UNIVERSITÀ DELLA CALABRIA

**Dipartimento di Elettronica,
Informatica e Sistemistica**

**Dottorato di Ricerca in
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XXI ciclo**

Tesi di Dottorato

Model Predictive Control Schemes for Linear Parameter Varying Systems

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Abstract

This dissertation presents several contributions inherently the control of constrained Linearly Parameter Varying (LPV) systems. First the basic analysis and synthesis tools needed to deal with the class of LPV systems are carried out and introduced. Some novel results are given, especially for what regards the use of scheduled control laws and stability conditions for LPV with slow parameter variations.

Then we moved on the problem of constrained control. Several new constrained stabilization results are proposed here for the first time and improvements in the procedures to build-up time-variant strategies able to deal with constrained LPV system are given. Moreover a new particular kind of control strategy based on the idea of exploiting the prediction set structure is introduced here for the first time in the LPV framework. It has been pointed out in which way those approaches can be arranged within Model Predictive Control (MPC) schemes to more efficiently deal with constrained LPV systems and two new fast-MPC algorithms for LPV system have been proposed. In this thesis some attention has been given to the analysis of LPV systems with slow parameter variations and some preliminary results are reported. Such a class of systems has many potentials but, due to the "hidden" nonlinearities it introduces, it is still not well understood and would deserves further careful investigations.

Contents

Introduction

Motivations and Goals

Nowadays practice suggests that amongst all possible deficiencies of current control design methods, the reduced capability to deal with accurate plant models is probably one of the most crucial, limiting the performance potentially achievable when applied to industrial processes. This is especially true when the satisfaction of prescribed constraints is a mandatory requirement.

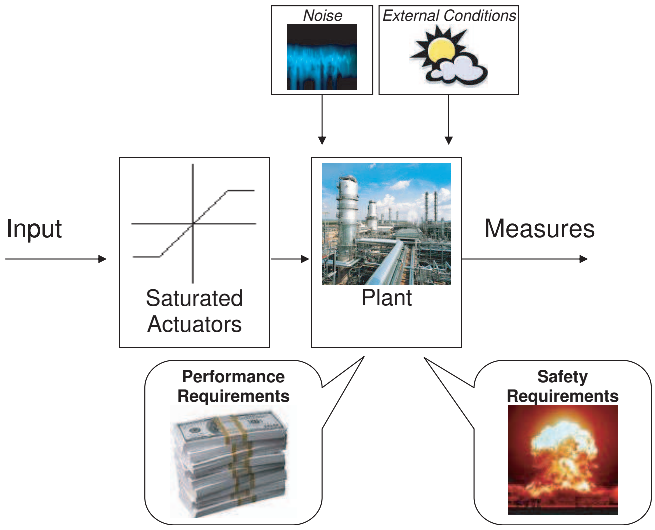

Fig. 0.1. Scheme of an Industrial Plant

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Such a situation is very common. In fact, accurate mathematical descriptions of "real-world" plants typically result in nonlinear systems subject to eventually time-varying uncertainties (due to environmental conditions, set points feature, unmodelled dynamics, noise, etc). Moreover, almost all real plants are subject to saturation phenomena (actuator limitations, physical restraints, etc) and to safety constraints which have to be enforced to guarantee a proper system behavior. Sometimes, when the required control performance are not stringent, some of the above aspects can be neglected. However, when a high performance control task is necessary, the designer has to carefully take them into account in order to avoid hazardous situations both for equipments and human beings.

In the last two decades a significant effort has been produced by the research community in the direction of developing formal tools able jointly to cope with constrained control problems and to take into account possibly time-varying uncertainties. In particular, because of its natural capabilities to handle constraints in a systematic way, Model Predictive Control (MPC) has proven to be a reliable choice for many applications and has gained a good reputation in the industrial world.

Originally developed to meet the control requirements of petroleum refineries and power plants, MPC refers to a family of optimization based control algorithms which make use of a process model to explicitly give plant forecasts. At each sampling time, an MPC algorithm optimizes future plant behavior by computing a sequence of virtual optimal input moves. Then, the first element of the computed sequence is applied to the plant and the entire procedure is repeated at the next sampling time. For such a reason this approach is sometimes referred to as Receding Horizon Control.

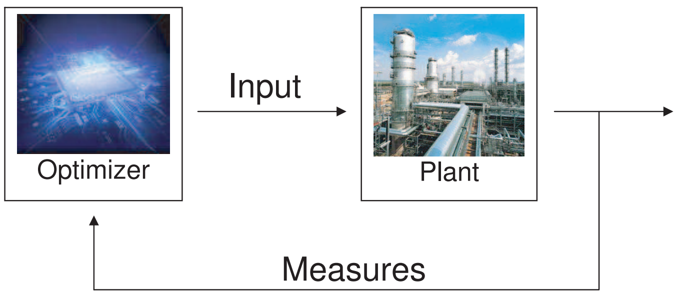

Fig. 0.2. Model Predictive Control Scheme

MPC literature is vast and it covers various system descriptions and frameworks including linear, nonlinear, uncertain and time-varying systems. As already pointed out, it is of industrial relevance the ability to properly deal with systems whose dynamic behavior is subject to non-stationary phenomena. For such a reason, in this dissertation we will focus on a particular class of (possibly) time varying systems: the so called Linear Parameter Varying (LPV) systems paradigm.

LPV systems are characterized by the dependence on a possibly time-varying parameter vector that is supposed to be measurable at each time step. Such a framework was introduced in the late 80's to formalize the ideas behind gain-scheduling control and, as we will in detail in this thesis, it has a wide range of applications in industrial and vehicular control. In recent

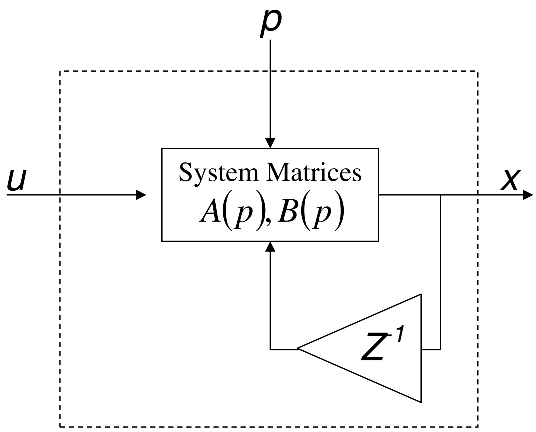

Fig. 0.3. Linear Parameter Varying System

years, several MPC algorithms have been proposed for the LPV framework yielding to interesting results. However, in our opinion, many of them are just a smart adaptation of previously strategies just proposed for uncertain systems and some of the properties of the LPV systems have not yet been fully exploited in MPC schemes. In this thesis, we will investigate the structural properties of the LPV framework in terms of prediction sets, stabilizability and constrained stabilizability results, providing some novel ideas, with the final aim at developing new better performing model predictive control strategies.

Thesis Outline

The thesis is divided into two parts.

In the first part, the LPV framework is described, together with the related control strategies. It will be shown how the use of scheduling control law may give rise to a more complex model structure, and how it is possible to overcome such a problem by means of convexification methods. Moreover, a new convexification method is proposed and a re-interpretation of linearly scheduled control laws applied to a LPV system is given. Finally, in the third Chapter the problem of finding a state feedback stabilizing control law is discussed. Several existing method, both using standard and nonstandard Lyapunov function are shown. Moreover, in the last part of Chapter 3, the use of nonstandard Lyapunov functions, allowed us to define new stability condition for LPV systems subject to bounded parameter variations.

The second part of the thesis is devoted to constrained control for LPV system. In Chapter 4 the constrained control design for LPV system is formulated and the concept of invariance introduced and specialized to the Linear Parameter Varying case. In Chapter 5 the problem of the state trajectories prediction for LPV system are discussed together with the possible strategies exploited to regulate its behavior. New approaches to the prediction problem are introduced and a class of control strategies based on these new prediction sets is introduced for the first time within this framework. Chapter 6 presents several way to solve the problem of constrained control design for the class of systems under exam. First, the problem of state-feedback constrained stabilization is detailed, the existing methods are recalled and some new approaches, based on nonstandard Lyapunov functions, are presented. Then the use of a time-varying control strategy is discussed and several different algorithms are proposed. In Chapter 7 the Model Predictive Control paradigm is introduced, underlining in which way the previous chapter results can be arranged in a Receding Horizon scheme. Feasibility and the stability issues are carefully discussed. The last Chapter presents two new low-computational demanding MPC algorithm.

The following novel results, published or currently under review, are proposed in this thesis:

- A new convexification method for LPV systems subject to linearly self-scheduled control laws, based on a refinement of existing techniques is shown in Subsection 2.2.3.
- An interpretation of an LPV system subject to linearly self-scheduled control laws as a particular class of uncertain system is proposed in Section 2.2.4.
- In Section 3.2 the problem of stabilizability for LPV systems subject to slowly varying systems is solved and several results are obtained. Stabilizability conditions for such a class of systems seems to complete miss in the discrete time LPV literature.
- In Section 4.2.3 some minor contributions to the definition of invariant set for LPV systems are shown.

- In Chapter 5 some slight improvements on the computation of prediction set are given.
- in Section 5.6 the Prediction Set Based Control Strategies are presented for the first time within the LPV framework.
- In the first part of Chapter 6, the constrained stabilization methods making use of standard Parameter Varying Lyapunov Functions have been proposed by the author in [45]. All the results based on non-standard Lyapunov, instead, are here presented for the first time.
- In the second part of Chapter 6, some minor novelties (cost computation, more flexible control strategies) are presented. Moreover the use of the the Prediction Set Based Control Strategies is adopted here for the first time to control constrained LPV systems.
- In Chapter 7, even if following classical ideas, new feasibility and stability results are given in a general way, able to include many MPC algorithm derived by using previous chapters results.
- Finally, in Chapter 8, two new fast-MPC algorithm are proposed using, respectively, invariant set ideas and viability arguments.

Part I

Linear Parameter Varying Systems

1

Discrete Time Linear Parameter Varying Systems

In this chapter we introduce the Linear Parameter Varying (LPV) systems framework, a class of possibly time-varying models whose system matrices depend linearly on a parameter which is supposed to be bounded and measurable at each time instant. This class of systems description is very interesting because enables us to deal in a simpler way with a wide class of processes, including particular nonlinear models whose state trajectory can be embedded inside a LPV "state evolution tube". The chapter is organized as follows, in the first Section, the framework is introduced together with the necessary notation. Then, in the second Section, we discuss some peculiarities of the LPV framework (some additional notation useful for the thesis remainder is also considered) and finally, in the third Section, examples of typical processes which can be described by means of the LPV framework are illustrated and its relationships with the Takagi-Sugeno Fuzzy descriptions are discussed.

1.1 Definitions

The introduction of Linear Parameter Varying systems framework can be dated to the end of the 80's (see [1]-[2]) with the aim of developing a formal framework to give soundness to gain-scheduling control strategies. As we will see in the following, LPV description has proved to be an interesting formalism to deal with a large class of real plants. In the literature both continuous and discrete-time LPV systems have been considered. Here we focus on discrete-time polytopic LPV systems (hereafter referred as LPV systems for simplicity), which can be defined as follows:

Definition 1.1. *A discrete-time LPV system is a possibly time-varying linear system*

$$x\,(k+1) = A\,(p(k))\,x\,(k) + B\,(p(k))\,u\,(k) \tag{1.1}$$

where $x(k) \in \mathcal{R}^n$ *denotes the state,* $u(k) \in \mathcal{R}^m$ *the input and* $A(p(k))$*,* $B(p(k))$ *are matrices of proper dimension depending linearly on a possibly time-varying* *parameter $p(k)$, constrained a-priori to lie in some known bounded real set and assumed to be measurable on-line.*

Definition 1.2. *A discrete-time LPV systems is said to be polytopic when the system matrices have the following structure*

$$[\,A\,(p(k))\;B\,(p(k))\,] = \sum_{i=1}^{l} p_i\,(k)\,[\,A_i\;B_i\,] \tag{1.2}$$

where $[A_i\;B_i]$, $i = 1,\ldots,l$ are the extremal realizations of the plants' family and the parameter vector $p(k) = [p_1(k),\;\ldots\;,p_l(k)]^T$ belongs to the l-dimensional unit simplex

$$\Sigma_l \triangleq \left\{ p \in \mathcal{R}^l \left| \sum_{i=1}^{l} p_i\,(k) = 1, p_i\,(k) \geq 0\;\; i = 1,...,l \right. \right\}. \tag{1.3}$$

The feature of the LPV framework that $p(k)$ is assumed to be measurable at each time step k is hereafter referred to as the ***LPV hypothesis****.*

Note that, in the above definition, no particular assumptions on the parameter behavior are given. This implies that, at least in principle, no particular dependence is supposed between $p(k)$ and its future evolutions $p(k+1),\;p(k+2),\ldots.$
The latter means that even switching-like behaviors, in which the parameter jumps from a point to another one of the simplex, are admissible. Though very general, such a freedom of the parameter behavior can be not adequate to provide a fitting description to some classes of time varying (or even nonlinear) plants. In fact, in a large number of relevant applications, the parameter $p\,(k)$ may represent slowly varying quantities (e.g. environmental conditions, physical parameters, unmodelled nonlinear dynamics,etc). In order to deal in a proper way with those kind of plants it is possible to sophisticate the LPV model by introducing some dependencies (or constraints) on the future occurrences of the parameter vector. The most natural way is the introduction of the so-called bounded parameter variation property:

Definition 1.3. *A discrete-time polytopic LPV system is said to be subject to bounded parameter variations if $p\,(k)$ belongs to the following possibly time-varying polytopic set*

$$\Upsilon(p(k-1)) \triangleq \{p \in \Sigma_l s.t. |p_i - p_i\,(k-1)| < \Delta p_i,\;\; i = 1,...,l\} \tag{1.4}$$

where $\Delta p_i > 0$ is the maximum increment of the i-th entry of the vector $p\,(k)$

Remark 1.4. Note that when $\Delta p_i = 1 \quad i = 1,\ldots,l$, one recovers the standard LPV system definition without any assumption on its rate of parameter changes.

1.2 LPV Information Vector

Here we introduce the notion of plant information vector and the related notation. As previously seen, LPV descriptions differ from uncertain plant representations due to the parameter which is supposed to be known (measurable or computable) at each time instant k. In order to describe all the information available at time k we introduce the information vector $\xi(k)$

$$\xi(k) \triangleq [x(k)^T, p(k)^T, k]^T \tag{1.5}$$

This definition is convenient because in general, at each time step k, all system decision variables (i.e. the manipulable input $u(k)$) can be chosen accordingly to the whole information vector

$$u\left(k\right) = u\left(\xi(k)\right) = u\left(x\left(k\right), p\left(k\right), k\right). \tag{1.6}$$

the input can be, in fact, a function of the parameter, the state and the actual time k. In what follows, we will distinguish between the two classes of dependence, namely:

- Time-Invariant **Control Laws** when the control law does not depend on k, i.e.
$$u(k) = u(\xi(k)) = u(x(k), p(k))$$
- Time-Varying **Control Laws** in the general case (1.6)

Remark 1.5. The information vector ξ acts as a sort of extended state for the system. Starting from this consideration, model (1.1) subject to bounded parameter variations (1.4) can be rewritten as follows

$$\begin{bmatrix} x\left(k+1\right) \\ p\left(k+1\right) \end{bmatrix} = \begin{bmatrix} \sum\limits_{i=1}^{l} p_i(k)\left(A_i x + B_i u\left(\xi\left(k\right)\right)\right) \\ p\left(k\right) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta p\left(k\right) \tag{1.7}$$

where $\Delta p(k)$, the variation on $p(k)$, can be regarded as an appropriate exogenous persistent disturbance which acts in such a way that the parameter variation constraints (1.4) are satisfied. The latter implies that $p(k+1)$ and $x(k+1)$ are not independent each other but both of them depend on $p(k)$. Interestingly enough, such a dependence disappears in the case of LPV without bounded parameter variations because $p(k+1)$ can be any value in the unit simplex Σ_l for any value of $x(k+1)$; i.e.:

$$\begin{bmatrix} x\left(k+1\right) \\ p\left(k+1\right) \end{bmatrix} = \begin{bmatrix} \sum\limits_{i=1}^{l} p_i(k)\left(A_i x + B_i u\left(\xi\left(k\right)\right)\right) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta p\left(k\right) \tag{1.8}$$

with $\Delta p(k) \in \Sigma_l$.

Remark 1.6. It may be interesting to note that if the control strategy is known, on the basis of (1.7) and (1.8), the information vector $\xi(k)$ can be replaced by an initial state $x(0)$ and a feasible sequence $p(\tau)$, $\tau = 0, \ldots, k$ without lack of information.

1.3 Applications and LPV-related frameworks

Some typical applications of the LPV framework are here briefly analyzed. In the first subsection we discuss the the LPV framework capability to deal with scheduling control strategies and, in the second, on the importance of the LPV paradigm to deal with particular classes of nonlinear systems. Finally, in the third subsection, we introduce the class of Takagi-Sugeno (T-S) Fuzzy systems. This class of systems is conceptually very close to the LPV framework and shares several properties with it. The differences between the two frameworks will be analyzed and it will be shown how results holding for the LPV framework reflect into those pertaining to T-S literature.

1.3.1 Scheduling Systems

In many real situations, a system may be subject to transitions among different conditions that can be associated to variations of some system parameter.

In some cases they arise because of modeling problems. In other cases they may be due to physical variables affecting the nominal behavior of the plant.

The idea of gain scheduling consists in using a parameterized control law with compensator parameters chosen online as a function of some plant parameters or states. Such an approach has been successfully used to solve many control problems since '70's (see for instance [5] and [6]) although, for a long time, in the absence of a rigorous analysis and without sound results on stability, robustness and performance. The LPV framework has been introduced by Shamma ([1]) and others ([4]) at the beginning of '90 as a rigorous approach to cover the above theoretical lacks.

1.3.2 Nonlinear Embedding

One of the main merits (see [7],[8],[9]) of the Linear Parameter Varying framework is its capability to allow the embedding the state trajectory of particular nonlinear systems inside a LPV tube of state evolutions. Let us consider a nonlinear system in the form

$$x(k+1) = f(x(k), u(k)) \tag{1.9}$$

where $x(k) \in X \subseteq \Re^n$ denotes the state and $u(k) \in U \subseteq \Re^m$ denotes the input and X, U are, respectively, the state and input prescribed domains. If we are able to rewrite (1.9) in a bilinear structure

$$x(k+1) = A(x(k))x(k) + B(x(k))u(k) \tag{1.10}$$

and if $(A(x(k)), B(x(k)))$, $\forall k$ can be embedded into a polytopic set of vertices $(A_i, B_i)\,, i = 1, \ldots, l$ i.e.

$$\{[A(x(k)), B(x(k))]\ s.t.\ x(k) \in X\} \subseteq conv\{\{[A_i, B_i]\}_i^l\}$$

then the system (1.10), for $x \in X$, is equivalent to

$$x(k+1) = \sum_{i=1}^{l} p_i\left(x(k)\right) A_i x(t) + B_i u(t) \qquad (1.11)$$

where the nonlinearity is transferred on a suitable realization of the parameter vector $p(x(k)) \in \Sigma_l$. The main idea of the LPV embedding is as follows:

- at each time k, $p(x(k))$ is known because $x(k)$ is fully available;
- the trajectory of the parameter $p(\cdot)$ does depend on the state trajectory. Because this relationship is nonlinear, we will relax it and suppose that $p(x(k))$ can be any value in the simplex Σ_l.

Such an embedding approach proved to be a powerful tool in the analysis and synthesis of nonlinear systems (see [10],[11],[12]).

1.3.3 Takagi-Sugeno Fuzzy Systems

In 1985 Takagi and Sugeno (see [13]) introduced a particular class of fuzzy systems, now known as "Takagi-Sugeno Fuzzy Systems". A Takagi-Sugeno fuzzy system is a particular LPV system in the form (1.1) where the value of the parameter vector $p(k) \in \Sigma_l$ depends on a set of normalized membership functions. Even if in this thesis we will not make use of such a formalism, it may be of interest because of similarities with the LPV framework. Moreover, many results presented in the vast T-S fuzzy literature can be successfully adopted into the LPV framework (and vice versa).

The typical discrete T-S fuzzy model description is given by a set of rules R_p^i for $i = 1, \ldots, l$.:

$$\begin{array}{l} R_p^i : IF\, x_1\left(k\right) \;\; is \;\; M_1^i \;\; and \;\; ... \;\; and \;\; x_n\left(k\right) \;\; is \;\; M_n^i \\ THEN \;\; x\left(k+1\right) = A_i x\left(k\right) + B_i u\left(k\right) \end{array}$$

where $x\left(k\right) = \left[\, x_1\left(k\right) \ldots \& x_n\left(k\right) \,\right]^T \in \Re^n$ denotes the state, $u\left(k\right) = \in \Re^m$ the input vector and $A_i \in \Re^{n \times n}, B_i \in \Re^{m \times n}$ the system matrices. $M_j^i\left(j = 1, ..., n\right)$ denotes fuzzy sets defined by some membership function. The membership function of x_j belonging to M_j^i is denoted by

$$\mu_j^i\left(x\left(k\right)\right).$$

If we introduce $M^i = M_1^i \times \ldots \times M_n^i$ and $\mu^i\left(x\left(k\right)\right) = \prod_{j=1}^{n} \mu_j^i\left(x_j\left(k\right)\right)$ denoting the grade of membership of $x\left(k\right)$ in M^i, we can rewrite the whole system as follows

$$x\left(k+1\right) = \sum_{i=1}^{l} h_i\left(x\left(k\right)\right)\left[A_i x\left(k\right) + B_i u\left(k\right)\right]$$

where

$$h_i(k) = \frac{\mu^i(x(k))}{\sum_{i=1}^{l} \mu^i(x(k))}.$$

Because functions $\mu^i(x(k))$ are assumed to be positive this implies

$$\begin{array}{l} 0 \leq h_i(x(k)) \leq 1, \\ \sum_{i=1}^{l} h_i(x(k)) = 1. \end{array}.$$

It is then possible to note that the T-S fuzzy systems can be seen as particular LPV systems where the parameter depends on some state function.

Chapter Summary

LPV formalism has been here introduced together with the information vector $\xi(k) = [x(k)^T, p(k)^T, k]$ representing all information available for control at time k. The main uses of the LPV framework have been summarized and the relationships with the Takagi-Sugeno Fuzzy System descriptions pointed out.

2

Scheduled Control Laws and LPV Systems

In this Chapter we define and discuss some of the typical scheduled control laws which are used in dealing with discrete-time LPV systems. Although those kind of control laws ensure better results w.r.t. non-scheduled control laws, the resulting closed-loop system structure is inherently more complicated and careful investigations are mandatory. This Chapter is organized as follows. First, the typical control laws adopted in the LPV framework are introduced. Then, some notes on the non-convex nature of the closed-loop state evolution sets are reported. Several convexification strategies are therefore proposed to deal with such a problem and a general interpretation of all the above convexification procedures is given.

2.1 Scheduled Control Laws

As already seen, the parameter $p(\cdot)$ is a relevant information to be used for improving the control action on the system. For such a reason hereafter we focus on the determination of control laws that explicitly depend on it.

2.1.1 State Affine Scheduled Control Laws

The simplest and the most common parameter dependent control laws are those which depend linearly from the parameter vector:

$$u\left(\xi(k)\right) = \left(\sum_{i=1}^{l} p_i\left(k\right) u_i\left(x\left(k\right)\right)\right) \tag{2.1}$$

where $u_i\left(x\left(k\right)\right),\ i = 1, \ldots, l$ are LTI affine state-dependent contol laws of the form

$$u_i(x(k)) = F_i x(k) + c_i, \quad i = 1, \ldots, l \tag{2.2}$$

with $F_i \in \mathcal{R}^{m \times n}, c_i \in \mathcal{R}^m$. Note that many classical gain scheduling control laws can be rewritten in this form. As detailed better later, the main disadvantages in using this kind of laws is that they introduces a quadratic dependence between the one-step ahead closed-loop state evolution and the parameter value. In this dissertation, we will mostly refer to this kind of control laws.

2.1.2 Non-standard Scheduled Control Laws

In recent literature, scheduled control laws alternative to (2.1) have been also proposed. Of particular interest here is the one proposed in [14], where a linearly parameter dependent state feedback control law is corrected by the inverse of a scheduled correction matrix:

$$u\left(\xi(k)\right) = \left(\sum_{i=1}^{l} p_i(k) F_i \left(\sum_{j=1}^{l} p_j(k) \Psi_j\right)^{-1}\right) x(k) \tag{2.3}$$

where $F_i \in \mathcal{R}^{m \times n}$ and the correction matrix elements $\Psi_j \in \mathcal{R}^{n \times n}$ act as an additional degree of freedom. The main merit of the above scheduled control law (2.3), as it will be better remarked later, is that it allows the derivation of less conservative conditions for closed-loop stability. Observe also that the state feedback control laws (2.1) can be seen as a special case of (2.3) with $\Psi_i = I, \ \ i = 1, \ldots, l$.

2.2 Non-convexity of the One-step Reachable Set

It is of particular interest here to understand what the closed-loop state evolutions become when a parameter scheduled control law (2.1) or (2.3) is employed. In fact, it is straightforward to note that, in general, even the simplest scheduled control law (2.1) introduces a parameter nonlinearity into the one-step ahead state evolution:

$$x(k+1) = \sum_{i=1}^{l} p_i(k) A_i x(k) + \sum_{i=1}^{l} p_i(k) B_i \sum_{j=1}^{l} p_j u_j\left(x\right)$$

As a consequence the set

$$X^{+} = \left\{ x(k+1) = \sum_{i=1}^{l} p_i A_i x(k) + \sum_{i=1}^{l} p_i B_i \sum_{j=1}^{l} p_j u_j\left(x\right) \middle| \, \forall p \in \Sigma_k \right\}$$

i.e. the set of all states reachable from $x(k)$ in one step for any possible parameter occurrence, is not convex in general. The above considerations imply that the use of parameter scheduled control laws leads up to more cumbersome analysis and synthesis tasks. On the other hands, the advantages of using scheduled control laws have been extensively remarked in literature and it is main reason of the LPV framework success.

A typical way to overcome the non-convexity nature of X^+ is that of exploiting a convenient (polytopic) outer approximation. In the following subsections, we will show and discuss some of the convexification procedures that can be used when control laws of the form (2.1) are used.

2.2.1 A Naive Convexification

A very simple way to deal with the quadratic dependence seen above, is given by the following procedure.

