sono superficie di genere zero e bigenere uno.* Memorie società dei XL, 24, 41-66, (1938).

[24] L. Giraldo, A. Lopez., R. Muñoz, *On the projective normality of Enriques surfaces (with an appendix by Angelo Felice Lopez and Alessandro Verra).* Math. Ann. 324, 135-158, (2002).

[25] L. Giraldo, A. Lopez., R. Muñoz, *On the existence of Enriques-Fano threefolds of index greater than one.* Journal of Algebraic Geometry 13, n.1, 143-166, (2004).

[26] L. Godeaux, *Sur le variétiés algebriques à trois dimensions dont les sections hyperplanes sont des surfaces de genre zéro et de bigenre un*, Bull. Acad. Belgique, Cl. des Sci., 14, 134-140, (1933).

[27] P. Griffiths, J. Harris, *Principles of algebraic geometry*, Reprint of the 1978 original. Wiley Classic Library. John Wiley & Sons, Inc. , New York, (1994).

[28] J. Harris, *Algebraic geometry. A first course*, Corrected reprint of the 1992 original. Graduate Texts in Mathematics, 133. Springer-Verlag, New York, (1995).

[29] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, (1977).

[30] V.A. Iskovskih, *Fano threefolds. I*, Math. USSR Izv., 11 (3): 485-527, (1977).

[31] V.A. Iskovskih, *Fano 3-folds II*, Math USSR Izv., 12 (3): 469-506, (1978).

[32] V.A. Iskovskikh, Yu. G. Prokhorov, *Fano varieties*, in A. N. Parshin; I. R. Shafarevich (eds.), Algebraic Geometry, V. Encyclopedia Math. Sci., 47, Springer-Verlag, 1-247, (1999).

[33] A.L. Knutsen, *On moduli spaces of polarized Enriques surfaces*, Journal de Mathématiques Pures et Appliquées (2020).

[34] A.L. Knutsen, A.F. Lopez, *A sharp vanishing theorem for line bundles on K3 or Enriques surfaces*, Proceedings of the American Mathematical Society 135, 3495-3498, (2007).

[35] A.L. Knutsen, A. F. Lopez, *Projective normality and the generation of the ideal of an Enriques surface*, Advances in Geometry 15, 339-348, (2015).

[36] A.L. Knutsen, A.F. Lopez, R. Muñoz, *On the extendability of projective surfaces and a genus bound for Enriques-Fano threefolds*, J. Diff. Geom. 88, 483-518, (2011).

[37] R. Lazarsfeld,*Positivity in Algebraic Geometry*, Vol. 1, Springer-Verlag, Berlin, Heidelberg, New York, (2004).

[38] V. Martello, *On Enriques-Fano threefolds and a conjecture of Castelnuovo*, soon to appear;

[39] V. Martello, *Simple isotropic decompositions of the curve sections of the known Enriques-Fano threefolds*, soon to appear;

[40] V. Martello, *Tangent cones to some Enriques-Fano threefolds of genus 9,13,17*, soon to appear (provisory title);

[41] V. Martello, *Configuration of the singularities of some Enriques-Fano threefolds of genus 8, 10*, soon to appear (provisory title);

[42] s. Mori, S. Mukai, *Classification of Fano threefolds with* $B_2 \geq 2$, Manuscr. Math. 36, 147-162, (1981).

[43] S. Mori, S. Mukai. *Erratum: "Classification of Fano 3-folds with* $B_2 \geq 2$*" [Manuscripta Math. 36 (1981/82), no. 2, 147-162]*, Manuscr. Math., 110:407, 2003.

[44] T. Minagawa, *Deformations of Q-Calabi-Yau 3-folds and Q-Fano 3-folds of Fano index 1*, J. Math. Sci. Univ. Tokyo, 6(2):397-414, (1999).

[45] L.Picco-Botta, A.Verra *The nonrationality of the Enriques threefold*, Composition Math., 48, 167-184, (1983).

[46] Yu. G. Prokhorov, *On Fano-Enriques varieties*, Mat. Sb. 198, 117-134, (2007).

[47] M. Reid, *Young person's guide to canonical singularities*, Proc. Symp. Pure Math. 46, 345-414, (1987).

[48] T. Sano, *On classification of non-Gorenstein* $\mathbb{Q}$*-Fano 3-folds of Fano index 1*, J. Math. Soc. Japan 47, 369-380, (1995).

[49] G. Staglianò, *A Macaulay2 package for computations with rational maps*, The Journal of Software for Algebra and Geometry, 8, (2018).