Let us rewrite (1.1) as follows

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{l} p_i(k) \left[A_i x(k) + B_i \sum_{j=1}^{l} p_j(k) u_j(x(k)) \right] = \\ &= \sum_{i=1}^{l} \sum_{j=1}^{l} p_i(k) p_j(k) \left[A_i x(k) + B_i u_j(x(k)) \right], \end{aligned} \tag{2.4}$$

$$\forall p(k) \in \Sigma_l$$

Observe that the set of all states $x(k+1)$ reachable from $x(k)$ for a given control law in form (2.1) by arbitrarily changing $p(k) \in \Sigma_k$ is not convex because of the products $p_i(k)p_j(k)$. A possible way to easily convexify the above set is by considering

$$x(k+1) = \sum_{i=1}^{l} \sum_{j=1}^{l} \bar{p}_{i,j}(k) \left[A_i x(k) + B_i u_j(x(k)) \right] \tag{2.5}$$

$$\forall \bar{p} \in \Sigma_l^2$$

were

$$\bar{p}(k) = \left[\bar{p}_{1,1}(k), \bar{p}_{1,2}(k), \ldots, \bar{p}_{1,l}(k), \bar{p}_{2,1}(k), \ldots, \bar{p}_{ll}(k) \right] \in \Re^{l^2} \tag{2.6}$$

The proof that (2.5) is a feasible convexification for (2.4) can be obtained by noting that for each $p(k)$ there exists a corresponding $\bar{p}(k)$ with entries

$$\bar{p}_{ij}(k) = p_i(k) p_j(k), \ i = 1, \ldots, l, j = 1, \ldots, l, \tag{2.7}$$

such that $\bar{p}(k) \in \Sigma_l$ because of

$$\begin{aligned} & p_i(k) p_j(k) \geq 0, \\ & \sum_{i=1}^{l} \sum_{j=1}^{l} p_i(k) p_j(k) = 1 \end{aligned} \tag{2.8}$$

While very simple to obtain, the above embedding is not very efficient and exhibits several weak spots. One of the main problems is that many advantages coming from the employment of a scheduled control law are lost. For instance, the following Lemma shows that, if we use this convexification procedure, scheduled control laws have the same stabilization capability of non-scheduled control laws, i.e. laws of the form (2.1) with $u_i(x(k)) = u(x(k)), i = 1, ..., l$:

Lemma 2.1. *A LPV system in the form (2.5) is stabilizable by a scheduled control law (2.1) if and only if it is stabilizable by a non-scheduled control law of the form*

$$u_j(x(k)) = u(x(k)), j = 1, ..., l$$

Proof. To prove the above result it is enough to consider a certain occurrence of $\tilde{p} \in \Sigma_{l^2}$ such that $\tilde{p}_{ij} = 1$ and all the other entries of $\tilde{p}$ are zero. Then, the autonomous system (2.5) becomes

$$x(k+1) = A_i x(k) + B_i u_j(x(k))$$

A necessary condition for (2.5) to be stable is that $u_j(x(k)$ asymptotically stabilizes $x(k+1) = A_i x(k) + B_i u(k)$. By repeating the above procedure for $i = 1, \ldots, l$, $j = 1, \ldots, l$, the statement is proved.

The above lemma implies that, by making use this kind of re-parametrization, the knowledge and use of $\tilde{p}$ for feedback does not improve the stabilization capabilities. The above considerations explain the weakness of this approach in exploiting the possible advantages of LPV systems w.r.t. other frameworks.

2.2.2 Half-sum Convexification

In order to overpass the limitations of the above naive procedure, a better convexification procedure has been recently proposed by Tanaka [15]. Let us rewrite (1.1), under the action of a whatsoever regulation strategy, (2.1) as follows

$$x\,(k+1) = \sum_{i=1}^{l}\sum_{j=1}^{l} p_i\,(k)\,p_j\,(k)\,[A_i x\,(k) + B_i u_j\,(x\,(k))] \tag{2.9}$$

By exploiting the symmetry of $p_i\,(k)\,p_j\,(k) = p_j\,(k)\,p_i\,(k)$, (2.9) can be rewritten as

$$\begin{aligned} x\,(k+1) = &\sum_{i=1}^{l} p_i^2\,(k)\,[A_i x\,(k) + B_i u_i\,(x(k))] \\ + &\sum_{\substack{i=1 \\ j=i+1}}^{l} 2p_i\,(k)\,p_j\,(k)\left[\tfrac{(A_i+A_j)x(k)+(B_i u_j(x(k))+B_j u_i(x(k)))}{2}\right] \end{aligned} \tag{2.10}$$

Observe that

$$\begin{array}{l} \sum\limits_{i=1}^{l} p_i^2(k) + \sum\limits_{\substack{i=1 \\ j=i+1}}^{l} 2p_i(k)\,p_j(k) = 1 \\ 2p_ip_j \geq 0, p_i^2 \geq 0 \end{array} \tag{2.11}$$

but again a non-convex set of one-step reachable states is achieved. Then, we can embed (2.10) into the following convex outer approximation

$$\begin{array}{l} x(k+1) = \sum\limits_{\substack{i=1 \\ j=i+1}}^{l} \bar{p}_{ij}(k) \left[\frac{(A_i+A_j)x(k)+(B_iu_j(x(k))+B_ju_i(x(k)))}{2} \right] \\ \bar{p}(k) = \left[\bar{p}_{11}(k)\ \bar{p}_{12}(k)\ \ldots\ \bar{p}_{ll}(k) \right]^T \in \Sigma_{l(l+1)/2}. \end{array} \tag{2.12}$$

where

$$\bar{p}_{ij}(k) = \begin{cases} p_i(k)^2 & if\ \ i=j \\ 2p_i(k)p_j(k) & otherwise \end{cases} \tag{2.13}$$

is the mapping between the parameter vector $p(k)$ and $\bar{p}(k)$. In order to eval-

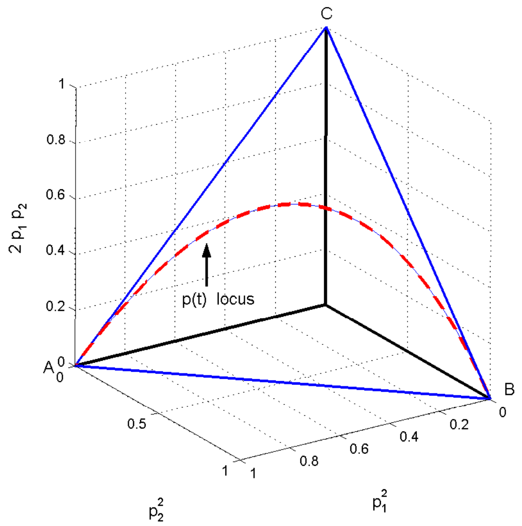

Fig. 2.1. $p(k)$ locus on the $(p_1^2, p_2^2, 2p_1p_2)$ space (Dashed line graph) and half-sum embedding

uate how this outer approximation is tight, one could consider the locus of values of the vector $\bar{p}$ obtained by (2.13) for all possible values of $p(k) \in \Sigma_k$ with respect to the whole simplex $\Sigma_{l(l+1)/2}$. To give a geometrical intuition, such a locus is depicted in Figure 2.1 for the case $l = 2$, where A,B,C are the vertices of the unitary simplex $\Sigma_{l(l+1)/2}$. Such a figure shows also how the Half-sum embedding exploits the symmetry of $p_ip_j = p_jp_i$ and could be eventually improved by means of geometrical considerations reported in the next subsection.

Remark 2.2. It is of interest to note that this reparamenterization is not only less conservative then the naive one, but also it generates less vertices.

2.2.3 Improving the Convexification

In this subsection we propose a new method to sharpen further on the outer approximating embedding seen in the previous subsection. To this end, consider the following preliminary result [16]:

Lemma 2.3. *Let $p(k) \in \Sigma_l$, where Σ_l denotes the l-dimensional unit simplex. Then, the following property holds true*

$$2\, p_i(k)\, p_j(k) \leq \frac{1}{2}, \;\; i = 1, \ldots, l, \; j = i+1, \ldots, l. \tag{2.14}$$

Proof. Consider the maximum allowable value of a single term $p_i\, p_j$ which is obtained when $p_k = 0$, $\forall k \neq i, j$. Under this condition and because $p \in \Sigma_l$, one has that $p_i + p_j = 1$. Therefore, it follows that $p_i\, p_j = (1-p_j)p_j = p_j - p_j^2$. As a consequence the maximum value of the above equation is attained when $p_j = \frac{1}{2}$ and (2.14) follows.

Lemma 2.3 implies that the parameter $p(k)$ can be embedded into a new convex region defined by the intersection of the unit simplex $\Sigma_{l(l+1)/2}$ and the family of half-spaces (2.14).

In order to make clear the idea via graphical arguments we consider first the case $l = 2$. Therefore, a general result will be presented. To this end, consider next Figure 2.2 which shows a geometrical representation of the Half-sum approach. Points ($A = [1, 0, 0]^T, B = [0, 1, 0]^T, C = [0, 0, 1]^T$) represent the vertices of the unit simplex $\Sigma_{l(l+1/2)}$ defined in (2.11) while the couple ($D_1 = [1/2, 0, 1/2]^T, D_2 = [0, 1/2, 1/2]$) describes the points of intersection between the unit simplex $\Sigma_{l(l+1)/2}$ and the hyperplane $2\, p_1 p_2 = 1/2$. The dashed line depicts the exact locus of the parameter defined by (2.13) for all possible values of $p(k) \in \Sigma_k$. Lemma 2.3 guarantees that such a locus can be then embedded by means of the convex combination of the points (A, B, D_1, D_2) Let us consider the Half-sum representation of the overall closed loop systems. Then, we have

$$\begin{aligned} &x(k+1) = \tilde{p}_1 \Phi_{1,1}(x(k)) + \tilde{p}_2 \Phi_{1,1}(x(k)) + \tilde{p}_{12} \Phi_{1,2}(x(k)) \\ &\tilde{p} = [\tilde{p}_1, \tilde{p}_2, \tilde{p}_{12}]^T \in \Sigma_3 \end{aligned} \tag{2.15}$$

where

$$\Phi_{i,j}(x(k)) = \frac{A_i x(k) + B_i u_j(x(k)) + A_j x(k) + B_j u_i(x(k))}{2}$$

Because of the above considerations, if we consider the convex hull of the points obtained by evaluating (2.15) for $\tilde{p} = A, B, D_1, D_2$, the following new tighter embedding is obtained

$$x(k+1) = \bar{p}_1(k)\Phi_{1,1}(x(k)) + \bar{p}_2(k)\Phi_{2,2} + \bar{p}_{1112}(k)\left(\tfrac{1}{2}\Phi_{1,1}(x(k)) + \tfrac{1}{2}\Phi_{1,2}(x(k))\right) + \\ + \bar{p}_{2212}(k)\left(\tfrac{1}{2}\Phi_{2,2}(x(k)) + \tfrac{1}{2}\Phi_{1,2}(x(k))\right)$$
$$\bar{p} \in \Sigma_4$$

with the original parameter vector $p(k)$ related to the new $\bar{p}(k)$ via the following relationship

$$\begin{bmatrix} p_1^2(k) \\ p_2^2(k) \\ 2p_1(k)p_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \bar{p}_1(k) \\ \bar{p}_2(k) \\ \bar{p}_{1112}(k) \\ \bar{p}_{2212}(k) \end{bmatrix} \tag{2.16}$$

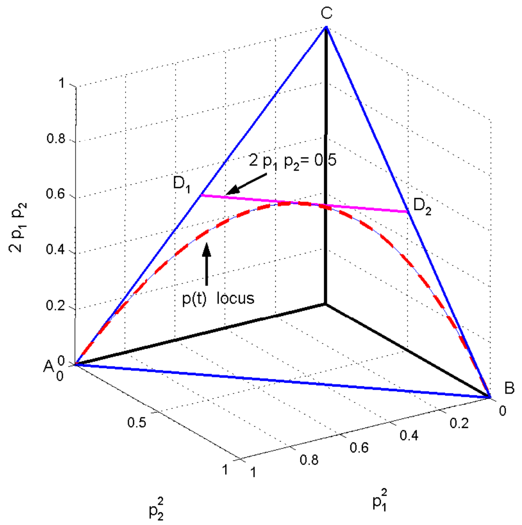

Fig. 2.2. $p(k)$ locus on the $(p_1^2, p_2^2, 2p_1p_2)$ space (Dashed line graph). $A - B - C$ polygon: Half-sum embedding. $A - D_1 - D_2 - B$ polygon: proposed embedding

The above ideas can be formalized and applied to any arbitrary dimension l. By generalizing the above approach, condition (2.14) consists of intersecting the Half-sum embedding via the hyperplanes

$$2\,p_i(k)\,p_j(k) = \frac{1}{2}, \quad i = 1, \ldots, l, \; j = i+1, \ldots, l$$

and to sharpen (2.12) as follows

$$
\begin{aligned}
x(k+1)=&\sum_{i=1}^{l} \bar{p}_i(k)\left[A_i x(k)+B_i u_i(x(k))\right]+\\
&\sum_{\substack{i=1,\\ w=1,\\ s=w+1}}^{l} \bar{p}_{iiws}(k)\left[\frac{(2A_i+A_w+A_s)x(k)+2B_i u_i(x(k))+B_w u_s(x(k))+B_s u_w(x(k))}{4}\right]+\\
&\sum_{\substack{i=1,\\ j=i+1,\\ s=j+1}}^{l} \bar{p}_{ijis}(k)\left[\frac{(2A_i+A_j+A_s)x(k)+B_i u_j(x(k))+(B_j+B_s)u_i(x(k))+B_i u_s(x(k))}{4}\right]+\\
&\sum_{\substack{i=1,\\ j=i+1,\\ w=i+1,\\ s=w+1}}^{l} \bar{p}_{ijws}(k)\left[\frac{(A_i+A_j+A_w+A_s)x(k)+B_i u_j(x(k))+B_j u_i(x(k))+B_w u_s(x(k))+B_s u_w(x(k))}{4}\right]
\end{aligned}
\tag{2.17}
$$

with

$$
\sum_{i=1}^{l} \bar{p}_i(k)+\sum_{\substack{i=1,\\ w=1,\\ s=w+1}}^{l} \bar{p}_{iiws}(k)+\sum_{\substack{i=1,\\ j=i+1,\\ s=j+1}}^{l} \bar{p}_{ijis}(k)+\sum_{\substack{i=1,\\ j=i+1,\\ w=i+1,\\ s=w+1}}^{l} \bar{p}_{ijws}(k)=1 \tag{2.18}
$$

where $\bar{p}_i(k) \geq 0$ and $\bar{p}_{ijks}(t) \geq 0$ are new suitable combinations of the parameter vector $p(k) \in \Sigma_l$.

Remark 2.4. The proposed convexification is obviously tighter than (2.12). The price to be paid is in an increased complexity: the number of vertices becomes $\frac{l^4+2l^3-5l^2+10l}{8}$ instead of $l(l+1)/2$.

Remark 2.5. Figure (2.2) suggests that further "cutting planes" can be used to obtain further tighter and tighter outer embedding of the parameter projection. This can be orderly obtained by geometrical considerations at a price of an increasingly number of vertices.

Example 2.6. In Figure 2.3 the three convexification methods presented in this Chapter are used to predict an outer approximation of the one-step ahead prediction set X^+. The following controlled plant is considered

$$
x(k)=\sum_{i=1}^{2} p_i(k)A_i x(k)+\sum_{i=1}^{2} p_i(k)B_i\sum_{j=1}^{2} p_j(k)u_j
$$

where

$$
\begin{aligned}
A_1&=\begin{pmatrix} 2 & -0.1\\ 0.5 & 1\end{pmatrix},\ B_1=[1\ \ -0.3]^T,\ u_1=1,\\
A_2&=\begin{pmatrix} 1 & 0.1\\ 2.5 & 1\end{pmatrix},\quad B_2=[0.7\ 0.1]^T,\quad u_2=10.
\end{aligned}
$$

and the initial state is $x(0) = [10 \ -2]^T$.

Dotted blue line represents the real X+; the area limited by the red line is the Naif convexification; magenta line encloses the Semi-Sum convexification set and the green line is the bound of the prediction set obtainable through the convexification method proposed in this Subsection.

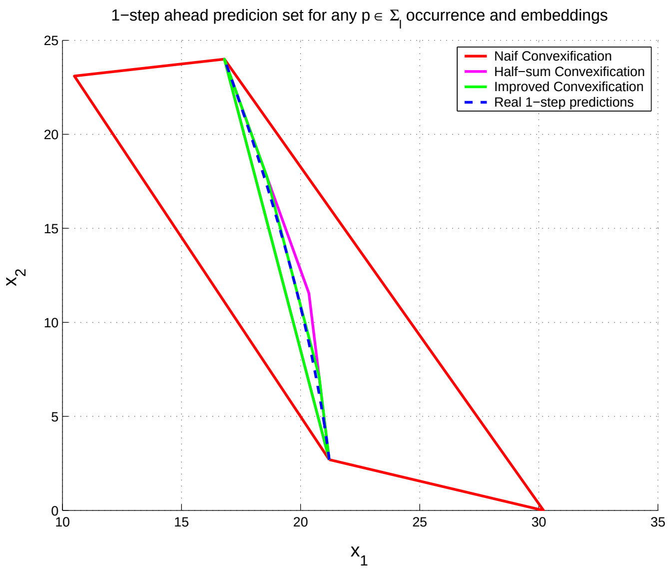

Fig. 2.3. An example of state prediction embedding

2.2.4 An interpretation of LPV convexification

It is interesting to furnish an interpretation of the above LPV convexification procedures. If the chosen control law is such that the input is a convex combination (depending on the parameter vector p) of $u_1, \ldots, u_l$ vectors, the LPV system can be regarded as an equivalent system described in the form

$$
\begin{aligned}
x(k+1) &= \sum_{i=1}^{l} p_i(k) A_i x(k) + \\
&+ \left[B_1 \ldots B_l \right] \begin{bmatrix} p_1(k) p_1(k) I_{m\times m} & \ldots & p_1(k) p_l(k) I_{m\times m} \\ \ldots & & \ldots \\ p_l(k) p_1(k) I_{m\times m} & \ldots & p_l(k) p_l(k) I_{m\times m} \end{bmatrix} \begin{bmatrix} u_1(k) \\ \ldots \\ u_l(k) \end{bmatrix} = \\
&= \sum_{i=1}^{l} p_i(k) A_i x(k) + \left[B_1 \ldots B_l \right] \left(p(k) p(k)^T \otimes I_{m\times m} \right) \begin{bmatrix} u_1(k) \\ \ldots \\ u_l(k) \end{bmatrix}
\end{aligned}
\tag{2.19}
$$

where $\otimes$ denotes the Kronecker product. Exploiting the fact that $p(k) \in \Sigma_l$, it is possible to rewrite (2.19) as follows,

$$\begin{array}{l} x(k+1) = \left(\left(1_n^T \otimes I_{nxn}\right) \left(p(k)\, p(k)^T \otimes I_{nxn}\right) \begin{bmatrix} A_1 \\ \dots \\ A_l \end{bmatrix} \right) x(k) + \\ \left(\begin{bmatrix} B_1 \dots B_l \end{bmatrix} \left(p(k)\, p(k)^T \otimes I_{mxm}\right) \right) \begin{bmatrix} u_1 \\ \dots \\ u_l \end{bmatrix} \\ p(k) \in \Sigma_l \end{array} \tag{2.20}$$

where $1_n = [1 \dots 1]^T \in \Re^n$. Such a system, with state $x(k)$ and input vector $\bar{u}^T = \begin{bmatrix} u_l^T \dots u_l^T \end{bmatrix}^T$, depends quadratically on $p(k)\, p^T(k)$ with $p(k) \in \Sigma_l$.

The idea here consists of embedding the matrix family $p(k)\, p^T(k)$, achieved for $p(k) \in \Sigma_l$, into a polytopic set of matrices defined by suitable vertices $\{\Pi_1, \dots, \Pi_{l_c}\}$. This interpretation allows us to embed our LPV system into the following uncertain polytopic system

$$x(k+1) = \sum_{i=1}^{l_c} \bar{p}_i(k) \left[\bar{A}_i x(k) + \bar{B}_i \bar{u}(x(k))\right] = \tag{2.21}$$

where

$$\bar{A}_i = \left(\left(1_n^T \otimes I_{nxn}\right) \left(\Pi_i \otimes I_{nxn}\right) \begin{bmatrix} A_1 \\ \dots \\ A_l \end{bmatrix} \right) \quad i = 1, \dots, l_c \tag{2.22}$$

$$\bar{B}_i = \left(\begin{bmatrix} B_1 \dots B_l \end{bmatrix} \left(\Pi_i \otimes I_{mxm}\right) \right) \quad i = 1, .., l_c \tag{2.23}$$

and $\bar{p}(k) \in \Sigma_{l_c}$ is the corresponding l_c-dimensional parameter vector. It is important to note that such a new vector $\bar{p}(k) \in \Sigma_{l_c}$ is related to $p(k) \in \Sigma_l$. In fact, by its definition, for each $p(k) \in \Sigma_l$ there exists (at least) one $\bar{p}(k) \in \Sigma_{l_c}$ such that

$$\sum_{i=1}^{l_c} \bar{p}_i(k)\, \Pi_i = p(k)\, p^T(k).$$

Then it is always possible to define a certain mapping function

$$\rho : \Sigma_l \rightarrow \Sigma_{l_c} \tag{2.24}$$

such that

$$\sum_{i=1}^{l_c} \left(\bar{\rho}(p(k))\right)_i \Pi_i = p(k)\, p^T(k).$$

Within this framework, the previously convexification procedures simply corresponds to three different ways to achieve convex outer approximations of matrices $p(k)\,p^T(k)$. In particular, if we denote by e_i for $i = 1, \ldots, l$ the i-th vector of the canonical basis of $\Re^l$, it is possible to show that:

1. The naive convexification procedure corresponds to the following embedding strategy
$$p(k)\,p^T(k) \subset conv\left\{\left\{e_i e_j^T\right\}_{\substack{i=1\\j=1}}^{l}\right\}$$
2. For the Half-sum convexification procedure, the symmetry of $p(k)\,p^T(k)$ is also taken into account and exploited. This corresponds to add new vertices to the above embedding for $p(k)\,p^T(k)$ and obtaining, as a consequence:
$$p(k)\,p^T(k) \subset conv\left\{\left\{e_i e_i^T\right\}_{i=1}^{l}, \left\{\frac{1}{2}\left(e_i e_j^T + e_j e_i^T\right)\right\}_{\substack{i=1\\j=i+1}}^{l}\right\}$$
3. In the new convexification procedure proposed in Subsection 2.2.3, the additional inequality $p_1(k)\,p_2^T(k) < 1/4$ is also imposed, which leads to

$$\begin{aligned} p(t)\,p^T(t) \in \mathrm{co}\Bigg\{ & \left\{e_i\,e_i^T\right\}_{i=1}^{l},\ \left\{\tfrac{1}{4}\left(e_i\,e_j^T + e_j\,e_i^T\right) + \tfrac{1}{2}e_i\,e_i^T\right\}_{\substack{i=1\\j=i+1}}^{l}, \\ & \left\{\tfrac{1}{4}\left(e_i\,e_j^T + e_j\,e_i^T\right) + \tfrac{1}{4}\left(e_i\,e_m^T + e_m\,e_i^T\right)\right\}_{\substack{i=1\\j=i+1\\m=j+1}}^{l}, \\ & \left\{\tfrac{1}{4}\left(e_i\,e_j^T + e_j\,e_i^T\right) + \tfrac{1}{4}\left(e_m\,e_n^T + e_n\,e_m^T\right)\right\}_{\substack{i=1\\j=i+1\\m=i+1\\n=m+1}}^{l} \Bigg\} \end{aligned} \tag{2.25}$$

Remark 2.7. It can be interesting to note that this reformulation allows one to look for tighter embedding procedures just focusing on the problem of approximate $p(k)\,p^T(k)$ within a polytope of matrices.

Chapter Summary

In this Chapter, scheduled control laws for the control of LPV systems have been introduced. Scheduled control law complicates the system dynamics because the set of states reachable from $x(k)$ in one-step for any possible parameter occurrence is not a convex set. The Half-sum convexification proposed in [15] has been introduced. As well known, it improves the naive one presented in Subsection 2.2.1. A further convexification procedure, based on a refinement of the Half-sum approach, is here presented for the first time, at the best of our knowledge, and it believed to be original. Finally, a new general interpretation of all described convexification procedures has been proposed which it is also believed original.

3

Stabilizability Results

The aim of this Chapter is to review existing stabilizability results for LPV systems and to propose novel and more refined conditions obtainable by exploiting different classes of Lyapunov functions and/or by using particular LMI tricks. The Chapter is organized into two Sections. First we focus on the stabilizability conditions for LPV systems with no hypothesis on their parameter variations. Many of the existing approaches in literature will be reviewed and discussed. In the second Section, the stabilizability problem of LPV systems subject to bounded parameter variations is addressed. There is a lack in literature regarding results on stabilizability conditions for such a class of systems and, up to our best knowledge, the results presented here are new.

Many of the above stabilizability conditions will result in the form of matrix inequalities not linear in the unknowns, which need to be satisfied for all values of the LPV parameter vector. Convex relaxations are then of paramount relevance for control design purposes. Several convexification procedures can be used and corresponding convex optimization formulations of the control design problem derived. In order to avoid a tedious enumeration of such procedures, in this Chapter we will focus on the main ideas underlying their application. An extensive though not exhaustive list of the convexification techniques proposed in literature will be provided in the final Appendix A

3.1 Stabilizability Results for LPV systems

In this section we introduce the main ideas and results about the stabilizability of LPV systems. We restrict our attention to scheduled state-feedback control laws of the form

$$u\left(\xi\left(k\right)\right)=F\left(p\left(k\right)\right)x\left(k\right) \tag{3.1}$$

where $F\left(p\left(k\right)\right)\in\Re^{m\times n}$ is a feedback matrix, having and arbitrary dependence on the parameter vector. In order to find stabilizing control laws we will use Lyapunov stability arguments. In particular, in the following subsections we will introduce different Lyapunov functions frameworks, the related basic ideas and stabilizability results.

3.1.1 Quadratic Lyapunov Function

The simplest Lyapunov function is the classical quadratic form

$$V\left(\xi\left(k\right)\right)=x^{T}\left(k\right)Px\left(k\right) \tag{3.2}$$

where $P\in\Re^{n\times n}$ is a positive definite matrix $P=P^{T}>0$. In order to impose asymptotic stability of the closed loop system

$$x\left(k+1\right)=\left[A\left(p\left(k\right)\right)+B\left(p\left(k\right)\right)F\left(p\left(k\right)\right)\right]x\left(k\right) \tag{3.3}$$

the gain $F(p(k))$ must be chosen such that the Lyapunov difference

$$V\left(\xi\left(k+1\right)\right)-V\left(\xi\left(k\right)\right) \tag{3.4}$$

is negative along the augmented trajectories $\xi(k)$. The above condition is equivalent to

$$x^{T}\left(k\right)\left(\left[A\left(p\left(k\right)\right)+B\left(p\left(k\right)\right)F\left(p\left(k\right)\right)\right]^{T}P\left[A\left(p\left(k\right)\right)+B\left(p\left(k\right)\right)F\left(p\left(k\right)\right)\right]-P\right)x\left(k\right)<0. \tag{3.5}$$

A sufficient condition which ensures that such an inequality can be satisfied for all the states $x(k)$, is obtained by imposing that the following matrix is negative definite

$$\left[A\left(p\left(k\right)\right)+B\left(p\left(k\right)\right)F\left(p\left(k\right)\right)\right]^{T}P\left[A\left(p\left(k\right)\right)+B\left(p\left(k\right)\right)F\left(p\left(k\right)\right)\right]-P<0 \tag{3.6}$$

Via Schur complements and standard congruence transformations, (3.6) can be shown to be equivalent to the fulfillment of

$$\begin{array}{l}\left[\begin{array}{cc}Q & *\\ \left[A\left(p\left(k\right)\right)Q+B\left(p\left(k\right)\right)Y\left(p\left(k\right)\right)\right] & Q\end{array}\right]>0\\ Q>0\end{array} \tag{3.7}$$

where $Q=P^{-1}$ and $Y\left(p\left(k\right)\right)=F\left(p\left(k\right)\right)Q$. Then, in order to find a stabilizing control law and an associated Lyapunov function, we can solve in principle the following problem

Problem 3.1. *Find $Y(p),Q$ such that*

$$\begin{array}{l}\left[\begin{array}{cc}Q & *\\ \left[A\left(p\right)Q+B\left(p\right)Y\left(p\right)\right] & Q\end{array}\right]>0,\forall p\in\Sigma_{l}\\ Q>0\end{array} \tag{3.8}$$

Note that due to the arbitrary structure of $F(p)$, the term $B(p)Y(p)$ could depend in a nonlinear way from the parameter p and, as a result, Problem 4.1 is not convex in general. A convex problem is obtained if some of the above conditions are relaxed and if particular structures to the allowable control laws are considered. Relaxations are generally achieved in two ways

- If linearly scheduled state-feedback control laws (2.1) are employed, i.e. $F(p) = \sum_{i=1}^{l} p_i F_i$, then $Y(p)$ can be parameterized as

$$Y(p) = \sum_{i=1}^{l} p_i Y_i, \quad Y_i = F_i Q \tag{3.9}$$

  Thanking to (3.9) we can easily exploit convexification procedures of the state evolution tube and its interpretation seen in Section 2.2 to obtain sufficient conditions able to ensure (3.8) (see [15], [17])
- In general, we can use other relaxation theories (see [18] and references therein).

If, for instance, the Half-sum convexification procedure is employed, the following lemma can be proved

Lemma 3.2. *Under control laws of the form (2.1) and by exploiting the Half-sum convexification procedure, the matrix inequality (3.8) can be relaxed to the following set of LMI conditions*

$$\begin{array}{l}\begin{bmatrix} Q & * \\ \left[\frac{A_iQ+A_jQ+B_iY_j+B_jY_i}{2}\right] & Q \end{bmatrix} > 0, i = 1, ..., l, j = i, ..., l \\ Q > 0\end{array} \tag{3.10}$$

which depend linearly on the unknowns $Y_i, i = 1, ..., l$ and Q.

Proof. The latter is obtained by relaxing (3.8) by means of the Half-sum convexification. □

3.1.2 Parameter Varying Lyapunov Functions

Parameter Varying Lyapunov functions have been introduced in the last decade (see [19]) and have been successfully used to deal with LPV systems. A Parameter Varying Lyapunov function is the following quadratic form

$$V(\xi(k)) = x^T(k) P(p(k)) x(k) = x^T(k) \left(\sum_{i=1}^{l} p_i(k) P_i \right) x(k) \tag{3.11}$$

where $P_i \in \Re^{n\times n}, \ i = 1, ..., l$ are positive definite matrices

$$P_i = P_i^T > 0, \ \ i = 1, ..., l$$

By using (3.11) as a Lyapunov candidate, the negativity condition on the difference $V(\xi(k+1)) - V(\xi(k))$ can be equivalently rewritten as

$$x^T(k)\Big([A(p(k))+B(p(k))F(p(k))]^T P(p(k+1))[A(p(k))+B(p(k))F(p(k))]-P(k)\Big)x(k)<0 \tag{3.12}$$

A sufficient condition ensuring the fulfillment of the above inequality is given by

$$[A(p(k))+B(p(k))F(p(k))]^T P(p(k+1))[A(p(k))+B(p(k))F(p(k))]-P(p(k))<0 \tag{3.13}$$

Note that the above condition needs to be satisfied $\forall p(k+1) \in \Sigma_l$. When standard LPV systems are considered, i.e. without any hypothesis on their parameters variation, (3.13) is equivalent to the following set of sufficient conditions

$$[A(p(k))+B(p(k))F(p(k))]^T P_i [A(p(k))+B(p(k))F(p(k))]-P(p(k))<0 \qquad i=1,\ldots,l \tag{3.14}$$

Next, via Schur complements, we obtain that (3.14) is implied by

$$\begin{bmatrix} P(p(k)) & * \\ [A(p(k))+B(p(k))F(p(k))] & P_i^{-1} \end{bmatrix} > 0 \qquad i=1,\ldots,l. \tag{3.15}$$

Now, let us introduce a square and invertible matrix $G \in \Re^{n\times n}$ as a new variable and by means of matrix inequalities congruence arguments we have

$$\begin{bmatrix} G & 0 \\ 0 & I_{n\times n} \end{bmatrix}^T \begin{bmatrix} P(p(k)) & * \\ [A(p(k))+B(p(k))F(p(k))] & P_i^{-1} \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & I_{n\times n} \end{bmatrix} > 0 \; i=1,\ldots,l \tag{3.16}$$

and we obtain

$$\begin{bmatrix} G^T P(p(k)) G & * \\ [A(p(k))+B(p(k))F(p(k))]G & P_i^{-1} \end{bmatrix} > 0 \qquad i=1,...,l \tag{3.17}$$

If new variables $Q_i = P_i^{-1}$ $i=1,...,l$, and $Y(p) = F(p)G$, are introduced, we can the find a stabilizing control law along with the associated Lyapunov function by solving, if possible, the following problem

Problem 3.3. *Find $Y(p), G, Q$ such that,*

$$\begin{array}{l} \begin{bmatrix} G^T P(p) G & * \\ [A(p)G+B(p)Y(p)] & Q_i \end{bmatrix} > 0, \forall p \in \Sigma_l,\; i=1,...,l \\ Q_i > 0 \qquad i=1,...,l \end{array} \tag{3.18}$$

where $P(p) = \left(\sum_{i=1}^{l} p_i Q_i^{-1}\right)$

Also in this case the above problem is not convex in general because of the possible existing nonlinearities in the terms $B(p)Y(p)$ and $G^T P(p) G$. The nonlinearity due to $B(p)Y(p)$ can be linearized by means of relaxation procedures shown in Appendix A once a particular form for the parameter dependent matrix $F(p)$ is chosen. Conversely, the nonlinearities concerning $G^T P(p) G$ are usually taken into account and relaxed by considering the following well known dilation technicality presented by [20]

Lemma 3.4. *Let $G \in \Re^{n\times n}$ a square and invertible matrix and $S = S' \in \Re^{n\times n}$ a symmetric matrix, then*

$$G^T S G \geq G + G^T - S^{-1}$$

As an example, we report in the following Lemma a convex design method in the form of a LMI problem which results from the joint application of the Dilation Lemma, the control law (2.1) and the Half-sum convexification procedure

Lemma 3.5. *Under the family of control laws (2.1) and by exploiting the Half-sum convexification procedure, the matrix inequality (3.18) can be relaxed to the following LMI conditions*

$$\begin{array}{l} \begin{bmatrix} G^T + G - \frac{Q_i+Q_j}{2} & * \\ \left[\frac{A_iG+A_jG+B_jY_i+B_iY_j}{2}\right] & Q_s \end{bmatrix} > 0, \begin{array}{r} i=1,...,l, \\ j=i,...,l, \\ s=1,...,l \end{array} \\ Q_i > 0 \qquad i=1,...,l \end{array} \tag{3.19}$$

where $P(p) = \left(\sum_{i=1}^{l} p_i Q_i^{-1}\right)$

Remark 3.6. Using a Parameter Varying Lyapunov Function yields in general less conservative stabilizability conditions with respect to the classical quadratic conditions. This advantage is typically paid in terms of the LMI problem size. A simple comparison between the conditions (3.8) and (3.18) shows an increased number of problem variables and LMI constraints to be tested.

3.1.3 Nonstandard Lyapunov Functions

In general any kind of Lyapunov candidate function can be introduced and used to characterize then stabilizability properties of LPV systems. However, not all of them yield to convex conditions. We will report here two nonstandard classes of Lyapunov functions introduced in [14] which give rise to control design methods based on convenient convex optimization problems.

The first class is the following positive definite quadratic form

$$V\left(\xi\left(k\right)\right)=x^{T}\left(k\right)\left(\sum_{i=1}^{l}p_{i}\left(k\right)Q_{i}\right)^{-1}x\left(k\right) \tag{3.20}$$

where $Q_i \in \Re^{n\times n},\ \ i=1,...,l$ are positive definite matrices

$$Q_i = Q_i^T > 0,\ \ i=1,...,l$$

By using this Lyapunov candidate, a sufficient condition for $V\left(\xi\left(k+1\right)\right)-V\left(\xi\left(k\right)\right)<0$ is given by

$$\begin{aligned}&\left[A\left(p\left(k\right)\right)+B\left(p\left(k\right)\right)F\left(p\left(k\right)\right)\right]^{T}\left(Q\left(p\left(k+1\right)\right)\right)^{-1}\left[A\left(p\left(k\right)\right)+B\left(p\left(k\right)\right)F\left(p\left(k\right)\right)\right]-\\&-\left(Q\left(p\left(k\right)\right)\right)^{-1}<0\end{aligned} \tag{3.21}$$

Then by Schur complements

$$\begin{bmatrix}\left(Q\left(p\left(k\right)\right)\right)^{-1} & *\\ \left[A\left(p\left(k\right)\right)+B\left(p\left(k\right)\right)F\left(p\left(k\right)\right)\right] & Q\left(p\left(k+1\right)\right)\end{bmatrix}>0 \tag{3.22}$$

and congruence transformation

$$\begin{bmatrix}Q\left(p\left(k\right)\right) & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\left(Q\left(p\left(k\right)\right)\right)^{-1} & *\\ \left[A\left(p\left(k\right)\right)+B\left(p\left(k\right)\right)F\left(p\left(k\right)\right)\right] & Q\left(p\left(k+1\right)\right)\end{bmatrix}\begin{bmatrix}Q\left(p\left(k\right)\right) & 0\\ 0 & I\end{bmatrix}>0 \tag{3.23}$$

can be converted into

$$\begin{bmatrix}Q\left(p\left(k\right)\right) & *\\ \left[A\left(p\left(k\right)\right)+B\left(p\left(k\right)\right)F\left(p\left(k\right)\right)Q\left(p\left(k\right)\right)\right] & Q\left(p\left(k+1\right)\right)\end{bmatrix}>0 \tag{3.24}$$

Note that if no bounds on the parameter variations are assumed, the above inequality needs to be checked $\forall p(k)\in\Sigma_l$, $\forall p(k+1)\in\Sigma_l$. Because $p(k+1)$ is linearly combined with $Q_i, i=1,...,l$, by convexity, we can equivalently test (3.24) only over the simplex vertices. The above observation allows to formulate the stabilizing control law design problem as

Problem 3.7. *Find* $Q_i, i=1,...,l, F\left(p\right)$ *such that,*

$$\begin{aligned}&\begin{bmatrix}Q\left(p\right) & *\\ \left[A(p)+B\left(p\right)F\left(p\right)\right]Q(p) & Q_i\end{bmatrix}>0, \forall p\in\Sigma_l,\ \ i=1,...,l\\&Q_i>0\ \ \ i=1,...,l\end{aligned} \tag{3.25}$$

In this case non-convexity may rise because of the term $A(p)Q(p)+B(p)F(p)Q(p)$ and can be relaxed in several ways. As shown in [14], this kind of approach reveals to be efficient when using the following class of control laws (2.3):

$$u\left(\xi\left(k\right)\right)=\left(\sum_{i=1}^{l}p_{i}\left(k\right)F_{i}\right)\left(\sum_{j=1}^{l}p_{i}\left(k\right)Q_{i}\right)^{-1}x\left(k\right)=\left(\tilde{F}(p(k))\right)\left(Q\left(p\left(k\right)\right)\right)^{-1}x\left(k\right) \tag{3.26}$$

In fact, (3.25) simplifies into

$$\begin{array}{l} \begin{bmatrix} Q(p) & * \\ \left[A(p)\,Q(p) + B(p)\,\tilde{F}(p)\right] & Q_i \end{bmatrix} > 0, \forall p \in \Sigma_l,\ i = 1,\ldots,l \\ Q_i > 0 \quad i = 1,\ldots,l \end{array} \tag{3.27}$$

by using (3.26) and the dependence on the parameter p becomes quadratic.

The above considerations allow to formulate relaxed stabilizability conditions which can be used to achieve less conservative control design methods w.r.t. those achievable by Lemma 3.5.

Lemma 3.8. *Under control laws of the form (3.26), the matrix inequality (3.25) can be relaxed by exploiting the Half-sum convexification procedure, to the following LMI conditions*

$$\begin{array}{l} \begin{bmatrix} \frac{Q_i+Q_j}{2} & * \\ \frac{A_iQ_j+A_jQ_i+B_iF_j+B_jF_i}{2} & Q_s \end{bmatrix} > 0, \begin{array}{r} i = 1,...,l, \\ j = i,...,l. \\ s = 1,...,l. \end{array} \\ Q_i > 0 \quad i = 1,...,l \end{array} \tag{3.28}$$

Proof. The result can be proved by applying the standard Half-sum convexification procedure to (3.25). □

A slightly more sophisticated version of the above Lyapunov candidate, also presented in [14], is given by

$$\begin{array}{l} V(\xi(k)) = x^T(k)\left(\sum_{i=1}^{l} p_i(k)\,G_i\right)^{-T}\left(\sum_{i=1}^{l} p_i(k)\,P_i\right)\left(\sum_{i=1}^{l} p_i(k)\,G_i\right)^{-1} x(k) = \\ x^T(k)\left[(G(p(k)))^{-T}(P(p(k)))(G(p(k)))^{-1}\right] x^T(k) \end{array} \tag{3.29}$$

where

$$\begin{array}{c} G_i \in \Re^n, P_i \in \Re^n, P_i = P_i^T > 0,\ i = 1,\ldots,l \\ rank\{G_i\} = n,\ i = \ldots,l \end{array} \tag{3.30}$$

Similarly to the previous case, a sufficient condition for $V(\xi(k+1)) - V(\xi(k)) < 0$ and (3.30) to hold true is that

$$\begin{array}{l} \begin{bmatrix} (G(p(k)))^{-T}P(p(k))(G(p(k)))^{-1} & * \\ \left[A(p(k)) + B(p(k))F(p(k))\right] & G(p(k+1))^T(P(p(k+1)))^{-1}G(p(k+1)) \end{bmatrix} > 0 \\ P_i > 0 \quad i = 1,\ldots,l \\ rank\{G_i\} = n \quad i = 1,\ldots,l \end{array} \tag{3.31}$$

hold true as well. By exploiting a congruence argument with the factor $diag\{G(p(k)), I\}$, we obtain

$$\begin{bmatrix} P\left(p\left(k\right)\right) & * \\ \left[A(p\left(k\right))+B\left(p\left(k\right)\right)F\left(p(k)\right)\right]G\left(p(k)\right) & G^T\left(p(k+1)\right)\left(P\left(p(k+1)\right)\right)G(p(k+1)) \end{bmatrix} > 0 \quad (3.32)$$
$$P_i > 0 \quad i = 1, ..., l$$

The possible nonlinearity in $G\left(p\left(k+1\right)\right)^T \left(P\left(p\left(k+1\right)\right)\right)^{-1} G\left(p\left(k+1\right)\right)$ can be relaxed by means of the Dilation Lemma 3.4. Then, a sufficient condition for (3.32) to hold true is given by

$$\begin{bmatrix} P\left(p\left(k\right)\right) & * \\ \left[A(p\left(k\right))+B\left(p\left(k\right)\right)F\left(p(k)\right)\right]G\left(p(k)\right) & G\left(p\left(k+1\right)\right)^T + G(p(k+1)) - \left(P(p(k+1))\right) \end{bmatrix} > 0$$
$$P_i > 0 \quad i = 1, ..., l \quad (3.33)$$

Note that using the Dilation Lemma 3.4 the rank constraint also disappears since $G^T + G - P > 0, P > 0$ implies that $rank\left\{G\right\} = n$. By following the same previous case technicalities, a corresponding control design method is given by

Problem 3.9. *Find $Q_i, i = 1, ..., l, F\left(p\right)$ such that,*

$$\begin{bmatrix} P\left(p\right) & * \\ \left[A(p) + B\left(p\right)F\left(p\right)\right]G\left(p\right) & G_i^T + G_i - P_i \end{bmatrix} > 0, \quad \begin{matrix} \forall p \in \Sigma_l, \\ i = 1, \dots, l \end{matrix} \quad (3.34)$$
$$P_i > 0 \quad i = 1, ..., l$$

As previously seen, the non-convexity here could appear in the term $A(p)G(p) + B(p)F(p)G(p)$ and its convexification can be obtained, as shown in [14], by adopting the following class of control laws (2.3) as a first step:

$$u\left(\xi\left(k\right)\right) = \left(\sum_{i=1}^{l} p_i\left(k\right)F_i\right)\left(\sum_{j=1}^{l} p_j G_i\right)^{-1} x\left(k\right) = \left(\tilde{F}(p(k))\right)\left(G\left(p\left(k\right)\right)\right)^{-1} x\left(k\right) \quad (3.35)$$

Thanks to (3.35), inequality (3.34) simplify into

$$\begin{bmatrix} P\left(p\right) & * \\ \left[A(p)G(p) + B(p)\tilde{F}(p)\right] & G_i^T + G_i - P_i \end{bmatrix} > 0, \quad \begin{matrix} \forall p \in \Sigma_l \\ i = 1, \dots, l \end{matrix} \quad (3.36)$$
$$P_i > 0 \quad i = 1, \dots, l$$

The latter, coupled with some of the convexification methods described in Appendix A, allows to obtain LMI conditions which can be exploited to compute a stabilizing control law. As an example, by using the Half-sum convexification procedure, we obtain

Lemma 3.10. *Under control laws of the form (3.35) and by exploiting the Half-sum convexification procedure, (3.34) can be relaxed to the following LMI conditions*

$$\begin{bmatrix} \frac{P_i + P_j}{2} & * \\ \frac{A_i G_j + A_j G_i + B_i F_j + B_j F_i}{2} & G_s^T + G_s - P_s \end{bmatrix} > 0, \quad \begin{matrix} i = 1, ..., l, \\ j = i, ..., l, \\ s = 1, ..., l \end{matrix} \quad (3.37)$$
$$P_i > 0 \quad i = 1, \dots, l$$

3.2 Stabilizability Results for LPV Systems subject to Bounded Parameter Variations

In this Section we propose stabilizability conditions for LPV systems subject to bounded parameter variations. While such results are known in the continuous-time case [21], there is a lack in the literature for the discrete-time case.

The Section is organized as follows. In the first subsection, we deal with the stabilization of slowly varying systems by means of non-scheduled control laws. Though obviously leading to more conservative results, their derivation are of help to understand the approach which will be used to determine similar conditions for scheduled control laws. Moreover, because no parameter knowledge is used for feedback, such results can be directly applied to the uncertain frameworks, where results on slowly varying systems are rare. In the second subsection, stabilizability results for scheduled control laws will be given, by exploiting some of the technicalities seen in the previous sections.

3.2.1 Non-scheduled Control Laws

Let us consider system (1.1) with bounded parameter variations and assume that we want to stabilize its behavior by means of a linear state feedback control law

$$u\left(k\right)=Fx\left(k\right) \tag{3.38}$$

where $F\in\Re^{n\times m}$. We will exploit the following Lyapunov candidate function (3.20):

$$V\left(\xi\left(k\right)\right)=x^{T}\left(k\right)\left(\sum_{i=1}^{l}p_{i}\left(k\right)Q_{i}\right)^{-1}x\left(k\right)=x^{T}\left(k\right)\left(Q\left(p\left(k\right)\right)\right)^{-1}x\left(k\right) \tag{3.39}$$

where $Q_i\in\Re^{n\times n},\;\; i=1,\ldots,l$ are positive definite matrices

$$Q_{i}=Q_{i}^{T}>0,\;\; i=1,\ldots,l \tag{3.40}$$

By using (3.39), the Lyapunov difference

$$V\left(\xi\left(k+1\right)\right)-V\left(\xi\left(k\right)\right)<0. \tag{3.41}$$

can be used to check the stability by ensuring that

$$[A(p(k))+B(p(k))F]^{T}(Q(p(k+1)))^{-1}[A\left(p(k)\right)+B\left(p(k)\right)F]-\left(Q(p(k))\right)^{-1}<0 \tag{3.42}$$

which, by Schur complements, can be rewritten as

$$\begin{bmatrix}\left(Q\left(p\left(k\right)\right)\right)^{-1} & *\\ \left[A\left(p\left(k\right)\right)+B\left(p\left(k\right)\right)F\right] & Q\left(p\left(k+1\right)\right)\end{bmatrix}>0 \tag{3.43}$$

and, by operating a congruence transformation operation over the previous inequality

$$\begin{bmatrix} G^T & * \\ 0 & I \end{bmatrix} \begin{bmatrix} (Q(p(k)))^{-1} & * \\ [A(p(k)) + B(p(k))F(p(k))] & Q(p(k+1)) \end{bmatrix} \begin{bmatrix} G & * \\ 0 & I \end{bmatrix} > 0 \quad (3.44)$$

At last, we obtain

$$\begin{array}{l} \begin{bmatrix} G^T Q^{-1}(p(k))G & * \\ [A(p(k))G + B(p(k))Y] & Q(p(t+1)) \end{bmatrix} > 0, \begin{array}{l} \forall p(k+1) \in \Psi(p(k)) \\ \forall p(k) \in \Sigma_l \end{array}, \\ Q_i > 0 \quad i = 1, \dots, l \end{array} \quad (3.45)$$

where $Y = FG$. By means of the Dilation Lemma, a sufficient condition for (3.45) to be true is

$$\begin{array}{l} \begin{bmatrix} G^T + G - Q(p(k)) & * \\ [A(p(k))G + B(p(k))Y] & Q(p(t+1)) \end{bmatrix} > 0, \begin{array}{l} \forall p(k+1) \in \Psi(p(k)) \\ \forall p(k) \in \Sigma_l \end{array} \\ Q_i > 0 \quad i = 1, \dots, l \end{array} \quad (3.46)$$

Let us focus now on $Q(p(k+1))$, since we can write $p(k+1) = p(k) + \Delta p(k)$, then

$$\begin{array}{l} Q(p(k+1)) = \left(\sum_{j=1}^{l} p_i(k+1) Q_i \right) = Q(p(k+1)) = \\ = \left(\sum_{j=1}^{l} [p_i(k) + \Delta p_i(k)] Q_i \right) = \left(\sum_{j=1}^{l} [p_i(k) Q_i] + \sum_{j=1}^{l} \Delta p_i(k) Q_i \right) = \\ = Q(p(k)) + Q(\Delta p(k)) \end{array} \quad (3.47)$$

Then, the inequality (3.46) becomes

$$\begin{array}{l} \begin{bmatrix} G^T + G - Q(p(k)) & * \\ [A(p(k))G + B(p(k))Y] & Q(p(k)) + Q(\Delta p(k)) \end{bmatrix} > 0, \begin{array}{l} \forall \Delta p(k) \in \Delta\Upsilon(p(k)) \\ \forall p(k) \in \Sigma_l \end{array}, \\ Q_i > 0 \quad i = 1, \dots, l \end{array} \quad (3.48)$$

where $\Delta\Upsilon(p(k))$ is a polytopic set containing all possible parameters variations

$$\Delta\Upsilon(p(k)) = \left\{ \Delta p(k) \in \Re^l \,\middle|\, \begin{pmatrix} 1^T \Delta p(k) = 0 \\ -\Delta p_i \le \Delta p_i(k) \le \Delta p_i \\ 0 \le \Delta p_i(k) + p(k) \le 1 \end{pmatrix} \right\} \quad (3.49)$$

The framework is here complicated because of the explicit dependence of this set on $p(k)$. A possible way do deal with it is to relax this dependence by defining a set $\Delta\Upsilon^{out} \supseteq \Delta\Upsilon(p(k))$, $\forall p(k) \in \Sigma_l$. This set could be defined as

$$\Delta\Upsilon^{out} = \left\{ \Delta p(k) \in \Re^l \,\middle|\, \begin{pmatrix} 1^T \Delta p(k) = 0 \\ -\Delta p_i \le \Delta p_i(k) \le \Delta p_i \end{pmatrix} \right\} \quad (3.50)$$

Such a relaxation allows to write the following sufficient conditions to satisfy the inequality (3.48)

$$\begin{array}{l}\begin{bmatrix} G+G^T-Q\left(p\left(k\right)\right) & * \\ \left[A\left(p\left(k\right)\right)G+B\left(p\left(k\right)\right)Y\right] & Q\left(p\left(k\right)\right)+Q\left(\Delta p\left(k\right)\right)\end{bmatrix} > 0, \begin{array}{r}\forall p\left(k\right)\in\Sigma_l\\ \forall\Delta p\left(k\right)\in\Delta\Upsilon^{out}\end{array}\\ Q_i>0 \quad i=1,\ldots,l\end{array} \tag{3.51}$$

which are equivalent to the fulfilment of the following LMIs:

$$\begin{array}{l}\begin{bmatrix} G+G^T-Q\left(p\left(k\right)\right) & * \\ \left[A\left(p(k)\right)G+B(p(k))Y\right] & Q\left(p\left(k\right)\right)+Q\left(\Delta p\left(k\right)\right)\end{bmatrix} > 0, \begin{array}{r}\forall p\left(k\right)\in vert\left\{\Sigma_l\right\}\\ \forall\Delta p(k)\in vert\{\Delta\Upsilon^{out}\}\end{array}\\ Q_i>0 \quad i=1,\ldots,l\end{array} \tag{3.52}$$

Finally, a possible way to find a stabilizing control law is given by the following design procedure

Problem 3.11. *Find $Q_i, i=1,\ldots,l, F$ such that ,*

$$\begin{array}{l}\begin{bmatrix} G+G^T-Q_i & * \\ \left[A_iG+B_iY\right] & Q_i+Q\left(\Delta p\right)\end{bmatrix} > 0, \begin{array}{c} i=1,...,l\\ \forall\Delta\in vert\{\Delta\Upsilon^{out}\}\end{array}\\ Q_i>0 \quad i=1,\ldots,l\end{array} \tag{3.53}$$

where $Y=FQ$

Remark 3.12. It is important to understand the implications behind the use of $\Delta\Upsilon^{out}\supseteq\Delta\Upsilon\left(p\left(k\right)\right), \forall p\left(k\right)\in\Sigma_l$. In fact, even if this choice turns out to be satisfactory for small increments $\Delta \mathrm{p}_i$, some conservativeness is nonetheless introduced.

Let us focus on the term

$$Q(p(k))+Q(\Delta(p(k))=Q(p(k)+\Delta p(k))$$

and consider the simple case $l=2$, $\Delta p_1=\Delta p_2=\Delta$. Because such a term is evaluated

$$\forall p(k)\in vert\{\Sigma_l\},\ \forall\Delta p(k)\in\Delta\Upsilon^{out}$$

then the vector $p(k+1)=p(k)+\Delta p(k)$ is defined as the convex combination of the following 4 vertices of the parameter set

$$p_{11}=\begin{bmatrix}1+\Delta\\ -\Delta\end{bmatrix}, p_{12}=\begin{bmatrix}1-\Delta\\ \Delta\end{bmatrix}, p_{21}=\begin{bmatrix}\Delta\\ 1-\Delta\end{bmatrix}, p_{22}=\begin{bmatrix}-\Delta\\ 1+\Delta\end{bmatrix}$$

where p_{11}, p_{22} clearly do not belong to the unitary simplex Σ_2. A geometrical intuition of the above phenomena is given in Figure 3.1 where it is seen that, the variation set $\Delta\Psi^{out}$ when applied on the simplex extremal points, is not contained in the unitary simplex.

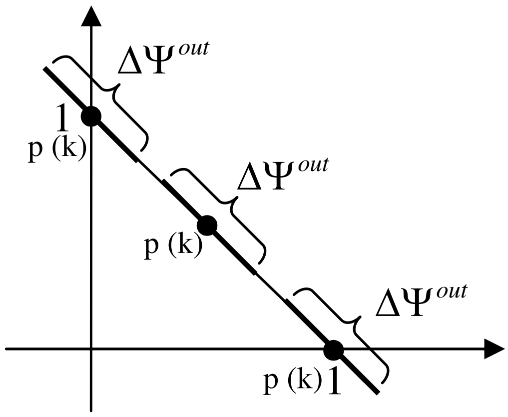

Fig. 3.1. Geometrical interpretation of $\Delta\Upsilon^{out}$)

Some ideas on how to overcome the above problem are under investigation. However, a hint for the $l = 2$ case is given in next Theorem 3.13. The main idea is to find a set of vertices for all the possible occurrences of the extended parameter vector $[p^T(k), p^T(k+1)]^T$. The first step is to add 2 further degrees of freedom to the definition of $p(k)$, by considering Σ_2 as the combination of the following 4 vertices (instead of the sole two extreme points)

$$\Sigma_2 = conv\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1-\Delta\\\Delta\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}\Delta\\1-\Delta\end{bmatrix}\right\} \tag{3.54}$$

Then, by applying all possible $\Delta\Upsilon(p(k))$ variations to the above four points, we obtain 8 vertices for $[p^T(k), p^T(k+1)]$. By considering the above discussion, we can prove the following result:

Theorem 3.13. *Consider the case $l = 2$. The set of all possible vectors $[p(k), p(k+1)]$ for $\forall p(k) \in \Sigma$, $\forall p(k+1) \in \Upsilon(p(k))$ can be expressed as the convex combination of 8 vertices, namely*

$$\begin{aligned}&\left\{\begin{bmatrix}p(k)\\p(k+1)\end{bmatrix} \middle| \;\forall p(k) \in \Sigma,\; \forall p(k+1) \in \Upsilon(p(k))\right\} = \\ &= conv\left\{\begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\\begin{bmatrix}1\\0\end{bmatrix}\end{bmatrix}, \begin{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\\begin{bmatrix}1-\Delta\\\Delta\end{bmatrix}\end{bmatrix}, \begin{bmatrix}\begin{bmatrix}1-\Delta\\\Delta\end{bmatrix}\\\begin{bmatrix}1\\0\end{bmatrix}\end{bmatrix}, \begin{bmatrix}\begin{bmatrix}1-\Delta\\\Delta\end{bmatrix}\\\begin{bmatrix}1-2\Delta\\2\Delta\end{bmatrix}\end{bmatrix},\right.\\ &\left.\begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\\\begin{bmatrix}\Delta\\1-\Delta\end{bmatrix}\end{bmatrix}, \begin{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\\\begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix}, \begin{bmatrix}\begin{bmatrix}\Delta\\1-\Delta\end{bmatrix}\\\begin{bmatrix}2\Delta\\1-2\Delta\end{bmatrix}\end{bmatrix}, \begin{bmatrix}\begin{bmatrix}\Delta\\1-\Delta\end{bmatrix}\\\begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix}\right\}\end{aligned} \tag{3.55}$$

Proof. See Appendix B.

The latter implies that this new formulation is equivalent to the original one but, if we use a convex combination of the above eight vertices to consider any occurrence $\forall p(k) \in \Sigma_l, \forall p(k+1) \in \Delta\Upsilon(p(k))$ into (3.46), the resulting stability conditions don't suffer of the problem depicted in Fig. 3.1 because all vertices are now all contained within the unitary simplex.

3.2.2 Scheduled Control Laws

Here we will see how it is possible to design parameter scheduled control laws able to stabilize LPV systems subject to bounded parameter variations by means of the nonstandard Lyapunov functions introduced in Subsection 3.1.3. Let us consider for instance (3.20) as a Lyapunov candidate function.

To this end, we should proceed by following the same mathematical lines presented in Section 3.1.3. In order to avoid repetitions, we will focus on equation (3.24) as a suitable starting point. Such an equation allows us to recast the stabilization problem as follows

Problem 3.14. *Find $Q_i, i = 1, \ldots, l, F(p(k))$ such that*

$$\begin{bmatrix} Q(p(k)) & * \\ [A(p(k)) + B(p(k))F(p(k))]Q(p(k)) & Q(p(k+1)) \end{bmatrix} > 0, \quad \begin{array}{l} \forall p(k+1) \in \Upsilon(p(k)) \\ \forall p(k) \in \Sigma_l \end{array} \tag{3.56}$$
$$Q_i > 0 \quad i = 1, ..., l$$

Then, similarly as done in the previous subsection

$$Q(p(k+1)) = Q(p(k)) + Q(\Delta(p(k))$$

and, in turn, (3.56) becomes

$$\begin{bmatrix} Q(p(k)) & * \\ [A(p(k)) + B(p(k))F(p(k))]Q(p(k)) & Q(p(t)) + Q(\Delta pk) \end{bmatrix} > 0, \quad \begin{array}{l} \forall \Delta p(k) \in \Delta\Upsilon(p(k)) \\ \forall p(k) \in \Sigma_l \end{array} \tag{3.57}$$
$$Q_i > 0 \quad i = 1, \ldots, l$$

Next, a sufficient condition for (3.57) to hold true is given by

$$\begin{bmatrix} Q(p(k)) & * \\ [A(p(k))Q(p(k)) + B(p(k))F(p(k))Q(p(k))] & Q(p(k)) + Q(\Delta p(k)) \end{bmatrix} > 0, \quad \begin{array}{l} \forall p(k) \in \Sigma_l \\ \Delta p(k) \in \Delta\Upsilon^{out} \end{array}$$
$$Q_i > 0 \quad i = 1, \ldots, l \tag{3.58}$$

which is equivalent to the following inequality

$$\begin{bmatrix} Q\left(p\right) & * \\ \left[A(p)Q(p)+B(p)F(p)Q(p)\right] & Q(p)+Q\left(\Delta p\right) \end{bmatrix} > 0, \quad \begin{array}{l} \forall p \in \Sigma_l \\ \forall \Delta p \in vert\left\{\Delta\Upsilon^{out}\right\} \end{array} \tag{3.59}$$
$$Q_i > 0 \quad i = 1, \ldots, l$$

Once a suitable structure of the control law is chosen and the term $A(p)Q(p)+B(p)F(p)Q(p)$ is relaxed, the latter yield to a convex formulation. In particular, if a control law of the form (3.26) is chosen and the Half-sum convexification procedure is used, the following result is achieved.

Lemma 3.15. *Under control laws of the form (3.26) and by exploiting the Half-sum convexification procedure, the inequality (3.59) can be relaxed to the following LMI conditions*

$$\begin{bmatrix} \frac{Q_i+Q_j}{2} & * \\ \frac{A_iQ+A_jQ+B_iF_j+B_jF_i}{2} & \frac{Q_i+Q_j}{2}+Q(\Delta p) \end{bmatrix} > 0, \quad \begin{array}{c} i=1,...,l, \\ j=i,...,l, \\ \forall \Delta p \in vert\left\{\Delta\Upsilon^{out}\right\}, \end{array} \tag{3.60}$$
$$Q_i > 0 \quad i = 1, ..., l.$$

Proof. By substituting (3.26) in (3.24) and applying Half-sum convexification techniques. □

The above described procedure can also be applied when more complex Lyapunov functions are used (3.29). In fact, consider again (3.33)

$$\begin{bmatrix} P\left(p\left(k\right)\right) & * \\ \left[A(p(k))+B(p(k))F(p(k))\right]G(p(k)) & G(p(k+1))^T+G(p(k+1))-(P(p(k+1))) \end{bmatrix} > 0$$
$$P_i > 0 \quad i = 1, ..., l \tag{3.61}$$

By exploiting the linearity we have

$$\begin{aligned} P(p(k+1)) &= P(p(k)+\Delta p(k)) = P(p(k))+P(\Delta p(k)) \\ G(p(k+1)) &= G(p(k)+\Delta p(k)) = G(p(k))+G(\Delta p(k)) \end{aligned}$$

Then, the problem of finding a parameter dependent stabilizing state feedback matrix $F(p)$ can be formalized as

Problem 3.16. *Find* $P_i, G_i, i = 1, \ldots, l, F\left(p\left(k\right)\right)$ *such that,*

$$\begin{bmatrix} P\left(p\right) & * \\ \left[A\left(p\right)+B\left(p\right)F\left(p\right)\right]G\left(p\right) & \Phi_{GP}\left(p\left(k\right)\right)+\Phi_{GP}\left(\Delta p\right) \end{bmatrix} > 0, \quad \begin{array}{l} \forall p \in \Sigma_l \\ \forall \Delta p \in vert\left\{\Delta\Upsilon^{out}\right\}, \end{array} \tag{3.62}$$
$$Q_i > 0 \quad i = 1, \ldots, l.$$

where $\Phi_{GP}\left(p\right) = G\left(p\right)+G^T\left(p\right)-P\left(p\right)$.

Again, by choosing an appropriate class of control laws and relaxing all unknowns nonlinearity couplings several alternative but equivalent sufficient convex conditions can be found that can be used for control design purposes. For instance, if the class of control laws (3.35) is used:

Lemma 3.17. *Under control laws of the form (3.35) and by exploiting the Half-sum convexification procedure, the inequality (3.62) can be relaxed to the following LMI conditions*

$$\begin{bmatrix} \frac{P_i+P_j}{2} & * \\ \frac{A_iG_j+A_jG_i+B_iF_j+B_jF_I}{2} & \frac{G_i+G_j+G_i^T+G_j^T-P_i-P_j}{2} + \Phi_{GP}\left(\Delta p\right) \end{bmatrix} > 0, \quad \begin{array}{l} i=1,...,l, \\ j=i,...,l, \\ \forall \Delta p \in vert\left\{\Delta \Upsilon^{out}\right\}, \end{array} \tag{3.63}$$

$$Q_i > 0 \quad i = 1, \ldots, l.$$

where $\Phi_{GP}\left(p\right) = G\left(p\right) + G^T\left(p\right) - P\left(p\right)$

Example 3.18. The aim of this Example is to compare both stabilizing capability and computational cost for the stabilizability results seen in this Chapter. Consider the following LPV system

$$x(k+1) = \sum_{i=1}^{2} p_i(k)A_ix(k) + \sum_{i=1}^{2} p_j(k)B_ju(k)$$

where

$$A_1 = \begin{pmatrix} 1 & -\beta \\ -1 & -0.5 \end{pmatrix}, B_1 = \begin{pmatrix} 5+\beta \\ 2\beta \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 1 & \beta \\ -1 & -0.5 \end{pmatrix}, B_2 = \begin{pmatrix} 5-\beta \\ -2\beta \end{pmatrix},$$

Such a benchmark system has been introduced in [14] to evaluate the performance of LPV stabilizability results. The system is built in order to become "harder" to be stabilized for growing β. The next Table summarizes the stability regions (w.r.t. β) for each of the presented stabilizability algorithm based on the semi-sum convexification procedures. Results can be refined further on by means of better convexification procedures. Because a raw comparison of the time needed to compute the control laws would be misleading (it strongly depends on the specific plant under consideration), here the number of LMI lines and scalar variables involved in the definition of each stabilization algorithm is reported as a more significant index of the computational cost.

Method	β	LMI Lines	Variables
Lemma 3.2	[-1.68,1.68]	$n+2nl_c$	n^2+lnm
Lemma 3.5	[-1.76,1.76]	$n+2nll_c$	$(l+1)n^2+lnm$
Lemma 3.8	[-1.87,1.87]	$n+2nll_c$	n^2+lnm
Lemma 3.10	[-2.04,2.04]	$n+2nll_c$	$2n^2+lnm$
Lemma 3.10, $\Delta\to 0$	[-2.15,2.15]	$n+2nll_c l_{\Delta\Upsilon}$	n^2+lnm
Lemma 3.17, $\Delta\to 0$	[-2.55,2.55]	$n+2nll_c l_{\Delta\Upsilon}$	$2n^2+lnm$
Non-scheduled (3.46), $\Delta=1$	[-1.67,1.67]	$n+2nl$	$mn+(l+1)n^2$
Non-scheduled (3.46), $\Delta\to 0$	[-1.80,1.80]	$n+2nll_{\Delta\Upsilon}$	$mn+(l+1)n^2$

Note that the use more complex stabilizability methods for general LPV systems introduces relevant performance improvements at the cost of a slight increased computational burdens. Finally let us remark the sensible improvements obtained for vanishing bounded parameter variation i.e. $\Delta \to 0$. In this case, anyway, the computational burdens grows more significatively because of the $l_{\Delta\Upsilon}$ terms, i.e. the number of vertices introduced by the one-step ahead parameter variation.

Chapter Summary

In the first part of this Chapter, an overview of the stabilizability methods existing in literature for LPV systems without any limitation on their parameter variations has been given. In the second part, new stabilizability results for LPV systems subject to bounded parameter variations have been proposed.

Part II

Model Predictive Control for LPV systems

4

Constrained Control - Definitions

In this Chapter some constrained control design problems for LPV systems are introduced together with the necessary mathematical tools.

The Chapter is organized as follows. In the first Section, constrained control problems for LPV systems are defined and the regulation problem to a desired set point is discussed and detailed. In the second Section, relevant set invariance notions are recalled and their use in constrained control design problems summarized. Finally, in the third Section, techniques for the determination of invariant sets for LPV systems are investigated.

4.1 Constrained Control – An Introduction

The problem of handling constraints in control systems is a fundamental issue to be taken into account in most real world problems. In fact, physical plants are inherently subject to constraints (input saturations, physical limitations and safety requirements) which need to be satisfied in order to ensure a correct dynamical behavior.

In the past, sub-optimal approaches have been used to deal with constraints satisfaction, mainly consisting in artificial limitations on system transients and steady-states. Even if in some cases successful (especially when dealing with non-critical plants and when there is no need to push up the system performance), this practice it is anyway a "risky business" in the sense that strong requirements are neglected and the plant cannot be able to recover from potentially avoidable dangerous situations.

Obviously such approaches fail when the control task requires large and fast system transients to achieve prescribed high performance. For those reasons in the last decade relevant efforts (see [22]-[23]-[24] for a survey) have been devoted to constrained control problems.

Typical limitations affecting a real plant can be bring back to input saturations and state constraints. Input saturations usually may arise from the fact that command signal input values always cannot go further prescribed limits that are related to the particular physical actuator taken into consideration. This phenomenon can be easily understood when thinking to standard actuators: voltage regulators have a maximum output voltage [25], engines can provide only a limited torque [26], valves in chemical plants have a maximum and a minimum displacements (full open and full closed) [27], surfaces and rudders in aeronautical and naval applications have only a limited angle of deflection [28], etc...

Limitations on state trajectories may depends on several causes, including physical saturation phenomena (e.g. the maximum grip between two surfaces [29], fluid phase transition curves [30], safety and plant integrity issues (e.g. maximum heat dissipation without components damaging [33], ranges of temperature/pressure that have to be guaranteed in chemical reactors to ensure a safe behavior [31], etc. . .) and comfort/human limitations reasons (e.g. maximum acceleration a pilot can stand without problems [34] or the range of roll and pitch a passenger can tolerate without feeling uncomfortable [35]).

In literature many aspects of the constrained control design problem have been addressed, including regulation (see for instance [36]) and tracking problems [37] for linear and nonlinear plants [38]-[39], robust formulations [40], and so on.

Here we focus on the regulation problem for LPV systems subject to symmetric convex input and state constraints. Namely, the problem we want to address is

Problem 4.1. (Constrained LPV) *Consider a polytopic LPV system*

$$x\left(k+1\right)=A\left(p\left(k\right)\right)x\left(k\right)+B\left(p\left(k\right)\right)u\left(k\right) \tag{4.1}$$

$$\left[\, A\left(p\left(k\right)\right)\; B\left(p\left(k\right)\right)\right]=\sum_{i=1}^{l}p_i\left(k\right)\left[\, A_i\; B_i\,\right] \tag{4.2}$$

where$\left[\, A_i\; B_i\,\right], i=1,...,l$ *are the the plant vertices. The parameter vector* $p\left(k\right)=\left[p_1\left(k\right)\,...\,p_l\left(k\right)\right]$ *belongs to the* l*-dimensional unit simplex* Σ_l *and it is possibly subject to bounded parameter variations i.e.*

$$p\left(k\right)\in\Upsilon\left(p\left(k-1\right)\right)\equiv\left\{p\in\Sigma_l\; s.t.\; \left|p_i-p_i\left(k-1\right)\right|<\Delta p_i\right\} \tag{4.3}$$

where $\Delta p_i>0$ *is the maximum one-step variation of the* i*-th entry of the parameter vector. The following input saturation/ state constraints are prescribed*

$$u\left(k\right)\in U,\quad k=0,1,2,\ldots \tag{4.4}$$
$$x\left(k\right)\in X,\quad k=1,2,.... \tag{4.5}$$

with

$$X=\left\{x\in\Re^n\right|\quad\left|\left(Cx\right)_i\right|\le y_{i,\max}\;\; i=1,...,p\} \tag{4.6}$$
$$U=\left\{u\in\Re^m\right|\quad\left|u_i\right|\le u_{i,\max}\;\; i=1,...,m\} \tag{4.7}$$

Given the information vector $\xi(0) = \left[x(0)^T \; p(0)^T \; 0 \right]^T$ *at time* $k = 0$*, compute an information vector based control strategy*

$$u(k) = u(\xi(k)) \tag{4.8}$$

such that constraints (4.4) and (4.5) are ensured, the closed loop system is asymptotically stable and a convenient performance index $J(x(k), u(\cdot))$ *is minimized.*

Remark 4.2. A performance index which will be used hereafter is the standard linear quadratic (LQ) cost

$$J(x(0), u(\cdot)) = \sum_{k=0}^{\infty} \|x(k)\|_{R_x}^2 + \|u(k)\|_{R_u}^2 \tag{4.9}$$

4.2 Set Invariance, Lyapunov Theory and Constrained Control

Since its introduction ([41]), the notion of invariant set has been exploited in many analysis and synthesis problems. In particular it has been shown to be a fundamental tool to deal with systems subject to state/input constraints (see [42] for a survey). For such a reason, this section is devoted to introduce the foundations of the set invariance control theory.

Definition 4.3. *Given an autonomous system*

$$x(k+1) = f(x(k), k), \quad x(k) \in \Re^n \tag{4.10}$$

the time-varying set $\mathcal{E}(k) \subset \Re^n$ *is said positively invariant if for all* $x(k) \in \mathcal{E}(k)$*, the solution* $x(k+\tau) \in \mathcal{E}(k+\tau), \tau > 0$

Definition 4.4. *Given the following system driven by an exogenous input*

$$x(k+1) = f(x(k), w(k), k), \quad x(k) \in \Re^n, \quad w(k) \in \Re^d \tag{4.11}$$

with $w(k) \in W$*, where* $W \subset \Re^d$ *denotes a closed and bounded set, the time-varying* $\mathcal{E}(k) \subset \Re^n$ *is said robustly positively invariant if for all* $x(k) \in \mathcal{E}(k)$*, and for all* $w(k+\tau) \in W$*,* $\tau \geq 0$ *the solution* $x(k+\tau) \in \mathcal{E}(k+\tau)$ *for* $\tau > 0$

The relevance of the above definitions for constrained control can be easily understood. Let us consider an autonomous dynamical system whose state variables are subject to state constraints

$$x(k) \in X, \quad k \geq 0 \tag{4.12}$$

Then (see [43]-[44]), constraint violations are avoided if and only if the state $x(0)$ at time $k = 0$ belongs to a positive invariant set $\mathcal{E}(k)$ such that

$\mathcal{E}(k) \subseteq X, k \geq 0$.

Given an initial state $x(0)$, we can then recast the constrained stabilization problem for

$$x(k+1) = f(x(k), k, u(k)) \tag{4.13}$$

subject to state constraints

$$x(k) \in X, \quad k \geq 0 \tag{4.14}$$

$$u(k) \in U, \quad k \geq 0 \tag{4.15}$$

as the problem of finding a stabilizing control strategy

$$u(k) = u(x(k), k) \tag{4.16}$$

such that a positive invariant set $\mathcal{E}$ for the resulting closed loop system $x(k+1) = f(x(k), k, u(x(k), k))$ exists and satisfies

$$x(0) \in \mathcal{E}(0) \tag{4.17}$$

$$\mathcal{E}(k) \subseteq X, \; k \geq 0 \tag{4.18}$$

$$U_{\mathcal{E}} \subseteq U \tag{4.19}$$

where $U_{\mathcal{E}} = \{u \in \Re^m | \, u = u(x, k), \forall x \in \mathcal{E}(k), k = 0, 1, \ldots\}$. The main difficulties behind the above approach is related to the derivation of efficient algorithms capable to compute, or at least suitably approximate at each time instant, $\mathcal{E}(k)$. In the next two subsections this problem will be addressed for two cases of interest.

4.2.1 Time-Invariant Control Laws

Let us consider the time-invariant system

$$x(k+1) = f(x(k), u(k)) \tag{4.20}$$

and a stationary control law

$$u(k) = u(x(k)) \tag{4.21}$$

A possible constructive way to derive $\mathcal{E}(k)$ is by resorting to Lyapunov theory which implicitly allows the definition of positive invariant sets. In fact, as well known, Lyapunov functions capture and generalize the concept of "energy of the system" and the stability results by showing that such an "energy" is monotonically decreasing along the system trajectories. As a consequence, given a Lyapunov function, its level curves represent the boundaries of positively invariant sets. More formally we can state the following result

Lemma 4.5. *Consider an autonomous (controlled) dynamical system*

$$x(k+1) = f(x(k), u(x(k))). \tag{4.22}$$

If there exists a Lyapunov function $V(x(k)) > 0, \quad \forall x \in \Re^n - \{0\}, V(x(k)) = 0, \quad x = 0$ *such that*

$$V(x(k+1)) - V(x(k)) \leq 0, \qquad \forall x(k) \in \Re^n, \tag{4.23}$$

then every set $\{x \in \Re^n \,|V(x) \leq \gamma\}$ *for* $\gamma \geq 0$ *is a positively invariant set for (4.22).*

Lemma 4.6. *Consider an autonomous (controlled) dynamical system*

$$x(k+1) = f(x(k), u(x(k)), w(k)). \tag{4.24}$$

where $w(k) \in W$ *denotes an exogenous input and* W *is a closed and bounded set. If there exists a Lyapunov function* $V(x(k)) > 0, \quad \forall x \in \Re^n - \{0\}, V(x(k)) = 0, \quad x = 0$ *such that*

$$V(x(k+1)) - V(x(k)) \leq 0, \qquad \forall x(k) \in \Re^n, \forall w(k) \in W \tag{4.25}$$

then every set $\{x \in \Re^n \,|V(x) \leq \gamma\}$ *for* $\gamma \geq 0$ *is a robust positively invariant set for (4.24).*

The problem of constrained stabilization can then be recast as follows

Problem 4.7. *Consider a system in form*

$$x(k+1) = f(x(k), u(k), w(k)) \tag{4.26}$$

subject to state and input saturation constraints

$$x(k) \in X, \quad k \geq 0 \tag{4.27}$$

$$u(k) \in U, \quad k \geq 0 \tag{4.28}$$

Find, if they exist, a control law $u(k) = u(x(k))$*, a Lyapunov function* $V(x(k))$ *and a scalar* $\gamma > 0$ *such that*

$$V(x(k+1)) - V(x(k)) \leq 0, \quad k \geq 0 \tag{4.29}$$

$$V(x(0)) \leq \gamma \tag{4.30}$$

$$\{x \in \Re^n \mid V(x) \leq \gamma\} \subseteq X \tag{4.31}$$

$$\{u \in \Re^m \,|\, u = u(x), \forall x : V(x) \leq \gamma\} \subseteq U \tag{4.32}$$

4.2.2 Time-Varying Control Laws

The second case of interest is when the system is time-invariant and the control strategy assumes the following structure

$$u(x(k), k) = \begin{cases} u(x(k), k) & k = 0, ..., N-1 \\ u(x(k)) & k \geq N \end{cases} \tag{4.33}$$

that is, it is allowed to be time-variant for the first N time steps, with N arbitrary but finite, and then is stationary for all subsequent instants. Let us define the following recursion of state prediction sets

$$\begin{aligned} &\hat{X}(0|0) = \{x(0)\} \\ &\hat{X}(k+1|0) = \left\{x^+ | x^+ = f(x, u(x,k), w), \forall x \in \hat{X}(k|0), \forall w \in W\right\} \end{aligned} \tag{4.34}$$

If a positive invariant set $\mathcal{E}$ for the autonomous system

$$x(k+1) = f(x(k), u(x(k)), w(k)) \tag{4.35}$$

exists and if $\hat{X}(N|0) \subseteq \mathcal{E}$, then the following time-varying set

$$\mathcal{E}(k) = \begin{cases} \hat{X}(k|0) & k = 0, 1, ..., N-1 \\ \mathcal{E} & k > N \end{cases} \tag{4.36}$$

is a robustly positive invariant set for the autonomous system

$$x(k+1) = f(x(k), u(x(k), k), w(k)). \tag{4.37}$$

As a consequence, the constrained control problem can be reformulated as follows

Problem 4.8. *Consider a system in form*

$$x(k+1) = f(x(k), u(k)) \tag{4.38}$$

subject to state and input saturation constraints

$$x(k) \in X, \quad k \geq 0 \tag{4.39}$$

$$u(k) \in U, \quad k \geq 0 \tag{4.40}$$

Find, if there exist, a control strategy (4.33), a Lyapunov function $V(x(k))$ and a scalar $\gamma > 0$ such that

$$\hat{X}(k|0) \subseteq X, \ k = 0, 1, ..., N \tag{4.41}$$

$$\left\{u | u = u(x,k), \forall x \in \hat{X}(k|0)\right\} \subseteq U, \ k = 0, 1, ..., N \tag{4.42}$$

$$V(x(k+1)) - V(x(k)) \leq 0, \quad k \geq N \tag{4.43}$$

$$V(x) \leq \gamma, \ \forall x \in \hat{X}(k+N|0) \tag{4.44}$$

$$\{x \in \Re^n \mid V(x) \leq \gamma\} \subseteq X \tag{4.45}$$

$$\{u \in \Re^m | \, u = u(x), \forall x : V(x) \leq \gamma\} \subseteq U \tag{4.46}$$

4.2.3 Set invariance and Lyapunov functions for LPV systems

Here the set invariance concept will be specialized for the LPV systems case (eq. (1.1)) in order to apply the previous section results.

Lemma 4.9. *Consider Lyapunov function of the form*

$$V\left(\xi\left(k\right)\right)=x\left(k\right)^{T}V\left(p\left(k\right)\right)x\left(k\right). \tag{4.47}$$

with $V\left(p\left(k\right)\right)\in\Re^{n}$ *is positive definite* $\forall p\left(k\right)\in\Sigma_{l}$ *and such that*

$$\begin{array}{l}V\left(\xi\left(k+1\right)\right)-V\left(\xi\left(k\right)\right)<0,\\ \qquad\forall x\left(k\right)\in\Re^{n}-\{0\},\forall p\left(k\right)\in\Sigma_{l},\forall p\left(k+1\right)\in\Upsilon\left(p\left(k\right)\right)\end{array} \tag{4.48}$$

If no bounded variations on $p\left(k\right)$ *are prescribed (i.e.* $\Upsilon\left(p\left(k\right)\right)=\Sigma_{l}$*), then, for each* $\gamma>0$*, the set*

$$\mathcal{E}=\left\{x\in\Re^{n}\,|\,x^{T}V\left(p\left(0\right)\right)x<\gamma\right\} \tag{4.49}$$

is a robustly positive invariant set.

Proof. Exploiting (4.48) we have recursively that

$$\begin{array}{l}x\left(k\right)^{T}V\left(p\left(k\right)\right)x\left(k\right)\leq x\left(0\right)^{T}V\left(p\left(0\right)\right)x\left(0\right)\\ \forall p\left(k\right)\in\Sigma_{l},k=0,1,...\end{array} \tag{4.50}$$

Since $p\left(k\right)$ can be any point into the unitary simplex Σ_{l} then also the following inequality holds true

$$x\left(k\right)^{T}V\left(p\left(0\right)\right)x\left(k\right)\leq x\left(0\right)^{T}V\left(p\left(0\right)\right)x\left(0\right)\quad k=0,1,2,.... \tag{4.51}$$

As a consequence if $x\left(0\right)\in\mathcal{E}\Rightarrow x\left(k\right)\in\mathcal{E},\ k=0,1,2,....$ □

Remark 4.10. Up to our best knowledge, the above definitions of invariant set for LPV systems under Parameter Dependent Lyapunov function seems new and here introduced for the first time. The approaches proposed in literature (see [45]) typically discards the knowledge of the initial parameter $p\left(0\right)$ by resorting to the invariant formulation defined for the uncertain framework [46]:

$$\mathcal{E}=\left\{x\in\Re^{n}\,|\,x^{T}V\left(p\right)x<\gamma,\forall p\in\Sigma\right\}. \tag{4.52}$$

It is worth to note that the same definition of invariant set does not hold under bounded parameter variations. In fact (4.50) would become

$$\begin{array}{l}x\left(k\right)^{T}V\left(p\left(k\right)\right)x\left(k\right)<x\left(0\right)^{T}V\left(p\left(0\right)\right)x\left(0\right),\\ \forall p\left(k+1\right)\in\Upsilon\left(p\left(k\right)\right),k=1,2,...\end{array} \tag{4.53}$$

and problems arise because in general $p\left(0\right)\notin\Upsilon\left(p\left(k\right)\right),k>2$. This property is depicted in Figure 4.1 for the case $l=2$ where dots represent possible

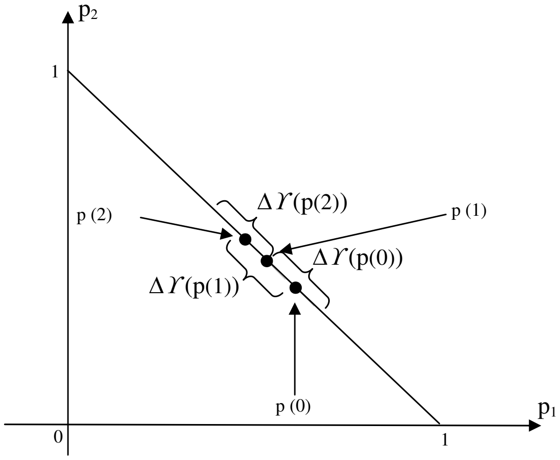

Fig. 4.1. Geometrical interpretation of set $\Upsilon(p(k)$ for LPV systems subject to bounded parameter variation

occurrences of parameters $p(0), p(1), p(2)$ whereas curly brackets represent $\Upsilon(p(i)), i = 0, 1, 2$. Clearly, $p(0) \notin \Upsilon(p(2))$. The reason is that, as continuously stressed in this dissertation, whenever dealing with bounded parameter variations, the vector $p(k)$ is not memoryless and it can be regarded as an additional state variable instead. A possible definition of an invariant set for this case is therefore given by

$$\mathcal{E} = \left\{ \left[x^T, p^T\right]^T \in \Re^n \times \Sigma_l \left| x^T V(p) x \leq \gamma \right. \right\} \tag{4.54}$$

The main drawback of the above definition is however that it does not seem to be easily employable within a convex optimization machinery. Further studies are in progress on the topic.

Chapter Summary

In this Chapter, the constrained control problem for LPV systems has been introduced and set invariance theory used to derive workable control design methods for the problems recalled in the first two Sections. Finally, in the last Section, the definition of invariant set for LPV systems has been introduced. Such a definition seems to be new in literature in the case of Parameter Varying Lyapunov functions.

5

Prediction Sets

In this chapter we introduce the notion of prediction sets for LPV systems and investigate how their structure becomes when the system is controlled by a specific class of control strategies to be used in MPC contexts.

The Chapter organization is as follows. In Section 1 and 2, the characterization of Information Vector Prediction Set is given for the family of control strategies of interest. Subsequent Sections deal with computational aspects Finally, in the last Section, a novel class of control strategies based on prediction set ideas is discussed.

5.1 Prediction Set Definition

The information characterizing the actual conditions of an LPV system includes both state and parameter values. For such a reason, to predict the set of all the possible future system states we cannot simply focus on the state vector but we need to take into account the trajectories of the whole information vector ξ, as defined in Section 1.2. More formally, we can define the information vector prediction set as follows:

Definition 5.1. *Let a LPV system (1.1) be given, possibly subject to bounded parameter rate of change i.e.* $p(k) \in \Upsilon(p(k-1))$, *with* $\Upsilon(\cdot)$ *defined in (1.4). Consider also a given control strategy* $u(k) = u(\xi(k)), k \geq 0$ *and let the value of the information vector* $\xi(0)$ *be known at time* $k = 0$. *Then, the* k*-step ahead information vector prediction set is recursively defined as follows*

$$\begin{aligned} &\hat{I}\,(0|0) \triangleq \{\xi\,(0)\} \\ &\hat{I}\,(k|0) \triangleq \left\{ \hat{\xi}(k|0) = \begin{bmatrix} \hat{x}(k|0) \\ \hat{p}(k|0) \\ k \end{bmatrix} \middle| \begin{array}{l} \hat{x}(k|0) = \sum\limits_{i=1}^{l} p_i\,(A_i x + B_i u(\xi)) \\ \hat{p}(k|0) \in \Upsilon(p) \\ \forall \xi = [p^T, x^T, k-1]^T \in \hat{I}(k-1|0) \end{array} \right\}, \quad (5.1) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 1, 2, \ldots \end{aligned}$$

On the basis of the above definition, it is of interest to define three particular prediction sets will be useful in future. First, we introduce the state prediction set as the projection of (5.4) on the state space :

$$\begin{array}{l} \hat{X}\,(0|0) \triangleq \{x\,(k)\} \\ \hat{X}\,(k|0) \triangleq \left\{ \hat{x}\,(k|0) \left| \begin{array}{l} \hat{x}\,(k|0) = \sum\limits_{i=1}^{l} p_i\,(A_i x + B_i u(\xi)) \\ \forall \xi = [x^T, p^T, k-1]^T \in \hat{I}(k-1|0) \end{array} \right. \right\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad k = 1, 2, \ldots \end{array} \tag{5.2}$$

Then, by exploiting the definition of $\Upsilon(p)$ in (1.4) we can also define the projection of $\hat{I}\,(k|0)$ on the parameter space that results

$$\hat{P}\,(k|0) \triangleq \{p \in \Sigma_l \mid |p_i - p_i\,(0)| < k\,\Delta p_i\ ,\ i = 1, \ldots, l\}\,,\ k = 0, 1, 2, \ldots \tag{5.3}$$

Finally, it is of interest to define the input prediction set, i.e. the set of possible future input

$$\begin{array}{l} \hat{U}\,(k|0) \triangleq \left\{ \hat{u}(k|0) = u\,(\xi) \left| \forall \xi \in \hat{I}(k|0) \right. \right\}, \\ \qquad\qquad\qquad\qquad\qquad k = 0, 1, \ldots \end{array} \tag{5.4}$$

It is evident that the adopted control strategy plays a fundamental role in the definitions of $\hat{I}(k|0)$, $\hat{X}(k|0)$ and $\hat{U}(k|0)$. On the contrary, the set $\hat{P}(k|0)$ is independent on the control strategy and assumes always the form (5.3). For such a reason, in the next Sections, we will focus on the computation $\hat{I}(k|0)$, $\hat{X}(k|0)$ and $\hat{U}(k|0)$.

Remark 5.2. Note that $\hat{I}\,(0|0)$ is by definition a singleton. As a consequence, $\hat{X}(1|0)$ is a singleton too and contains the vector

$$\hat{x}\,(k+1|k) = \sum_{i=1}^{l} p_i\,(k)\,A_i x\,(k) + \sum_{i=1}^{l} p_i\,(k)\,B_i u\,(\xi(k))\,. \tag{5.5}$$

The exact knowledge of the one-step ahead state prediction is one of the peculiar and interesting properties of the LPV framework and will be often recalled. Moreover, note that one-step ahead information set $\hat{I}\,(k+1|k)$ is given by:

$$\hat{I}(1|0) = \left\{ \hat{\xi}(1|0) = \begin{bmatrix} \hat{x}(1|0) \\ \hat{p}(1|0) \\ 1 \end{bmatrix} \left| \hat{p}(1|0) \in \hat{P}(1|0) \right. \right\}$$

Because $\hat{I}\,(k|0)\,, k > 0$ is a nonsingular set, any set depending on it will be in general nonsingular as well. This means that both $\hat{X}\,(k|0)\,, k > 1$ and $\hat{U}\,(k|0)\,, k > 0$ are nonsingular.

5.2 Control Strategies

Hereafter, we consider control strategies based on the whole information vector $\xi(k)$, i.e. the command $u(t)$ is not only computed on the basis of the state and parameter vectors $x(k)$ and $p(k)$ but also on the actual time k.

Even if many possible control strategies can be defined, only the ones mostly used in LPV-MPC literature (see [47], [48],[49], etc...) are here of interest, which fall in the following family

$$u\left(\xi\left(k\right)\right)=\begin{cases}u^{k}\left(x\left(k\right),p\left(k\right)\right) & k=0,...,N-1\\ u^{N}\left(x\left(k\right),p\left(k\right)\right) & k\geq N\end{cases} \tag{5.6}$$

where N is the control horizon. Observe that the first N control actions can be expressed by control laws belonging to the family (2.1)

$$\begin{aligned}u_{\tau}\left(x\left(k\right),p\left(k\right)\right)&=\sum_{i=1}^{l}p_{i}\left(k\right)u_{i}^{\tau}\left(x\left(k\right)\right)= \\ &=\sum_{i=1}^{l}p_{i}\left(k\right)\left[F_{i}^{k}x\left(k\right)+c_{i}^{k}\right]\end{aligned} \tag{5.7}$$

whereas the last one, referred to as the terminal controller in MPC literature, is a state-feedback control law in the form (2.1) or eventually (2.3).

For reasons which will be clarified later, it is of interest to have a convex description of the prediction set $\hat{I}\left(k|0\right),\ \ k=0,...,N$ or, at least, to be able to compute suitably convex outer approximations. This problem will be faced in the next Sections. In order to proceed systematically and for the sake of clarity we will treat separately the following three relevant possible cases.

1. Non-scheduled control laws, i.e.

$$F_{i}^{k}=F^{k},c_{i}^{k}=c^{k},\ i=1,...,l,\ \ k=0,...,N-1 \tag{5.8}$$

2. Scheduled control laws for LPV systems without bounded parameter variations
3. Scheduled control laws for LPV systems subject to bounded parameter variations

5.3 Nonscheduled Control Strategies

Consider the following control strategy (5.6)-(5.7)

$$u\left(\xi\left(k\right)\right)=\begin{cases}F^{k}x(k)+c^{k} & k=0,...,N-1\\ u^{N}\left(x\left(k\right),p\left(k\right)\right) & k\geq N\end{cases} \tag{5.9}$$

to be applied to a LPV system possibly subject to bounded parameter variations. The closed-loop one-step ahead state prediction is given by

$$x\,(k+1) = \sum_{i=1}^{l} p_i\,(k)\left[A_i\,x\,(k) + B_i\left(F^k x\,(k) + c^k\right)\right] = f\,(\xi\,(k)) \tag{5.10}$$

where $f\,(\xi\,(k))$ is introduced to be used as a shorthand hereafter.

In Section 1.2, LPV systems subject to bounded parameter variation have been shown to hide a nonlinearity which introduce a sort of "parameter memory effect". Such a nonlinearity strongly complicates the structure of the information set $\hat{I}(k|0)$ and the machineries to compute exactly the prediction sets. Then, in order to obtain manageable sets, we need to look for convex outer polytopic approximations of $\hat{I}(k|0)$.

A suitable way to obtain such an outer approximation is by relaxing the dependence between the state vector x and the parameter vector p. Then, the following set results:

$$\hat{\hat{I}}(k|0) = \left\{ \xi(k|0) = \begin{bmatrix} \hat{x}(k|0) \\ \hat{p}(k|0) \\ k \end{bmatrix} \middle| \begin{matrix} \hat{x}(k|0) \in \hat{\hat{X}}(k|0) \\ \hat{p}(k|0) \in \hat{P}(k|0) \end{matrix} \right\} \tag{5.11}$$

where $\hat{P}(k|0)$ is the set defined in (5.3) and $\hat{\hat{X}}(k|0)$ is an outer approximation of $\hat{X}(k|0)$ recursively defined as follow

$$\begin{aligned} &\hat{\hat{X}}(0|0) = \{x(0)\} \\ &\hat{\hat{X}}(k|0) = \left\{ \hat{x} = f([x^T, p^T, k-1]^T) \middle| \begin{matrix} x \in \hat{\hat{X}}(k-1|0) \\ p \in \hat{P}(k-1|0) \end{matrix} \right\} \end{aligned} \tag{5.12}$$

Remark 5.3. Note that, in the case of parameter variations not subject to bounds, the parameter vector and the state vector are already independent, then $\hat{\hat{I}}(k|0)$ exactly coincides with $\hat{I}(k|0)$.

Because of (5.10), $\hat{\hat{X}}$ results to be a polytope whose vertices can be easily obtained in a recursive fashion. In fact, if we denote by $\hat{P}_{k,i_k} \in \Re^l, i_k = 1, ..., l_k$ the l_k vertices of $\hat{P}(k|0)$, then the vertices of $\hat{\hat{X}}(1|0), \hat{\hat{X}}(2|0), ..., \hat{\hat{X}}(k|0)$ are given by:

$$\begin{aligned} &\hat{x}_(1|0) = f\left(\left[\hat{x}^T(0), p^T(0), 0\right]^T\right), \\ &\hat{x}_{i_1}(2|0) = f\left(\left[\hat{x}^T(1|0), \hat{P}^T_{1,i_1}, 1\right]^T\right), i_1 = 1, ..., l_1 \\ &... \\ &\hat{x}_{i_1,i_2,...,i_{k-1}}(k|0) = f\left(\left[\hat{x}^T_{i_1,i_2,...,i_{k-2}}, \hat{P}^T_{k-1,i_{k-1}}, k-1\right]^T\right), \quad \begin{matrix} i_1 = 1, ..., l_1, \\ ... \\ i_{k-1} = 1, ..., l_{k-1} \end{matrix} \end{aligned} \tag{5.13}$$

The above vertices formulation will be very useful when time-varying control strategies will be applied to LPV systems subject to constraints. Moreover, if we define the outer approximation to the input prediction set $U(k|0)$ as follows

$$\hat{\hat{U}}(k|0) = \left\{ \hat{u}(k|0) \left| \begin{array}{l} \hat{u}(k|0) = F^k \hat{x}(k|0) + c^k \\ \forall \hat{x}(k|0) \in \hat{\hat{X}}(k|0) \end{array} \right. \right\} \tag{5.14}$$

Then, vertices (5.13) allows us to rewrite $\hat{\hat{U}}(k|0)$ as a convex combination of vertices

$$\hat{\hat{U}}(k|0) = conv \left\{ \left\{ \hat{u}_{i_1,\ldots,i_{k-1}} \right\}_{\substack{i_1 = 1 \\ \ldots \\ i_{k-1} = 1}}^{\substack{l_1 \\ \ldots \\ l_{k-1}}} \right\} \tag{5.15}$$

where $\hat{u}_{i_1,i_2,\ldots,i_{k-1}}(k|0)$ are

$$\hat{u}_{i_1,i_2,\ldots,i_{k-1}}(k|0) = F^k \hat{x}_{i_1,i_2,\ldots,i_{k-1}}(k|0) + c^k, \quad \begin{array}{l} i_1 = 1, \ldots, l_1 \\ \ldots \\ i_{k-1} = 1, \ldots, l_{k-1} \end{array} \tag{5.16}$$

Finally note that, by simply applying the above definitions, it is possible to obtain the relationships between the prediction set computed with the information vector available at time 0 and those computed in the subsequent time instants. Those relationship are resumed by the following lemma

Lemma 5.4. *Set inclusions $\hat{\hat{X}}(k|1) \subseteq \hat{\hat{X}}(k|0)$, $\hat{\hat{U}}(k|1) \subseteq \hat{\hat{U}}(k|0)$ result for the above sets. Moreover, if $p(1)$ is known, the vertices of $\hat{\hat{X}}(k|1)$ and $\hat{\hat{U}}(k|1)$ can be obtained from the vertices of $\hat{\hat{X}}(k|0)$ and $\hat{\hat{U}}(k|0)$ as follows*

$$\hat{x}_{i_2,\ldots,i_{k-1}}(k|0) = \sum_{i_1=1}^{l_1} \theta_{i_1} \hat{x}_{i_1,i_2,\ldots,i_{k-1}}(k|0), \quad \begin{array}{c} i_2 = 1, \ldots, l_2, \\ \ldots \\ i_{k-1} = 1, \ldots, l_{k-1}. \end{array} \tag{5.17}$$

$$\hat{u}_{i_2,\ldots,i_{k-1}}(k|0) = \sum_{i_1=1}^{l_1} \theta_{i_1} \hat{u}_{i_1,i_2,\ldots,i_{k-1}}(k|0), \quad \begin{array}{c} i_2 = 1, \ldots, l_2, \\ \ldots \\ i_{k-1} = 1, \ldots, l_{k-1}. \end{array} \tag{5.18}$$

where $\theta = [\theta_1, \ldots, \theta_{l_1}]^T \in \Sigma_{l_1}$ is such that

$$\sum_{i_1=1}^{l_1} \theta_{i_1} \hat{P}_{1,i_1} = p(1) \tag{5.19}$$

5.4 Scheduled Control Strategies - No bounds on parameter variations

If a scheduled control strategy (5.6)-(5.7) is used, the closed-loop one-step ahead predictions for the system (1.1) becomes

$$x(k+1) = \sum_{i=1}^{l} p_i(k) \left[A_i\, x(k) + B_i \sum_{j=1}^{l} u_j^{\tau}(x(k)) \right] = f(\xi(k)) \tag{5.20}$$

where $f(\xi(k))$ is used hereafter as a shorthand. As already seen in Section 2.2, the set of states reachable from $x(k)$ for any possible parameter occurrence is in general nonconvex. As a consequence, if a scheduled control strategy is employed, even in the case that no bounds on the parameter variations are prescribed, $\hat{I}(k|0)$ is not convex for $k \geq 2$.

For many interesting applications shown successively in this thesis, we would like to deal with polytopic prediction sets (possibly expressed in the form of convex hulls of its vertices) and use scheduled control strategies, because their use strongly improves the control performances. Because this is not directly achievable, suitable outer approximations of the exact prediction sets have to be introduced.

To proceed in this direction, let us focus on the information vector prediction set. If no hypotheses on the parameter variations are assumed, then the information vector prediction set is given by

$$\begin{aligned} \hat{I}(0|0) &\triangleq \{\xi(0)\} \\ \hat{I}(k|0) &\triangleq \left\{ \hat{\xi}(k|0) = \begin{bmatrix} \hat{x}(k|0) \\ \hat{p}(k|0) \\ k \end{bmatrix} \middle| \begin{array}{c} \hat{x}(k|0) = f(\xi) \\ \hat{p}(k|0) \in \Sigma_l \\ \xi \in \hat{I}(k-1|0) \end{array} \right\}, \\ &\qquad k = 1, 2, \ldots, N \end{aligned} \tag{5.21}$$

Because $\hat{x}(k|0)$ and $\hat{p}(k|0)$ are independent each other, the above set can be furtherer simplified into

$$\begin{aligned} \hat{I}(0|0) &\triangleq \{\xi(k)\} \\ \hat{I}(k|0) &\triangleq \left\{ \hat{\xi}(k|0) = \begin{bmatrix} \hat{x}(k|0) \\ \hat{p}(k|0) \\ k \end{bmatrix} \middle| \begin{array}{c} \hat{x}(k|0) \in \hat{X}(k|0) \\ \hat{p}(k|0) = e_i, i = 1, ..., l \end{array} \right\}, \\ &\qquad k = 1, 2, \ldots, N \end{aligned} \tag{5.22}$$

where e_i represents the i-th vector of the canonical basis of $\Re^l$ and $\hat{X}(k|0)$ is recursively defined as

$$\hat{X}(1|0) = \left\{ \hat{x}(1|0) = f([x(0)^T, p(0)^T, 0]^T) \right\} \tag{5.23}$$

$$\hat{X}(k|0) = \left\{ \hat{x}(k|0) \middle| \begin{array}{c} \hat{x}(k|0) = f([x^T, p^T, k]) \\ x \in \hat{X}(k|0), p \in \Sigma_l \end{array} \right\} \tag{5.24}$$

Then, the problem of the convexification of $\hat{I}(k|0)$ simplifies into the problem of finding a polytopic outer approximation for $\hat{X}(k+\tau|k)$. By taking advantage of the convexifications seen in subsection 2.2.4, an outer approximations can be obtained by considering the following one-step ahead variation

$$x(k+1) = \sum_{i=1}^{l_c} \bar{p}_i(k) \left[\bar{A}_i x(k) + \bar{B}_i \bar{u}^k (x(k)) \right] = \\ = \bar{f}(x(k), \bar{p}(k), \tau)$$

where $\bar{A}_i \in \Re^{n \times n}, \bar{B}_i \in \Re^{n \times (lm)}, i = 1, ..., l_c$ are proper matrices, $\bar{u}^\tau = [(\bar{u}_1^\tau)^T, ..., (\bar{u}_l^\tau)^T]^T$ and $\bar{p} \in \Sigma_{l_c}$ is the new parameter vector resulting from the convexification. $\bar{f}(x(k), \bar{p}(k), \tau)$ is again introduced as a shorthand. Then, an outer polytopic approximation for $\hat{X}(k|0)$ can be defined as follows

$$\hat{\hat{X}}(1|0) = \hat{X}(1|0) \tag{5.25}$$

$$\hat{\hat{X}}(k|0) = \left\{ \hat{x}(k|0) \middle| \begin{array}{l} \hat{x}(k|0) = \bar{f}(x, p, k)) \\ x \in \hat{\hat{X}}(k|0), p \in \Sigma_l \end{array} \right\} \tag{5.26}$$

whose vertices can be recursively defined as follows

$$\begin{array}{l} \hat{x}(1|0) = f(\xi(0)) \\ \hat{x}_{i_1}(2|0) = \bar{f}(\hat{x}(1|0), \bar{e}_{i_1}, 1), \ \ i_1 = 1, \ldots, l_c \\ \hat{x}_{i_1,i_2}(3|0) = \bar{f}(\hat{x}_{i_1}(2|0), \bar{e}_{i_2}, 2), i_1 = 1, \ldots, l_c, i_2 = 1, \ldots, l_c \\ \ldots \\ \hat{x}_{i_1,\ldots,i_{k-1}}(k|0) = \bar{f}\left(\hat{x}_{i_1,i_2,\ldots,i_{k-2}}(k-1|0), \bar{e}_{i_{k-1}}, k-1\right), i_1 = 1, \ldots, l_c, \ldots, i_{k-1} = 1, \ldots, l_c \end{array} \tag{5.27}$$

where by $\bar{e}_i, i = 1, ..., l_c$ are denoted the l_c vectors of the standard basis of $\Re^{l_c}$. For reasons which will be clear in the next Chapter, is convenient to formulate, on the basis of the above vertices formulation, an outer approximation for the input prediction set $U(k|0)$ that is "chorded" with $\hat{X}(k|0)$ vertices. To this end, let us define

$$\hat{\hat{U}}(k|0) = \left\{ \hat{u}(k|0) \middle| \begin{array}{l} \hat{u}(k|0) = \sum\limits_{i=1}^{l_c} \hat{\bar{p}}_i(k|0) M_i \begin{bmatrix} F_1 \hat{x}(k|0) + c_1^k \\ \ldots \\ F_l \hat{x}(k|0) + c_l^k \end{bmatrix} \\ \forall \hat{\bar{p}}(k|0) \in \Sigma_{l_c} \\ \forall \hat{x}(k|0) \in \hat{\hat{X}}(k|0) \end{array} \right\} \tag{5.28}$$

where, with reference to (2.22)-(2.23) , $\bar{M}_i \in \Re^{m \times (ml)}, i = 1, ..., l_c$ are the mapping matrices between the extended input $\bar{u} \in \Re^{ml}$ and the real input $u \in \Re^m$

$$\bar{M}_i = \left[I_{m \times m} \ ... \ I_{m \times m} \right] (\Pi_i \otimes I_{m \times m}), \ \ i = 1, ..., l_c. \tag{5.29}$$

The latter, coupled with vertices (5.27) allow us to easily obtain

$$\hat{\hat{U}}(k|0) = conv \left\{ \{\hat{u}_{i_1,\ldots,i_k}\}_{\substack{i_1 = 1 \\ \ldots \\ i_k = 1}}^{\substack{l_c \\ \ldots \\ l_c}} \right\} \tag{5.30}$$

where vertices $\hat{u}_{i_1,i_2,...,i_k}(k|0)$ are

$$\hat{u}_{i_1,i_2,...,i_k}(k|0) = M_{i_k} \begin{bmatrix} F_1\hat{x}_{i_1,i_2,...,i_{k-1}}(k|0) + c_1^k \\ \dots \\ F_l\hat{x}_{i_1,i_2,...,i_{k-1}}(k|0) + c_l^k, \end{bmatrix} \quad \begin{array}{l} i_1 = 1, ..., l_c, \\ ... \\ i_k = 1, ..., l_c. \end{array} \tag{5.31}$$

Finally note that, by simply exploit the above prediction set structures, we can state the following results

Lemma 5.5. *Set inclusions $\hat{\hat{X}}(k|1) \subseteq \hat{\hat{X}}(k|0)$, $\hat{\hat{U}}(k|1) \subseteq \hat{\hat{U}}(k|0)$ result for the above sets. Moreover, if $p(1)$ is known, the vertices of $\hat{\hat{X}}(k|1)$ and $\hat{\hat{U}}(k|1)$ can be obtained from the vertices of $\hat{\hat{X}}(k|0)$ and $\hat{\hat{U}}(k|0)$ as follows*

$$\hat{x}_{i_2,...,i_{k-1}}(k|0) = \sum_{i_1=1}^{l_c} \theta_{i_1}\hat{x}_{i_1,i_2,...,i_{k-1}}(k|0), \quad \begin{array}{l} i_2 = 1, ..., l_c, \\ ... \\ i_{k-1} = 1, ..., l_c. \end{array} \tag{5.32}$$

$$\hat{u}_{i_2,...,i_k}(k|0) = \sum_{i_1=1}^{l_c} \theta_{i_1}\hat{u}_{i_1,i_2,...,i_k}(k|0), \quad \begin{array}{l} i_2 = 1, ..., l_c, \\ ... \\ i_k = 1, ..., l_c. \end{array} \tag{5.33}$$

where $\theta = [\theta_1, ..., \theta_{l_1}]^T \in \Sigma_{l_c}$ is such that

$$\sum_{i_1=1}^{l_1} \theta_{i_1}\bar{e}_{i_1} = \rho(p(1)) \tag{5.34}$$

and $\rho(p)$ is the mapping function defined in (2.24)

5.5 Scheduled Control Strategies - Bounded parameter variations

The problem of computing the information vector prediction set strongly complicates when we apply a parameter dependent control strategy to an LPV systems subject to bounded parameter variations. This is due to both the nonlinearities coming from the "parameter memory effect" introduced when dealing with bounded parameter variations and to the quadratic dependencies arising with self-scheduled control strategies. Then, in order to obtain polytopic outer approximation for the prediction set, we need to relax both the nonlinearities.

The quadratic parameter dependence can be relaxed by means of the one-step ahead state prediction outer approximation introduced in Chapter 2 obtaining

$$\begin{aligned} x\,(k+1) &= \sum_{i=1}^{l_c} \bar{p}_i\,(k)\left[\bar{A}_i x\,(k) + \bar{B}_i \bar{u}^{\tau}\,(x\,(k))\right] = \\ &= \bar{f}\,(x\,(k)\,, \bar{p}\,(k), k) \end{aligned} \tag{5.35}$$

where the parameter vector is mapped through $\bar{p}(k) = \bar{\rho}(p(k))$ as defined in (2.24). Then, by using the latter one-step ahead state prediction machinery and by relaxing state-parameter dependence, we can define the following outer approximation for the information vector prediction set:

$$\begin{array}{l} \hat{\hat{I}}(0|0) = \{\xi(0)\} \\ \hat{\hat{I}}(k|0) = \left\{ \hat{\xi}(k|0) \left| \begin{array}{c} \hat{\xi}(k|0) = [\hat{x}(k|0)^T, \hat{p}(k|0), k] \\ \hat{x}(k|0) \in \hat{\hat{P}}(k|0) \hat{p}(k|0) \in \hat{P}(k|0) \end{array} \right. \right\} \end{array} \tag{5.36}$$

with $\hat{P}(k|0)$ defined in (5.3) and $\hat{\hat{X}}(k|0)$ defined as follows

$$\begin{array}{l} \hat{\hat{X}}(0|0) = \{x(0)\} \\ \hat{\hat{X}}(1|0) = \left\{\bar{x}(1) = f([x(0)^T, p(0)^T, 0])\right\} \\ \hat{\hat{X}}(k|0) = \left\{ \bar{x}(k) \left| \begin{array}{c} \bar{x}(k) = f(x^T, \bar{p}, k) \\ x \in \hat{\hat{X}}(k-1|0) \\ \bar{p} \in \bar{\Upsilon}(P(k|0)) \end{array} \right. \right\} \end{array} \tag{5.37}$$

where $f(\cdot)$ is defined in (5.20), $\bar{f}(\cdot)$ is defined in (5.35) and $\bar{\Upsilon}(P(k|0))$ is a polytopic set such that

$$\forall p \in P(k|0) \Rightarrow \rho(p) \in \bar{\Upsilon}(P(k|0)) \tag{5.38}$$

mapping the set $P(k|0)$ into Σ_{l_c}.

Note that, if we can explicitly compute the $\Upsilon(P(k|0))$ vertices, denoted hereafter as $\Upsilon_{k,i}, i = 1, ..., l_{c,k}$, the vertices of $\hat{X}(k|0)$ can be recursively obtained as follows

$$\begin{array}{l} \hat{x}(1|0) = f(\xi(0)) \\ \hat{x}_{i_1}(2|0) = \bar{f}\left(\hat{x}(1|0), \bar{\Upsilon}_{1,i_1}, 1\right), \ \ i_1 = 1, \ldots, l_{c,1} \\ \hat{x}_{i_1,i_2}(3|0) = \bar{f}\left(\hat{x}_{i_1}(2|0), \bar{\Upsilon}_{2,i_2}, 2\right), \begin{array}{l} i_1 = 1, \ldots, l_{c1}, \\ i_2 = 1, \ldots, l_{c,2} \end{array} \\ \ldots \\ \hat{x}_{i_1,i_2,\ldots i_{k-1}}(k|0) = \bar{f}\left(\hat{x}_{i_1,i_2,\ldots,i_{k-2}}(k-1|0), \bar{\Upsilon}_{k-1,i_{k-1}}, k-1\right), \begin{array}{l} i_1 = 1, \ldots, l_{c,1}, \\ \ldots \\ i_{k-1} = 1, \ldots, l_{c,k-1}. \end{array} \end{array} \tag{5.39}$$

If we define an outer approximation of the input prediction set as follows:

$$\hat{\hat{U}}(k|0) = \left\{ \hat{u}(k|0) \left| \begin{array}{l} \hat{u}(k|0) = \sum_{i=1}^{l_c} \hat{\bar{p}}_i(k|0) M_i \begin{bmatrix} F_1 \hat{x}(k|0) + c_1^k \\ \ldots \\ F_l \hat{x}(k|0) + c_l^k \end{bmatrix} \\ \forall \hat{\bar{p}}(k|0) \in \bar{\Upsilon}(\hat{P}(k|0)) \\ \forall \hat{x}(k|0) \in \hat{\hat{X}}(k|0) \end{array} \right. \right\} \tag{5.40}$$

where $M_i, i = 1, ..., l_c$ are defined in (5.29), vertices (5.39) allows one to reformulate the input prediction set as a convex combination of vertices

$$\hat{\hat{U}}(k|0) = conv\left\{ \left\{\hat{u}_{i_1,...,i_{k-1}}\right\}_{\substack{i_1=1\\ \cdots \\ i_k=1}}^{\substack{l_{c,1}\\ \cdots \\ l_{c,k}}} \right\}. \tag{5.41}$$

where $\hat{u}_{i_1,i_2,...,i_k}(k|0)$ can be obtained as follows

$$\hat{u}_{i_1,i_2,...,i_k}(k|0) = \sum_{j=1}^{l_c} (\bar{\Upsilon}_{k,i_k})_j M_j \begin{bmatrix} F_1\hat{x}_{i_1,i_2,...,i_{k-1}}(k|0) + c_1^k \\ \cdots \\ F_l\hat{x}_{i_1,i_2,...,i_{k-1}}(k|0) + c_l^k \end{bmatrix} \begin{array}{l} i_1 = 1, ..., l_{c,1} \\ ... \\ i_{k-1} = 1, ..., l_{c,k-1}, \\ i_k = 1, ..., l_k. \end{array} \tag{5.42}$$

and $(\bar{\Upsilon}_{k,i_k})_j$ denotes the j-th entry of the i_k-th vertex of $\bar{\Upsilon}(\hat{P}(k|0))$. Finally note that Lemma 5.6 characterizing $\hat{\hat{X}}(k|1)$ and $\hat{\hat{U}}(k|1)$ can be easily rewritten as follows

Lemma 5.6. *Set inclusions $\hat{\hat{X}}(k|1) \subseteq \hat{\hat{X}}(k|0)$, $\hat{\hat{U}}(k|1) \subseteq \hat{\hat{U}}(k|0)$ result for the above sets. Moreover, if $p(1)$ is known, the vertices of $\hat{\hat{X}}(k|1)$ and $\hat{\hat{U}}(k|1)$ can be obtained from the vertices of $\hat{\hat{X}}(k|0)$ and $\hat{\hat{U}}(k|0)$ as follows*

$$\hat{x}_{i_2,...,i_{k-1}}(k|0) = \sum_{i_1=1}^{l_c} \theta_{i_1}\hat{x}_{i_1,i_2,...,i_{k-1}}(k|0), \quad \begin{array}{l} i_2 = 1, ..., l_c, \\ ... \\ i_{k-1} = 1, ..., l_c. \end{array} \tag{5.43}$$

$$\hat{u}_{i_2,...,i_k}(k|0) = \sum_{i_1=1}^{l_c} \theta_{i_1}\hat{u}_{i_1,i_2,...,i_k}(k|0), \quad \begin{array}{l} i_2 = 1, ..., l_c, \\ ... \\ i_k = 1, ..., l_c. \end{array} \tag{5.44}$$

where $\theta = [\theta_1, ..., \theta_{l_1}]^T \in \Sigma_{l_c}$ is such that

$$\sum_{i_1=1}^{l_1} \theta_{i_1}\bar{\Upsilon}_{k,i_k} = \rho(p(1)) \tag{5.45}$$

and $\rho(p)$ is the mapping function defined in (2.24)

Example 5.7. In order to give a geometrical intuition of the parameter manipulations shown in this section let us suppose to have an LPV plant with $l = 2$. Let $p(k) = [0.5\ 0.5]^T$ and $\Delta p_i = 0.1,\ i = 1, 2$. The prediction set of the parameter vector will be

$$\begin{aligned} &\hat{P}(k+1|k) = \\ &= \left\{ p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \in R^2 \left| \begin{bmatrix} p_1(k) - \Delta p_1 \\ p_2(k) - \Delta p_2 \end{bmatrix} \geq \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} > \begin{bmatrix} p_1(k) + \Delta p_1 \\ p_2(k) + \Delta p_2 \end{bmatrix} \right. \right\} \cap \Sigma_2 = \\ &= conv\left\{ \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}, \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \right\} \end{aligned}$$

as depicted in Figure 5.1. As a consequence $\bar{\Upsilon}(P(k+1|k))$ can be written (for instance) as follows

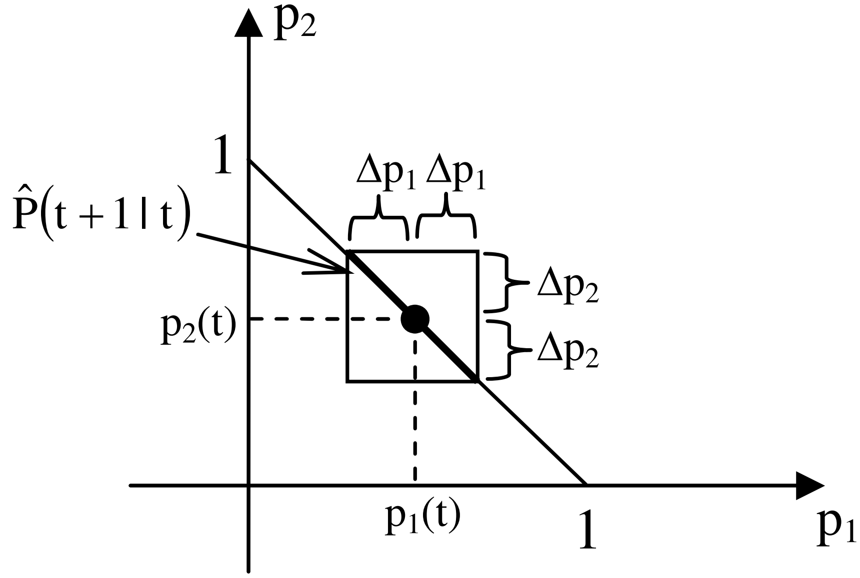

Fig. 5.1.

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$$\bar{\Upsilon}\left(\hat{P}\,(k+1|k)\right)=$$
$$\left\{\bar{p}=\begin{bmatrix}\bar{p}_{11}\\ \bar{p}_{22}\\ \bar{p}_{12}\end{bmatrix}\middle|\begin{bmatrix}\bar{p}_{11}\,(k)-\Delta p_1^2\\ \bar{p}_{22}\,(k)-\Delta p_2^2\\ \bar{p}_{12}\,(k)-2\Delta p_1\Delta p_2\end{bmatrix}\leq\begin{bmatrix}\bar{p}_{11} & 0 & 0\\ 0 & \bar{p}_{22} & 0\\ 0 & 0 & \bar{p}_{12}\end{bmatrix}\leq\begin{bmatrix}\bar{p}_{11}\,(k)+\Delta p_1^2\\ \bar{p}_{22}\,(k)+\Delta p_2^2\\ \bar{p}_{12}\,(k)+2\Delta p_1\Delta p_2\end{bmatrix}\right\}\cap\Sigma_3$$

Performing this intersection, it results that its solution is a polytope composed by $l_1 = 5$ vertices:

$$\bar{\Upsilon}\left(\hat{P}\,(k+1|k)\right)=$$
$$=conv\left\{\bar{\Upsilon}_{1,1}=\begin{bmatrix}0.16\\0.16\\0.68\end{bmatrix},\bar{\Upsilon}_{1,2}=\begin{bmatrix}0.16\\0.36\\0.48\end{bmatrix},\bar{\Upsilon}_{1,3}=\begin{bmatrix}0.32\\0.36\\0.32\end{bmatrix},\bar{\Upsilon}_{1,4}=\begin{bmatrix}0.36\\0.16\\0.48\end{bmatrix},\bar{\Upsilon}_{1,5}=\begin{bmatrix}0.36\\0.32\\0.32\end{bmatrix}\right\}$$

as depicted in Figure 5.2.

Remark 5.8. In Figure 5.2 it has been reported the real locus of $\hat{P}\,(k+1|k)$ into the $(\bar{p}_{11}, \bar{p}_{22}, \bar{p}_{12})$ space. It is worth to note that such a reparameterization is very conservative and can be refined by the intersection with further condition in order to have a tighter outer approximation of $\hat{P}(t+1|t)$ projection. In the above case, for instance, it would allow one to discard vertices $\bar{\Upsilon}_{1,3}$ and $\bar{\Upsilon}_{1,4}$.

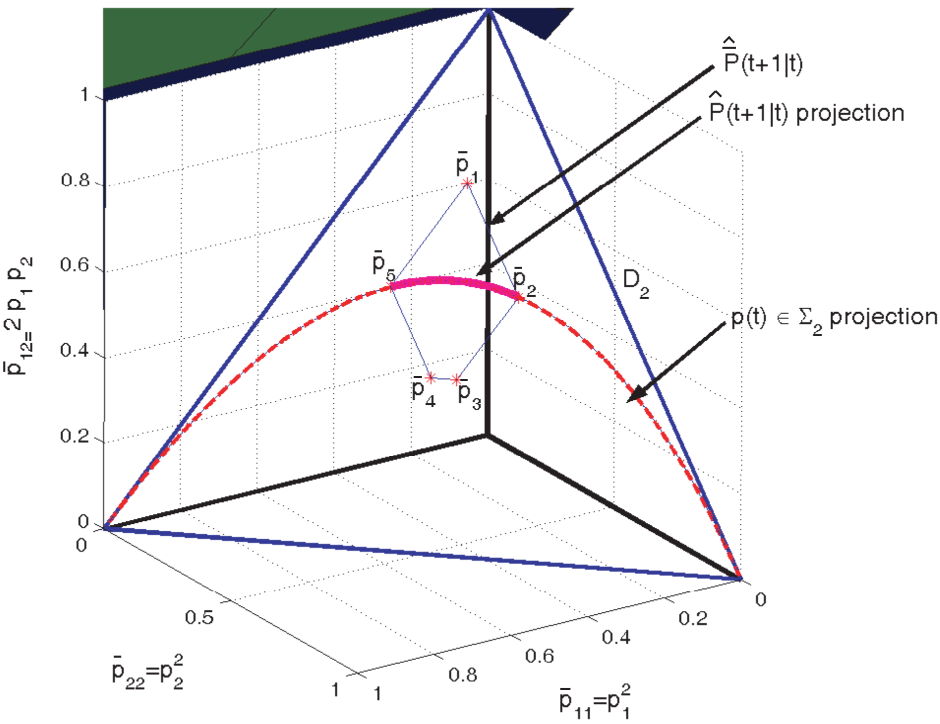

Fig. 5.2.

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5.6 Prediction Set based Control Strategies

In this Section we introduce a different class of control strategy based on the idea to exploit the prediction set structure seen in the previous Sections. Such a kind control strategy was first introduced by Pluymers in [50] for the robust polytopic framework. Here a generalization is presented by adding the capability to deal with bounded parameter variations and to exploit the actual parameter knowledge. Up to our best knowledge, this class of control strategies has never been used within the LPV framework.

Hereafter, for the sake of clarity, we will first introduce and discuss nonscheduled prediction set based control strategies, then we will move to the more complex case of scheduled control laws.

5.6.1 Nonscheduled case

In Section 5.3 it has been shown that an outer approximation to the state prediction set con be obtained by means a convex combination of vertices

$$\hat{\hat{X}}(k|0) = conv\left\{ \left\{ \hat{x}_{i_1,\ldots,i_{k-1}}(k|0) \right\}_{\substack{i_1=1\\ \cdots \\ i_{k-1}=1}}^{\substack{l_1\\ \cdots \\ l_{k-1}}} \right\}, k = 1, \ldots, N \tag{5.46}$$

where each vertex $\hat{x}_{i_1,\ldots,i_{k-1}}(k|0)$ (see (5.13)) is given by

$$\begin{aligned}&\hat{x}_{i_1,\ldots,i_{k-1}}(k|0) = \\ &= \sum_{j=1}^{l} \left(\hat{P}_{k-1,i_{k-1}} \right)_j \left[A_j \hat{x}_{i_1,\ldots,i_{k-2}}(k-1|0) + B_j u^k \left(\hat{x}_{i_1,\ldots,i_{k-2}}(k-1|0) \right) \right], \\ &\qquad k = 1, \ldots, N\end{aligned} \tag{5.47}$$

where by $(\hat{P}_{k-1,i_{k-1}})_j$ is denoted the j-th entry of the i_{k-1}-th vertex of $\hat{P}(k-1|0)$.

The idea behind prediction set based control strategy is that of substituting, for the first N time steps, the state dependent input $u^\tau\left(\hat{x}_{i_1,\ldots,i_{\tau-1}}(k+\tau|k)\right)$ in (5.47) with a static input chorded with indices $i_1, \ldots, i_{\tau-1}$ obtaining the following new vertices

$$\begin{aligned}&\hat{x}_{i_1,\ldots,i_{k-1}}(k|0) = \\ &= \sum_{j=1}^{l} \left(\hat{P}_{k-1,i_{k-1}} \right)_j \left[A_j \hat{x}_{i_1,\ldots,i_{k-2}}(k-1|0) + B_j \hat{u}_{i_1,\ldots,i_{k-2}}(k-1|0) \right], \\ &\qquad k = 1, \ldots, N\end{aligned} \tag{5.48}$$

By resorting to Section 5.3 results, the latter is clearly equivalent to a control strategy defined by means of the input prediction set $\hat{\hat{U}}(k|0)$ vertices for $k = 0, \ldots, N-1$ that, by exploiting Lemma 5.4, assumes the form of a control strategy whose first N moves depend on the parameter occurrence history:

$$u(k) = \begin{cases} \sum_{i_1=1}^{l_1} \theta_{1,i_1} \cdots \sum_{i_{k-1}=1}^{l_{k-1}} \theta_{k-1,i_{k-1}} \hat{u}_{i_1,i_2,\ldots,i_{k-1}}(k|0), k = 0, .., N-1 \\ u^N(x(k), p(k)), k \geq N \end{cases} \tag{5.49}$$

where $\theta_\tau = [\theta_{\tau,1}, \ldots, \theta_{\tau,l_\tau}]^T \in \Sigma_{l_\tau}$ is such that

$$\sum_{i_\tau=1}^{l_\tau} \theta_{\tau,i_\tau} \hat{P}_{\tau,i_\tau} = p(\tau), \tau = 0, \ldots, k-1$$

Such a control strategy has proven to yield to very interesting results in terms of control system performances, because, as it follows from its definition, it also contains control strategies in form (5.9). The main drawback is in terms of the computational effort needed to manage it: the number of its variables, in fact, grows with the number of the vertices of the prediction sets.

5.6.2 Scheduled case

To build a prediction set based control strategy capable to make use of the parameter knowledge, we have to complicate a bit the above definition. By referring to Section 5.5 notations and results, also in this case the polytopic outer approximation of the state prediction set $\hat{\hat{X}}(k|0)$ can be seen as the convex combination of vertices. By combining equations (5.20) and (5.39), those vertices assume the form

$$\hat{x}_{i_1,...,i_{k-1}}(k|0) = \sum_{i=1}^{l_c} \bar{p}_i(k) \left[\bar{A}_i x_{i_1,...,i_{k-2}}(k-1|0) + \bar{B}_i \bar{u}^\tau \left(x_{i_1,...,i_{k-2}}(k-1|0) \right) \right] \tag{5.50}$$

Then to get a prediction set based control strategy we can substitute the term $\bar{u}^\tau \left(x_{i_1,...,i_{k-2}}(k-1|0) \right)$ with a chorded vector $\hat{u}_{i_1,...,i_{k-2}}(k-1|0)$, obtaining

$$\hat{x}_{i_1,...,i_{k-1}}(k|0) = \sum_{i=1}^{l_c} \bar{p}_i(k) \left[\bar{A}_i x_{i_1,...,i_{k-2}}(k-1|0) + \bar{B}_i \hat{\hat{u}}_{i_1,...,i_{k-2}}(k-1|0) \right]. \tag{5.51}$$

The main conceptual difference between this case and the nonscheduled one is that, because of $\bar{u}(k) = [u_1^T(k), ..., u_l^T(k)]^T$, this prediction strategy is not based on the vertices of the "real" input prediction set but it makes use of the extended input vector introduced in Section 2.2.4. If we denote by $\left(\hat{\hat{u}}_{i_1,...,i_{k-2}}(k-1|0) \right)_{[j]} \in \Re^m$ the j-th m-dimensional vector composing $\hat{\hat{u}}_{i_1,...,i_{k-2}}(k-1|0)$, an explicit representation of the scheduled prediction set based control strategy is given by

$$u(k) = \begin{cases} \sum\limits_{i_1=1}^{l_1} \theta_{1,i_1} \cdots \sum\limits_{i_{k-1}=1}^{l_{k-1}} \theta_{k-1,i_{k-1}} \sum\limits_{j=1}^{l} p_j(k) \left(\hat{\hat{u}}_{i_1,i_2,...,i_{k-1}} \right)_{[j]}, k = 0, .., N-1 \\ u^N(x(k), p(k)), k \geq N \end{cases} \tag{5.52}$$

where $\theta_\tau = [\theta_{\tau,1}, ..., \theta_{\tau,l_\tau}]^T \in \Sigma_{l_\tau}$ is such that

$$\sum_{i_\tau=1}^{l_\tau} \theta_{\tau,i_\tau} \hat{P}_{\tau,i_\tau} = p(\tau), \tau = 0, ..., k-1$$

Remark 5.9. Note that the elements of the above control law are strictly linked with the input prediction set vertices. In fact, by recalling equation (5.42), the vertices of the input prediction set can be obtained by means of the following linear transformation

$$\hat{u}_{i_1,...,i_k}(k|0) = \sum_{j=1}^{l_c} (\bar{\Upsilon}_{k,i_k})_j M_j \hat{\hat{u}}_{i_1,...,i_k}, \; i_1 = 1, ..., l_{c,1}. \ldots, i_1 = 1, ..., l_{c,k}$$

where $M_i, i = 1, ..., l_c$ is defined in (5.29)

Chapter Summary

In this Chapter, the notion state information prediction set has been introduced and applied to the several classes of control strategies. It has been show that the use of bounded parameter variations complicates the construction of the prediction sets. Convexification approaches to relax the problem and arrive to convex formulations have been shown. Finally, the prediction set based control strategy proposed by Pluymers in [50] has been generalized, for the first time at the best of our knowledge, to the LPV case.

6

Constrained Control of LPV systems

In this Chapter we focus on the control problem for LPV systems and we show how it is possible to reformulate many constrained LPV control design methods as convex optimization problems solvable in polynomial time.

The Chapter is divided into two Sections. Both of them deal with constrained regulation: the first via the use of time-invariant control laws whereas the second by means of time-depending control strategies.

6.1 Time-invariant Control Laws

By exploiting the set invariance properties of Chapter 4, a quite general reformulation of the Constrained LPV design problem is given in the following Problem 6.1 in the case of unbounded parameter variations and for a general class of time-invariant control laws.

Problem 6.1. Let a LPV system (1.1) be given. Then, determine on the basis of the actual information vector $\xi(0)$ a time-invariant control law

$$u\left(k\right)=u(\xi(k))=u\left(x(k),p(k)\right),k\geq 0,$$

a Lyapunov function

$$V\left(\xi(k)\right),k\geq 0$$

and a robustly positive invariant set $\mathcal{E}$ such as the following conditions are satisfied

a) Lyapunov stability. $V\left(\xi\left(k+1\right)\right)-V\left(\xi\left(k\right)\right)\leq 0, \quad k\geq 0$
b) The initial state $x\left(0\right)$ belongs to the invariant set, i.e. $x(0)\in\mathcal{E}$
c) $\left\{u\in\Re^{m}\right|u=u\left(x,p\right),\forall x\in\mathcal{E},\forall p\in\Sigma_{l}\}\subseteq U$
d) $\{x^{+}\in\Re^{n}\left|x^{+}=\left(A\left(p\right)+B\left(p\right)F\left(p\right)\right)x,.\ \forall x\in\mathcal{E}\right.,\forall p\in\Sigma_{l}\}\subseteq X$
e) A convenient upper-bound to the cost $J(x(k),u(\cdot))$ defined in (4.9) is minimized.

In the next subsections we will show how the above conditions can be translated into several convex conditions by making use of the Lyapunov function machineries introduced in Chapter 3. While the reformulations for the case of standard quadratic Lyapunov functions are well known in literature (see [47] and [48]), workable derivations for Parameter Varying Lyapunov functions have received less attention and an approach presented in [45] is detailed. Finally, novel and original design methods achievable by the use of nonstandard Lyapunov functions are presented here, at the best of our knowledge, for the first time.

In order to proceed systematically the following preliminary result, common to all the approaches that will be presented, is introduced.

Theorem 6.2. *If parameter-depending state-feedback control laws*

$$u\left(\xi(k)\right) = F\left(p(k)\right)x(k), k \geq 0 \tag{6.1}$$

and Lyapunov functions

$$V\left(\xi\left(k\right)\right) = x^T\left(k\right)V\left(p\left(k\right)\right)x\left(k\right), k \geq 0 \tag{6.2}$$

are employed, then providing a solution to Problem 6.1 consists in determining (if exist) a parameter dependent feedback matrix $F(p) \in \Re^{n\times m}$, a parameter dependent square matrix $V(p) \in \Re^{n\times n}$, a square matrix $X \in \Re^{m\times m}$ and a positive scalar γ that solve the following optimization problem

$$\min_{\gamma,V(\cdot),F(\cdot),X} \gamma$$

subject to matrix inequalities

$$\begin{aligned} &\left[A\left(p\right)+B\left(p\right)F\left(p\right)\right]^T V\left(p^+\right)\left[A\left(p\right)+B\left(p\right)F\left(p\right)\right]-V\left(p\right)\leq \\ &\qquad \leq -\left[R_x+F\left(p\right)^T R_u F\left(p\right)\right],\ \forall p\in\Sigma_l, \forall p^+\in\Sigma_l \end{aligned} \tag{6.3}$$

$$V\left(p\right) > 0, \forall p \in \Sigma_l \tag{6.4}$$

$$x^T\left(0\right)V\left(p\left(0\right)\right)x\left(0\right) \leq \gamma, \forall p \in \Sigma_l \tag{6.5}$$

$$\begin{aligned} &\left(F\left(p\right)\left(\tfrac{1}{\gamma}V\left(p\left(0\right)\right)\right)^{-1}F\left(p\right)^T\right) \leq X,\ \ \forall p\in\Sigma_l \\ &X>0,\ \ X_{jj}\leq u_{j,\max}^2\ \ j=1,...,m \end{aligned} \tag{6.6}$$

$$\begin{aligned} &\left(C_i\left[A\left(p\right)+B\left(p\right)F(p)\right]\left(\tfrac{1}{\gamma}V\left(p\left(0\right)\right)\right)^{-1}\left[A^t\left(p\right)+F^T(p)P^T(p)\right]C_i^T\right)\leq y_{i,\max}^2 \\ &\qquad\qquad i=1,\ldots,n_y,\ \ \forall p\in\Sigma_l \end{aligned} \tag{6.7}$$

Proof. We need to prove that the above conditions (6.3)-(6.7) satisfy a),b),c),d),e) in Problem 6.1.
Conditions a) b) and e)
Let (6.4) and the following inequality

$$\begin{aligned}&x\left(p\left(k+1\right)\right)^{T}V\left(p\left(k+1\right)\right)x\left(p\left(+1\right)\right)-x\left(p\left(k\right)\right)^{T}V\left(p\left(\right)\right)x\left(p\left(k\right)\right)\leq\\&-\left[x^{T}\left(k\right)R_{x}x\left(k\right)+x^{T}\left(k\right)F^{T}\left(p\left(k\right)\right)R_{u}F\left(p\left(k\right)\right)x\left(k\right)\right]\\&\forall p\left(k\right)\in\Sigma_{l},\forall p\left(k+1\right)\in\Sigma_{l}\end{aligned}\tag{6.8}$$

hold true. $R_u > 0, R_x > 0$, (6.4) and (6.8) are sufficient conditions for asymptotical stability. Then, because $\lim_{k\to\infty} x(k) = 0$, if we sum all the terms of (6.8) for $i = 0, \ldots, \infty$ we obtain

$$\max_{p(k)\in\Sigma_l}\sum_{k=0}^{\infty}\|x(k)\|_{R_x}^2+\|u(k)\|_{R_u}^2<x(0)^T V(p(0))\,x(0)\tag{6.9}$$

Then, $x(0)^T V(p(0))\,x(0)$ is an upper-bound to the cost (4.9). By introducing a slack variable γ, the minimization of such an upper-bound is achieved by imposing that

$$\min\gamma\tag{6.10}$$

Subject to matrix inequalities (6.3), (6.4), (6.5)

where (6.3) is obtained by direct manipulation of (6.8). Finally, note that condition (6.5) coincides with the state inclusion into the invariant set $\mathcal{E} = \{x \in \Re^n | x^T V(p(0))\, x \leq \gamma\}$ defined by the Lyapunov function.

Condition c)
If the set $\mathcal{E}$ is a robustly positive invariant set for the closed-loop system and $x(0) \in \mathcal{E}$, then the following inequalities hold true

$$\begin{aligned}&\max_{k\geq0}\|u_j(k|0)\|^2=\max_{k\geq0}\left\|\left(F\left(p\left(k\right)\right)x\left(k\right)\right)_j\right\|^2\leq\\&\leq\max_{\substack{x\,:\,x^T\left(\frac{1}{\gamma}V\left(p\left(0\right)\right)\right)x\,<\,1\\p\,\in\,\Sigma_l}}\left\|\left(F\left(p\right)x\right)_j\right\|^2\end{aligned}\tag{6.11}$$

Via simple manipulations we also obtain

$$\begin{aligned}&\max_{k\geq0}\|u_j(k|0)\|^2\leq\\&\leq\max_{\substack{x\,:\,x^T\left(\frac{1}{\gamma}V\left(p\left(0\right)\right)\right)x\,<\,1\\p\,\in\,\Sigma_l}}\left\|\left(F\left(p\right)\left(\tfrac{1}{\gamma}V\left(p\left(0\right)\right)\right)^{-1/2}\left(\tfrac{1}{\gamma}V\left(p\left(0k\right)\right)\right)^{1/2}x\right)_j\right\|^2\leq\\&\max_{p\in\Sigma_l}\left\|\left(F\left(p\right)\left(\tfrac{1}{\gamma}V\left(p\left(0\right)\right)\right)^{-1/2}\right)_j\right\|_2^2=\\&=\left(F\left(p\right)\left(\tfrac{1}{\gamma}V\left(p\left(0\right)\right)\right)^{-1}F^T\left(p\right)\right)_{jj}\leq u_{j,\max}^2\;\;p\in\Sigma_l\end{aligned}\tag{6.12}$$

If matrix $X = X^T > 0$ is introduced such that $X_{jj} \leq u_{j,\max}^2$, then we achieve the conditions (6.6) ensuring

$$\max_{i\geq 0}\|u_j(k|0)\|^2 \leq y^2_{j_{\max}},\ j=1,...,m\ \ k=0,1,2,....$$

Condition d)
The state constraints we are considering here are in the form

$$\|(Cx(k))_i\|_2 \leq y_{i,\max},\quad k=1,2,...,\infty,\quad i=1,...,n_y \tag{6.13}$$

Let us focus on the term

$$(Cx(k))_i = \left(\begin{bmatrix} C_1 \\ ... \\ C_p \end{bmatrix} x(k)\right)_i = C_i x(k) \tag{6.14}$$

where $C_i \in \Re^{1\times n}$ is a vector. Inequality (6.13) can be rewritten as

$$\|C_i x(k)\|_2^2 \leq y^2_{i,\max},\quad k=1,2,...,\infty,\ \ i=1,...,n_y \tag{6.15}$$

Let us consider a single constraint. If we shift k, (6.15) can be rewritten as

$$\begin{aligned}\|C_i[A(p(k)) + B(p(k))F(p(k))]x(k)\|_2^2 \leq y^2_{i,\max}\\ k=0,1,...,\infty.\end{aligned} \tag{6.16}$$

Because $x(k)\in\mathcal{E},\quad k=0,1,\ldots,\infty,$ we can use the same procedure as in the input case

$$\begin{aligned}
&\max_{k\geq 0}\|u_j(k|0)\|^2 \\
&\leq \max_{x:x^TV(p(0))x<\gamma}\|C_i[A(p(k)) + B(p(k))F(p(k))]x\|_2^2 \\
&\leq \max_{\substack{x\,:\,x^T\left(\frac{1}{\gamma}V(p(0))\right)x<1\\ p\,\in\,\Sigma_l}}\|C_i[A(p) + B(p)F(p)]x\|^2 = \\
&= \max_{\substack{x\,:\,x^T\left(\frac{1}{\gamma}V(p(0))\right)x<1\\ p\,\in\,\Sigma_l}}\left\|\left(C_i[A(p)+B(p)F(p)]\left(\tfrac{1}{\gamma}V(p(0))\right)^{-1/2}\left(\tfrac{1}{\gamma}V(p(0))\right)^{1/2}x\right)_j\right\|^2 \leq \\
&\max_{p\in\Sigma_l}\left\|C_i[A(p) + B(p)F(p)]\left(\tfrac{1}{\gamma}V(p(k))\right)^{-1/2}\right\|_2^2 = \\
&\left(C_i[A(p)+B(p)F(p)]\left(\tfrac{1}{\gamma}V(p(k))\right)^{-1}\left[A^T(p) + F^T(p)P^T(p)\right]C_i^T\right) \leq y^2_{i,\max} \\
&\hfill \forall p\in\Sigma_l.
\end{aligned}$$

Remark 6.3. It is important to note that the above conditions are considered for invariant sets in the form (4.49). Often, for technical reasons, we will make use of the invariant sets as in (4.52). In such a case, the above conditions can be simply translated by substituting $p(0)$ with $p, \forall p\in\Sigma_l$.

Hereafter, we customize the above general results for the various specific Lyapunov functions of interest. Because from any of the approaches we present in the sequel it is possible to derive different formulations, we will present first a quite general prototype design method, not necessarily convex, which will be successively customized by adding further specifications on the form and structure of the selected control laws and Lyapunov functions. As a result, several convex formulations of control design problems will be obtained and presented in form of Lemmas.

6.1.1 Quadratic Lyapunov Functions

If quadratic Lyapunov functions

$$V\left(\xi(k)\right)) = x^T(k)Px(k), k \geq 0 \tag{6.17}$$

and scheduled state-feedback control laws

$$u(k) = F(p(k))x(k), k \geq 0 \tag{6.18}$$

are employed, where $F(p(k))$ depends somehow, not necessarily in a linear or affine way, on the parameter vector, then Theorem 6.2 translates into the following result

Theorem 6.4. *The control design problem underlying Theorem 6.2 is solvable, for control laws (6.18) and Lyapunov functions (6.17), if exist a square matrix $Q \in \Re^{n\times n}$, a parameter dependent matrix $Y(p) \in \Re^{m\times n}$ depending on the parameter vectors $p \in \Sigma_l$, a square matrix $X \in \Re^{m\times m}$ and a scalar γ which jointly solve the following not necessarily convex optimization problem*

$$\min_{\gamma, Q, Y(\cdot), X} \gamma \tag{6.19}$$

subject to matrix inequalities

$$\begin{bmatrix} Q & * & * & * \\ A(p)P^{-1} + B(p)Y(p) & Q & * & * \\ R_x^{1/2}Q & 0 & \gamma I & * \\ R_x^{1/2}Y(p) & 0 & 0 & \gamma I \end{bmatrix} \geq 0, \ \ \forall p \in \Sigma \qquad Q > 0 \tag{6.20}$$

$$\begin{bmatrix} 1 & x(0)^T \\ x(0) & Q \end{bmatrix} > 0 \tag{6.21}$$

$$\begin{bmatrix} X & Y(p) \\ Y^T(p) & Q \end{bmatrix} > 0, \ \ p \in \Sigma_l \qquad X > 0, \ \ X_{jj} \leq u_{j,\max}^2 \ \ j = 1, ..., m \tag{6.22}$$

$$\begin{bmatrix} y_{i,\max}^2 & C_i\left[A(p)Q + B(p)Y(p)\right] \\ * & Q \end{bmatrix} > 0, \ \ p \in \Sigma_l, i = 1, ..., n_y \tag{6.23}$$

where $Q = \gamma P^{-1}$ and $Y(p) = F(p)Q$

Proof. The proof is divided in three parts.
1. Condition (6.20)
If $V(p(k)) = P$, (6.3) is given by

$$P - \Phi^T(p) P \Phi(p) - R_x - F(p)^T R_u F(p) \geq 0 \tag{6.24}$$

where $\Phi(p) = A(p) + B(p)F(p)$. By Schur transformations, it becomes

$$\begin{bmatrix} P - \Phi^T(p) P \Phi(p) - R_x & F(p)^T R_u^{1/2} \\ R_u^{1/2} F(p) & I \end{bmatrix} \geq 0 \tag{6.25}$$

Because the term $P - \Phi^T(p) P \Phi(p) - R_x$ has to be positive definite, by iteratively applying Schur complements one arrives to

$$\begin{bmatrix} P & * & * & * \\ \Phi(p) & P^{-1} & * & * \\ R_x^{1/2} & 0 & I & * \\ R_x^{1/2} F(p) & 0 & 0 & I \end{bmatrix} \geq 0 \tag{6.26}$$

Finally by the congruence transformation $diag\{\gamma^{1/2}P^{-1}, \gamma^{1/2}I, \gamma^{1/2}I, \gamma^{1/2}I\}$, we obtain (6.20).

2. Condition (6.21)
Equation (6.5) can be written as

$$x(0)^T P x(0) \leq \gamma \tag{6.27}$$

which finally becomes the LMI (6.21) by Schur complements

3. Conditions (6.22) and (6.23)
Equation (6.6) can be rewritten

$$\left(F(p) \left(\frac{1}{\gamma} P \right)^{-1} F^T(p) \right) \leq X, \, p \in \Sigma_l \tag{6.28}$$

Then, via Schur complements

$$\begin{bmatrix} X & F(p) \\ F^T(p) & \frac{1}{\gamma} P \end{bmatrix} > 0, \quad p \in \Sigma_l. \tag{6.29}$$

Finally, by means of the congruence transformation $diag\left\{I, \gamma P^{-1}\right\}$, (6.22) is obtained. By following the same lines, also (6.23) is obtained.

Notice that if a nonscheduled control laws

$$u(k) = Fx(k), k \geq 0$$

is chosen, the above problem become an LMIs optimization problem. Otherwise, if a scheduled control laws is used, relaxation methods shown in Appendix A can be used to obtain convex formulations. As an example, the following convex optimization problem is achieved by using the Half-sum convexification procedure

Lemma 6.5. *Let the initial state $x(0)$ be given and a scheduled control law in the form*

$$u(k) = \sum_{i=1}^{l} p_i(k) F_i x(k), k > 0 \tag{6.30}$$

be chosen. Then, the control design method of Theorem 6.4 can be relaxed into the following LMI optimization problem to be solved by determining, if exist, a square matrix $Q \in \Re^{n\times n}$, a set of matrices $Y_i \in \Re^{n\times m}, i = 1, ..., l$, a square matrix $X \in \Re^{m\times m}$ and a scalar γ such that

$$\min_{\gamma,Q,Y_i,X,i=1,...,l} \gamma \tag{6.31}$$

subject to matrix inequalities

$$\begin{bmatrix} Q & * & * & * \\ \frac{A_i+A_j}{2}Q + \frac{B_iY_j+B_jY_i}{2} & Q & * & * \\ R_x^{1/2}Q & 0 & \gamma I & * \\ R_x^{1/2}\frac{Y_i+Y_j}{2} & 0 & 0 & \gamma I \end{bmatrix} \geq 0 \begin{array}{l} i = 1,\ldots,l \\ j = i,\ldots,l \end{array} \tag{6.32}$$

$$Q > 0$$

$$\begin{bmatrix} 1 & x(0)^T \\ x(0) & Q \end{bmatrix} > 0 \tag{6.33}$$

$$\begin{bmatrix} X & Y_i \\ Y_i^T & Q \end{bmatrix} > 0, \; i = 1, ..., l$$
$$X > 0, \; X_{ww} \leq u_{w,\max}^2 \; w = 1, ..., m \tag{6.34}$$

$$\begin{bmatrix} y_{s,\max}^2 & C_s\left[\frac{A_i+A_j}{2}Q + \frac{B_iY_j+B_jY_i}{2}\right] \\ * & Q \end{bmatrix} > 0 \begin{array}{l} i = 1,\ldots,l \\ j = i,\ldots,l \\ s = 1,\ldots,n_y \end{array} \tag{6.35}$$

where $Q = \gamma P^{-1}$ and $Y_i = F_iQ$

Proof. This is directly achievable by using Half-sum convexification procedure.

Remark 6.6. Because, as reported in Appendix A, there are many ways to relax the quadratic dependence on the parameter, many other possible convex reformulations are possible. Moreover, at least in principle, other classes of scheduled control laws could be used. However, this freedom does not seem to yield to interesting results.

6.1.2 Parameter Dependent Lyapunov Functions

As a first result of this Section we show how it is possible to formulate Theorem (6.2) in the case that we make use of a Parameter Dependent Lyapunov function, i.e. a Lyapunov function in form (6.2) with

$$V(p(k)) = P(k) = \sum_{i=1}^{l} p_i(k) P_i \tag{6.36}$$

and a scheduled control law (6.18) is chosen.

Theorem 6.7. *The control design problem underlying Theorem 6.2 is solvable, for control laws (6.18) and Lyapunov functions $V(x(k)) = x^T(k)V(p(k)x(k)$ with $V(p(k))$ as in (6.36), if exist square matrices $P_i \in R^{n\times n}, i = 1,...,l, G \in \Re^{n\times n}$, a parameter dependent state feedback matrix $F(p) \in \Re^{n\times m}$, a square matrix $X \in \Re^{m\times m}$ and a scalar γ which jointly solve the following not necessarily convex optimization problem*

$$\min_{\gamma, P_1,...,P_l, F(\cdot), G, X} \gamma$$

subject to matrix inequalities

$$\begin{bmatrix} G^T P(p) G & * & * & * \\ A(p) G + B(p) F(p) G & P_i^{-1} & * & * \\ R_x^{1/2} G & 0 & I & * \\ R_u^{1/2} F(p) G & 0 & 0 & I \end{bmatrix} \geq 0, \quad \forall p \in \Sigma, i = 1,...,l$$
$$P_i > 0 \quad i = 1,...,l \tag{6.37}$$

$$\begin{bmatrix} \gamma_i & x(0)^T \\ x(0) & \frac{1}{p_i(0)} P_i^{-1} \end{bmatrix} > 0 \quad i = 1,...,l \tag{6.38}$$

$$\sum_{i=1}^{l} \gamma_i \leq \gamma \tag{6.39}$$

$$\begin{bmatrix} X & F(p) G \\ G^T F^T(p) G^T & \left(\frac{1}{\gamma} P(p(0))\right) G \end{bmatrix} > 0, \quad \forall p \in \Sigma_l$$
$$X > 0, \;\; X_{jj} \leq u_{j,\max}^2 \;\; j = 1,...,m \tag{6.40}$$

$$\begin{bmatrix} y_{i,\max}^2 & [C_i [A(p) + B(p) F(p)]] G \\ * & G^T \left(\frac{1}{\gamma} P(p(0))\right) G \end{bmatrix} > 0, \quad \forall p \in \Sigma_l, i = 1,...,n_y \tag{6.41}$$

Proof. The proof is divided in three parts.
1. Condition (6.37)
Matrix inequality (6.37) can be obtained by applying iteratively Schur complements to (6.3) and by using the congruence transformation $diag\{G, I, I, I\}$.

2. Condition (6.38) and (6.39)
Matrix inequality (6.5) becomes

$$x(0)^T P(p(0))\, x(0) = \sum_{i=1}^{l} p_i(0) \left(x(0)^T P_i x(0) \right) \leq \gamma. \qquad (6.42)$$

By introducing slack variables $\gamma_i, i = 1, ..., l$, the latter inequality can be reformulated as follows.

$$p_i(0) \left(x(0)^T P_i x(0) \right) \leq \gamma_i \qquad (6.43)$$

$$\sum_{i=1}^{l} \gamma_i \leq \gamma \qquad (6.44)$$

Finally by using Schur complements (6.38) is achieved.

3. Conditions (6.40) and (6.41)
Because $V(p)$ assumes form (6.36), matrix inequality (6.6) becomes

$$\left(F(p) \left(\frac{1}{\gamma} P(p(0)) \right)^{-1} F^T(p) \right) \leq X. \qquad (6.45)$$

By applying Schur complements we have:

$$\begin{bmatrix} X & F(p) \\ F^T(p) & \frac{1}{\gamma} P(p(0)) \end{bmatrix} > 0, \;\; p \in \Sigma_l \qquad (6.46)$$

Finally, by using the congruence transformation $diag\{I, \gamma G\}$ inequality (6.40) is obtained. After similar lines used above we arrive to (6.41) from (6.7).

Note that, once a control law is chosen, to obtain convex optimization formulations we need to relax $G^T P(p)\, G$. A possible way to do that is via the Dilation result presented in Lemma 3.4. Moreover, the quadratic parameter dependencies $B(p)\, F(p)$ have to be relaxed too.
As an example, if a control law of the form

$$u(k) = \sum_{i=1}^{l} p_i(k) F_i x(k), k > 0 \qquad (6.47)$$

is selected, the control design method of Theorem 6.7 can be relaxed into the following quasi-LMI optimization problem

Lemma 6.8. *Let the initial state $x(0)$ be given and a scheduled control law in the form (6.47) be chosen. Then, the control design method of Theorem 6.7 can be relaxed into the following quasi-LMI optimization problem to be solved by determining, if exist, square matrices $Q_i \in \Re^{n \times n}, i = 1, ..., l, G \in \Re^{n \times n}$, a* *set of matrices* $Y_i \in \Re^{n\times m}, i = 1,...,l,$ *a square matrix* $X \in \Re^{m\times m}$ *and scalars* $\gamma_i \in \Re, i = 1,...,l, \gamma \in \Re$ *such that*

$$\min_{\substack{\gamma,\,\gamma_i,\,Q_i,\,Y_i,\,G\\ i=1,\dots,l}} \gamma$$

Subject to

$$\begin{bmatrix} G^T + G - \frac{Q_i+Q_j}{2} & * & * & * \\ \frac{A_i+A_j}{2}G + \frac{B_iY_j+B_jY_i}{2} & Q_s & * & * \\ R_x^{1/2}G & 0 & I\gamma & * \\ R_x^{1/2}\frac{F_i+F_j}{2}G & 0 & 0 & I\gamma \end{bmatrix} \geq 0, \quad \begin{array}{l} i=1,...,l \\ j=i,...,l \\ s=1,...,l \end{array} \tag{6.48}$$
$$Q_i > 0 \quad i = 1,...,l$$

$$\begin{bmatrix} \gamma\gamma_i & \gamma x(0)^T \\ \gamma x(0) & \frac{1}{p_i(0)}Q_i \end{bmatrix} > 0 \quad i = 1,...,l \tag{6.49}$$

$$\sum_{i=1}^{l} \gamma_i \leq \gamma \tag{6.50}$$

$$\begin{bmatrix} X & Y_i \\ Y_i^T & G^T + G - \sum_{s=1}^{l} p_s(0)\,Q_s \end{bmatrix} > 0, \quad i = 1,..,l \tag{6.51}$$
$$X > 0, \quad X_{jj} \leq u_{j,\max}^2 \quad j = 1,...,m$$

$$\begin{bmatrix} y_{w,\max}^2 & \left[C_w\left[\frac{A_i+A_j}{2}G + \frac{B_iY_j+B_jY_i}{2}\right]\right] \\ * & G^T + G - \sum_{s=1}^{l} p_s(0)\,Q_s \end{bmatrix} > 0, \quad \begin{array}{l} i=1,...,l \\ j=i,...,l \\ w=1,...,n_y \end{array} \tag{6.52}$$

where

$$Q_i = \gamma P_i^{-1}, Y_i = F_i G$$

Proof. Consider inequality (6.37). By using the congruence transformation

$$diag\left\{\gamma^{-1/2}, \gamma^{1/2}, \gamma^{1/2}, \gamma^{1/2}\right\}$$

and the Half-sum convexification procedure we obtain

$$\begin{bmatrix} \frac{1}{2}G^T\gamma^{-1}P_iG + \frac{1}{2}G^T\gamma^{-1}P_jG & * & * & * \\ \frac{A_i+A_j}{2}G + \frac{B_iF_j+B_jF_i}{2}G & \gamma P_s^{-1} & * & * \\ R_x^{1/2}G & 0 & I\gamma & * \\ R_u^{1/2}\frac{F_i+F_j}{2}G & 0 & 0 & I\gamma \end{bmatrix} \geq 0, \quad \begin{array}{l} i=1,...,l \\ j=i,...,l \\ s=1,...,l \end{array}$$
$$P_i > 0 \quad i = 1,...,l$$

Then, by a direct application of the Dilation lemma, we obtain (6.48). Inequality (6.49) is derived by applying congruence transformation $diag\{\gamma^{1/2}, \gamma^{1/2}I\}$ to (6.38) Finally, the families of Matrix Inequalities (6.51), (6.52) can be trivially obtained by applying the Dilation lemma to (6.40) and (6.41), respectively.

Though solvable in polynomial time, the use of the above quasi-LMI can result in several numerical problems. A possible way to further simplify things and obtain an LMI formulation, is by making use of the invariant set (4.52) instead of (4.49), used in [45]. The following further result can be proved

Lemma 6.9. *Let the initial state $x(0)$ be given and a scheduled control law in the form (6.47) be chosen. Then, the control design method of Theorem 6.7 can be relaxed into the following LMI optimization problem to be solved by determining, if exist, square matrices $Q_i \in \Re^{n\times n}, i = 1, ..., l, G \in \Re^{n\times n}$, a set of matrices $Y_i \in \Re^{n\times m}, i = 1, ..., l$, a square matrix $X \in \Re^{m\times m}$ and a scalar γ such that*

$$\min_{\gamma, Q_i, Y_i, G} \gamma$$

Subject to

$$\begin{bmatrix} G^T + G - \frac{Q_i+Q_j}{2} & * & * & * \\ \frac{A_i+A_j}{2}G + \frac{B_iY_j+B_jY_i}{2} & Q_s & * & * \\ R_x^{1/2}G & 0 & I\gamma & * \\ R_u^{1/2}\frac{F_i+F_j}{2}G & 0 & 0 & I\gamma \end{bmatrix} \geq 0, \quad \begin{matrix} i = 1, ..., l \\ j = i, ..., l \\ s = 1, ..., l \end{matrix} \tag{6.53}$$

$$Q_i > 0 \quad i = 1, ..., l$$

$$\begin{bmatrix} 1 & x(0)^T \\ x(0) & Q_i \end{bmatrix} > 0 \quad i = 1, ..., l \tag{6.54}$$

$$\begin{bmatrix} X & Y_i \\ Y_i^T & G^T + G - Q_i \end{bmatrix} > 0, \quad i = 1, .., l$$
$$X > 0, \;\; X_{jj} \leq u_{j,\max}^2 \;\; j = 1, ..., m \tag{6.55}$$

$$\begin{bmatrix} y_{w,\max}^2 & \left[C_w\left[\frac{A_i+A_j}{2}G + \frac{B_iY_j+B_jY_i}{2}\right]\right] \\ * & G^T + G - \frac{Q_i+Q_j}{2} \end{bmatrix} > 0, \;\; , w = 1, ..., n_y \tag{6.56}$$

where

$$Q_i = \gamma P_i^{-1}, Y_i = F_iG$$

Proof. If we choose as a new invariant set

$$\mathcal{E} = \left\{x \in \Re^n | x^T P_i x \leq \gamma, \;\; i = 1, ..., l\right\}$$

it is enough to note that (6.38) becomes

$$\begin{bmatrix} \gamma & x(0) \\ x(0) & P_i^{-1} \end{bmatrix} \geq 0, \; i = 1,...,l$$

Then, by a congruence transformation based on $diag\left\{\gamma^{-1/2}, \gamma^{1/2}I\right\}$ LMI (6.54) is obtained. Inequalities (6.55) and (6.55) result by following the same lines.

6.1.3 Nonstandard Lyapunov Functions - 1

In this subsection, the class of nonnstandad Lyapunov functions (6.2) are considered where

$$V\left(p\left(k\right)\right)=\left(\sum_{i=1}^{l}p_{i}\left(k\right)Q_{i}\right)^{-1}=\left(Q\left(p\left(k\right)\right)\right)^{-1} \tag{6.57}$$

Then, Theorem 6.2 becomes

Theorem 6.10. *The control design problem underlying Theorem 6.2 is solvable, for control laws (6.18) and Lyapunov functions* $V(x(k))=x^T(k)V(p(k)x(k)$ *such that* $V(p(k))$ *is as in (6.57), if exist square matrices* $Q_i\in\Re^{m\times m}, i=1,...,l$*, a parameter dependent state feedback matrix* $F(p)$ *depending on the parameter vector* $p\in\Sigma_l$*, a square matrix* $X\in Re^{m\times m}$ *and a scalar* γ *which jointly solve the following not necessarily convex optimization problem*

$$\min_{\gamma,Q(\cdot),F(\cdot),X}\gamma$$

subject to matrix inequalities

$$\begin{bmatrix} Q\left(p\right) & * & * & * \\ A\left(p\right)Q\left(p\right)+B\left(p\right)F\left(p\right)Q\left(p\right) & Q\left(p^{+}\right) & * & * \\ R_x^{1/2}Q\left(p\right) & 0 & I & * \\ R_x^{1/2}F\left(p\right)Q\left(p\right) & 0 & 0 & I \end{bmatrix}\geq 0,\ \ \forall p\in\Sigma,\forall p^{+}\in\Sigma \tag{6.58}$$

$$Q_i>0,\ \ i=1,...,l$$

$$\begin{bmatrix} \gamma & x\left(0\right)^T \\ x\left(0\right) & Q\left(p\left(0\right)\right) \end{bmatrix}>0 \tag{6.59}$$

$$\begin{bmatrix} X & \gamma F\left(p\right)Q\left(p\left(0\right)\right) \\ \gamma Q\left(p\left(0\right)\right)F^{T}\left(p\right) & \gamma Q\left(p\left(0\right)\right) \end{bmatrix}>0,\ \ \forall p\in\Sigma_l \tag{6.60}$$

$$X>0,\ \ X_{jj}\leq u_{j,\max}^2\ \ j=1,...,m$$

$$\begin{bmatrix} y_{i,\max}^2 & \gamma C_i\left[A\left(p\right)+B\left(p\right)F\left(p\right)\right]Q\left(p\left(0\right)\right) \\ * & \gamma Q\left(p\left(0\right)\right) \end{bmatrix}>0,\ \ \forall p\in\Sigma_l, i=1,...,q \tag{6.61}$$

Proof. This proof is divided in three parts.
1. Condition (6.58)
By substituting the prescribed Lyapunov function into (6.3)and by iteratively applying Schur complements the following inequality is reached

$$\begin{bmatrix} Q\left(p\right)^{-1} & * & * & * \\ A\left(p\right)+B\left(p\right)F\left(p\right) & Q\left(p^{+}\right) & * & * \\ R_x^{1/2} & 0 & I & * \\ R_x^{1/2}F\left(p\right) & 0 & 0 & I \end{bmatrix}\geq 0,\ \ \forall p\in\Sigma,\forall p^{+}\in\Sigma \tag{6.62}$$

By congruence transformation $diag\left\{Q\left(p\right),I,I,I\right\}$, the latter becomes (6.58).

2. Condition (6.59)
The inclusion of $x(0)$ into the invariant set can be written as

$$x(0)^T (Q(p(0)))^{-1} x(0) \leq \gamma \tag{6.63}$$

By applying Schur complements, inequality (6.59) results.

3. Conditions (6.60) and (6.61)
If the prescribed Lyapunov function is used, (6.6) becomes

$$F(p(0))\, \gamma Q(p(0))\, F^T(p) > 0, \;\; p \in \Sigma_l \tag{6.64}$$

By applying Schur complements, the latter is equivalent to

$$\begin{bmatrix} X & F(p) \\ F^T(p) & \frac{1}{\gamma} Q(p(0))^{-1} \end{bmatrix} > 0, \;\; p \in \Sigma_l \tag{6.65}$$

Finally, by making use of the congruence transformation $diag\{I, \gamma Q(p(0))\}$, inequality (6.60) is obtained. The same procedure enables us to obtain inequality (6.61).

Even if many way to relax the above non convex optimization problem can be applied, the most convenient way is by making use of the following control law

$$u(k) = \tilde{F}(p(k))\, Q(p(k))^{-1} x(k) \tag{6.66}$$

coupled with the more conservative invariant set (4.52). In this case the above problem can be relaxed into the following

Lemma 6.11. *Let the initial state $x(0)$ be given and a scheduled control law in the form (6.66) be chosen. Then, the control design method of Theorem 6.10 can be relaxed into the following LMI optimization problem to be solved by determining, if exist, square matrices $\bar{Q}_i \in \Re^{n \times n}, i = 1, ..., l$, a set of matrices $\bar{F}_i \in \Re^{n \times m}, i = 1, ...,$ a square matrix $X \in \Re^{m \times m}$ and a scalar γ such that*

$$\min_{\substack{\gamma, \bar{Q}_i, \bar{F}_i, X \\ i = 1, \dots, l}} \gamma \tag{6.67}$$

subject to matrix inequalities

$$\begin{bmatrix} \frac{\bar{Q}_i+\bar{Q}_j}{2} & * & * & * \\ \frac{A_i\bar{Q}_j+A_j\bar{Q}_i+B_i\bar{F}_j+B_j\bar{F}_i}{2} & \bar{Q}_s & * & * \\ R_x^{1/2}\frac{\bar{Q}_i+\bar{Q}_j}{2} & 0 & I & * \\ R_x^{1/2}\frac{\bar{F}_i+\bar{F}_j}{2} & 0 & 0 & I \end{bmatrix} \geq 0, \begin{array}{r} i=1,...,l \\ j=i,...,l \\ s=1,...,l \end{array} \qquad (6.68)$$
$$\bar{Q}_i > 0, \quad i=1,...,l$$

$$\begin{bmatrix} 1 & x(0)^T \\ x(0) & \bar{Q}_i \end{bmatrix} > 0, \quad i=1,...,l \qquad (6.69)$$

$$\begin{bmatrix} X & \bar{F}_i \\ * & \bar{Q}_i \end{bmatrix} > 0, \quad i=1,\dots,l$$
$$X > 0, \quad X_{jj} \leq u_{j,\max}^2 \;\; j=1,...,m \qquad (6.70)$$

$$\begin{bmatrix} y_{w,\max}^2 & \gamma C_w \left[\frac{A_i\bar{Q}_j+A_j\bar{Q}_i+B_i\bar{F}_j+B_j\bar{F}_i}{2}\right] \\ * & \frac{\bar{Q}_i+\bar{Q}_j}{2} \end{bmatrix} > 0, \begin{array}{r} i=1,...,l \\ j=i,...,l \\ w=1,...,n_y \end{array} \qquad (6.71)$$

where $\bar{F}_i = \gamma\tilde{F}_i$, $\bar{Q}_i = \gamma Q_i$

Proof. If the invariant set (4.52) is employed, then the dependencies on $p(0)$ in Theorem 6.10 have to be substituted with $p, \forall p \in \Sigma_l$.
Because of the use of control laws (6.66), terms $F(p)Q(p)$ simplifies into

$$F(p)Q(p) = \tilde{F}(p)(Q(p))^{-1}Q(p) = \tilde{F}(p).$$

By substituting the latter into (6.59)-(6.61) a quadratic dependence on the parameter vector results. Then, by applying Semi-sum convexifications and opportune congruence transformations, inequalities (6.68)-(6.71) follow.

Remark 6.12. Exactly like in the other cases, other convexifications may apply by using, for instance, different quadratic relaxations as those reported in Appendix A.

6.1.4 Nonstandard Lyapunov Function - 2

If a Lyapunov function (6.2) such that

$$\begin{array}{l} V(p(k)) = \left(\sum_{i=1}^{l} p_i G_i\right)^{-T} \left(\sum_{i=1}^{l} p_i P_i\right) \left(\sum_{i=1}^{l} p_i G_i\right)^{-1} = \\ = (G(p(k)))^{-T}(P(p(k)))(G(p(k)))^{-1} \end{array} \qquad (6.72)$$

is employed then the following holds true

Theorem 6.13. *The control design problem underlying Theorem 6.2 is solvable, for control laws (6.18) and Lyapunov functions* $V(x(k)) = x^T(k)V(p(k)x(k)$ *such that* $V(p(k))$ *is (6.72), if exist square matrices* $G_i \in \Re^{n\times n}, P_i \in \Re^{n\times n}$*, a parameter dependent state feedback matrix* $F(p)$ *depending on the parameter* *vector* $p \in \Sigma_l$*, a square matrix* $X \in \Re^{m \times m}$ *and a scalar* γ *which jointly solve the following not necessarily convex optimization problem*

$$\min_{\gamma, P(\cdot), G(\cdot), F(\cdot), X} \gamma \tag{6.73}$$

$$rank\left\{G_i\right\} = n, \quad i = 1, ..., l \tag{6.74}$$

$$P_i > 0, \;\; i = 1, ..., l \tag{6.75}$$

$$\begin{bmatrix} P(p) & * & * \; * \\ A(p)G(p) + B(p)F(p)G(p) & \left(G(p^+)\right)^T \left(P(p^+)\right)^{-1} G(p^+) & * \; * \\ R_x^{1/2} G(p) & 0 & I \; * \\ R_x^{1/2} F(p) G(p) & 0 & 0 \; I \end{bmatrix} \geq 0 \quad \forall p \in \Sigma, \forall p^+ \in \Sigma \tag{6.76}$$

$$\begin{bmatrix} \gamma & x(0)^T \\ x(0) & G^T(p(0)) P^{-1}(p(0)) G(p(0)) \end{bmatrix} > 0 \tag{6.77}$$

$$\begin{gathered} \begin{bmatrix} X & F(p) G(p(0)) \\ G^T(p(0)) F^T(p) & \frac{1}{\gamma} P(p(0)) \end{bmatrix} > 0, \;\; \forall p \in \Sigma_l \\ X > 0, \;\; X_{jj} \leq u_{j,\max}^2 \;\; j = 1, ..., m \end{gathered} \tag{6.78}$$

$$\begin{bmatrix} y_{i,\max}^2 & [C_i[A(p) + B(p)F(p)]]G(p(0)) \\ * & \frac{1}{\gamma} P(p(k)) \end{bmatrix} > 0, \forall p \in \Sigma_l, \;\; i = 1, ..., n_y \tag{6.79}$$

Proof. This proof is divided in four parts.
1. Condition (6.74) and (6.75)
By resorting to (6.4) we need to guarantee

$$V(p) = G^{-T}(p) P(p) G^{-1}(p) > 0, \forall p \in \Sigma_l.$$

The latter straightforwardly implies (6.74) and (6.75).

2. Condition (6.76)
Let substitute Lyapunov function (6.72) into (6.3). By iteratively apply Schur complements inequality (6.76) is obtained.

3. Condition (6.77)
The inclusion $x(0) \in \mathcal{E}$ is equivalent to

$$x(0)^T G(p(0))^{-T} P(p(0)) G(p(0))^{-1} x(0) \leq \gamma. \tag{6.80}$$

By applying Schur complements to the latter, inequality (6.77) is achieved.

4. Conditions (6.78) and (6.79)
By substituting Lyapunov function (6.72) into (6.6), we get the following inequality

$$\left(F\left(p\right)\left(\frac{1}{\gamma}\left(G\left(p\left(0\right)\right)\right)^{-T}\left(P\left(p\left(0\right)\right)\right)\left(G\left(p\left(0\right)\right)\right)^{-1}\right)^{-1}F\left(p\right)^{T}\right)\leq X,\;\; p\in\Sigma_{l} \tag{6.81}$$

By using Schur complement, the latter is equivalent to

$$\begin{bmatrix} X & F\left(p\right) \\ F^{T}\left(p\right) & \frac{1}{\gamma}G\left(p\left(0\right)\right)^{-T}P\left(p\left(0\right)\right)G^{-1}\left(p\left(0\right)\right)\end{bmatrix}>0,\;\;\forall p\in\Sigma_{l} \tag{6.82}$$

If we finally use the following congruence transformation

$$\begin{bmatrix} I & 0 \\ 0 & G^{T}\end{bmatrix}\begin{bmatrix} X & F\left(p\right) \\ F^{T}\left(p\right) & \frac{1}{\gamma}G\left(p\left(0\right)\right)^{-T}P\left(p\left(0\right)\right)G^{-1}\left(p\left(k\right)\right)\end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & G\end{bmatrix}>0,\;\;\forall p\in\Sigma_{l} \tag{6.83}$$

(6.78) is obtained. Inequalities (6.79) can be achieved by following the same lines.

There exists many way to turns the above optimization into a convex optimization problem. The fist step is , by making use of dilation Lemma 3.4, to relax $G^{T}(p)P(p)G(p)$ nonlinearities relaxed and the rank constraint (6.74). The following simpler (but still not convex) problem is obtained.

$$\min_{\gamma,P(\cdot),G(\cdot),F(\cdot),X}\gamma$$

$$\begin{bmatrix} \gamma P\left(p\right) & * & * & * \\ \gamma(A(p)\,G(p)\,\gamma+B(p)\,F(p)G\left(p\right)) & \gamma\left(G\left(p^{+}\right)^{T}+G\left(p^{+}\right)-P\left(p^{+}\right)\right) & * & * \\ R_{x}^{1/2}\gamma G\left(p\right) & 0 & I\gamma & * \\ R_{x}^{1/2}\gamma F\left(p\right)G\left(p\right) & 0 & 0 & I\gamma \end{bmatrix}\geq 0 \tag{6.84}$$

$$\forall p\in\Sigma,\forall p^{+}\in\Sigma$$

$$P_{i}>0,\;\; i=1,...,l$$

$$\begin{bmatrix} 1 & x\left(0\right)^{T} \\ x\left(0\right) & \gamma G^{T}\left(p\left(0\right)\right)+\gamma G\left(p\left(0\right)\right)-\gamma P\left(p\left(0\right)\right)\end{bmatrix}>0 \tag{6.85}$$

$$\begin{bmatrix} X & \gamma F\left(p\right)G\left(p\left(0\right)\right) \\ \gamma G^{T}\left(p\left(0\right)\right)F^{T}\left(p\right) & \gamma P\left(p\left(0\right)\right)\end{bmatrix}>0,\forall p\in\Sigma_{l} \tag{6.86}$$

$$X>0,\;\; X_{jj}\leq u_{j,\max}^{2}\;\; j=1,...,m$$

$$\begin{bmatrix} y_{i,\max}^{2} & \gamma\left[C_{i}\left[A(p)+B(p)F(p)\right]\!\right]G\left(p(0)\right) \\ * & \gamma P\left(p\left(0\right)\right)\end{bmatrix}>0,\forall p\in\Sigma_{l}, i=1,...,n_{y} \tag{6.87}$$

While many relaxation are possible we want to give two interesting way to proceed.

The first is obtained by making use of the following special control law

$$u\left(k\right)=\tilde{F}\left(p\left(k\right)\right)G\left(p\left(k\right)\right)^{-1}x\left(k\right) \tag{6.88}$$

coupled with the more conservative invariant set (4.52). It allows us to obtain

Lemma 6.14. *Let the initial state $x(0)$ be given and a scheduled control law in the form (6.88) be chosen. Then, the control design method of Theorem 6.13 can be relaxed into the following LMI optimization problem to be solved by determining, if exist, square matrices $\bar{P}_i \in \Re^{n\times n}, \bar{G}_i \in \Re^{n\times n} i = 1, ..., n$, a set of matrices $\bar{F}_i \in \Re^{n\times m}, i = 1, ...,$ a square matrix $X \in \Re^{m\times m}$ and a scalar γ such that*

$$\min_{\substack{\gamma,\, \bar{P}_i,\, \bar{G}_i,\, \bar{F}_i,\, X \\ i\,=\,1,\,\ldots,\,l}} \gamma$$

$$\begin{bmatrix} \frac{\bar{P}_i+\bar{P}_j}{2} & * & * & * \\ \frac{A_i\bar{G}_j+A_j\bar{G}_i+B_i\bar{F}_j+B_j\bar{F}_i}{2} & \bar{G}_s^T+\bar{G}_s-\bar{P}_s & * & * \\ R_x^{1/2}\frac{\bar{G}_i+\bar{G}_j}{2} & 0 & \gamma I & * \\ R_u^{1/2}\frac{\bar{F}_i+\bar{F}_j}{2} & 0 & 0 & \gamma I \end{bmatrix} \geq 0, \quad \begin{array}{l} i = 1, ..., l \\ j = i, ..., l, \\ s = 1, ..., l \end{array} \tag{6.89}$$

$$\bar{P}_i > 0, \;\; i = 1, ..., l$$

$$\begin{bmatrix} 1 & x(0)^T \\ x(0) & \bar{G}_i^T+\bar{G}_i-\bar{P}_i \end{bmatrix} > 0, \;\; i = 1, ..., l \tag{6.90}$$

$$\begin{bmatrix} X & \bar{F}_i \\ * & \bar{P}_i \end{bmatrix} > 0, \;\; i = 1, ..., l \qquad X > 0, \;\; X_{jj} \leq u_{j,\max}^2 \;\; j = 1, ..., m \tag{6.91}$$

$$\begin{bmatrix} y_{i,\max}^2 & C_w\left[\frac{A_i\bar{G}_j+A_j\bar{G}_i+B_i\bar{F}_j+B_j\bar{F}_i}{2}\right] \\ * & \frac{\bar{P}_i+\bar{P}_j}{2} \end{bmatrix} > 0, \;\; \begin{array}{l} i = 1, ..., l \\ j = i, ..., l \\ w = 1, ..., n_y \end{array} \tag{6.92}$$

where

$$\bar{G} = \gamma G, \bar{F}_i = \gamma\tilde{F}_i, \bar{P}_i = \gamma P_i, i = 1, ..., l$$

Proof. If the invariant set (4.52) is employed, then the dependencies on $p(0)$ in Theorem 6.10 have to be substituted with $p, \forall p \in \Sigma_l$.
Because of the use of control laws (6.88), terms $F(p)G(p)$ simplifies into

$$F(p)G(p) = \tilde{F}(p)(G(p))^{-1}G(p) = \tilde{F}(p).$$

By substituting the latter into inequalities (6.84)-(6.87) a simple quadratic dependence on the parameter vector results. Then, by simply applying Semi-sum convexifications, (6.68)-(6.71) are obtained.

The second method we want to propose here is using control law (6.88), and by imposing $G_1 = G_2 = ... = G_l = G$ that implies $G(p) = G$. Such an assumption allows us to use invariant set (4.49) in the following way

Lemma 6.15. *Let the initial state $x(0)$ be given and a scheduled control law in the form (6.88) be chosen. Then, the control design method of Theorem*

6.13 can be relaxed into the following LMI optimization problem to be solved by determining, if exist, square matrices $\bar{P}_i \in \Re^{n\times n}, i = 1, ..., n, \bar{G} \in \Re^{n\times n}$*, a set of matrices* $\bar{F}_i \in \Re^{n\times m}, i = 1, ...,$ *a square matrix* $X \in \Re^{m\times m}$ *and a scalar* γ *such that*

$$\min_{\substack{\gamma,\, \bar{P}_i,\, \bar{G},\, \bar{F}_i,\, X \\ i\,=\,1,\,...,\,l}} \gamma$$

$$\begin{bmatrix} \frac{\bar{P}_i+\bar{P}_j}{2} & * & * & * \\ \frac{A_i\bar{G}+A_j\bar{G}+B_i\bar{F}_j+B_j\bar{F}_i}{2} & \bar{G}^T+\bar{G}-\bar{P}_s & * & * \\ R_x^{1/2}\frac{\bar{G}+\bar{G}}{2} & 0 & \gamma I & * \\ R_u^{1/2}\frac{\bar{F}_i+\bar{F}_j}{2} & 0 & 0 & \gamma I \end{bmatrix} \geq 0, \quad \begin{matrix} i = 1, ..., l \\ j = i, ..., l, \\ s = 1, ..., l \end{matrix} \tag{6.93}$$

$$\bar{P}_i > 0, \;\; i = 1, ..., l$$

$$\begin{bmatrix} 1 & x(0)^T \\ x(0) & \bar{G}^T+\bar{G}-\bar{P}(p(0)) \end{bmatrix} > 0, \;\; i = 1, ..., l \tag{6.94}$$

$$\begin{bmatrix} X & \bar{F}_i \\ * & \bar{P}(p(0)) \end{bmatrix} > 0, \;\; i = 1, ..., l \qquad X > 0, \;\; X_{jj} \leq u_{j,\max}^2 \;\; j = 1, ..., m \tag{6.95}$$

$$\begin{bmatrix} y_{i,\max}^2 & C_w\left[\frac{A_i\bar{G}+A_j\bar{G}+B_i\bar{F}_j+B_j\bar{F}_i}{2}\right] \\ * & \bar{P}(p(0)) \end{bmatrix} > 0, \;\; \begin{matrix} i = 1, ..., l \\ j = i, ..., l \\ w = 1, ..., n_y \end{matrix} \tag{6.96}$$

where

$$\bar{G} = \gamma G_i, \bar{F}_i = \gamma \tilde{F}_i, \bar{P}_i = \gamma P_i, i = 1, ..., l$$

Proof. By substituting control law (6.88) into (6.84)-(6.87) and by assuming, $G_1 = G_2 = ... = G_l = G$, the following simplifications apply:

$$\begin{aligned} G(p(0)) &= G \\ F(p)G(p) &= \tilde{F}(p) \\ F(p)G(p(0)) &= \tilde{F}(p) \end{aligned}$$

then by simply resorting Semi-sum convexification, the statement is proved.

Example 6.16. In this Example we will compare four different approaches to constrained stabilizabiliy proposed in this Chapter. Namely we will compare the use of the LMI methods obtained in:

- Lemma 6.5 *via* Standard Quadratic Lyapunov Function,
- Lemma 6.9 *via* Parameter Varying Lyapunov Function,
- Lemma 6.11 *via* Nonstandard Lyapunov Function (6.57),
- Lemma 6.15 *via* Nonstandard Lyapunov Function (6.72).

Hereafter, for the sake of simplicity, we will refer to the above methods by means of the Lemmas in which they are defined.
The following two vertices LPV system is considered

$$x(k+1) = \sum_{i=1}^{2} p_i(k) A_i x(k) + \sum_{i=1}^{2} p_j(k) B_j u(k)$$

where

$$A_1 = \begin{pmatrix} 2 & -0.1 \\ 0.5 & 1 \end{pmatrix},\ B_1 = \begin{pmatrix} 1 \\ -0.3\beta \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 1 & 0.1 \\ 2.5 & 1 \end{pmatrix},\quad B_2 = \begin{pmatrix} 0.7 \\ 0.1 \end{pmatrix}.$$

The input signal is constrained to be

$$|u(k)| < 1, k = 0, ..., \infty$$

and weighting matrices $R_x = I, R_u = 1$ are introduced. In Figure 6.1 the

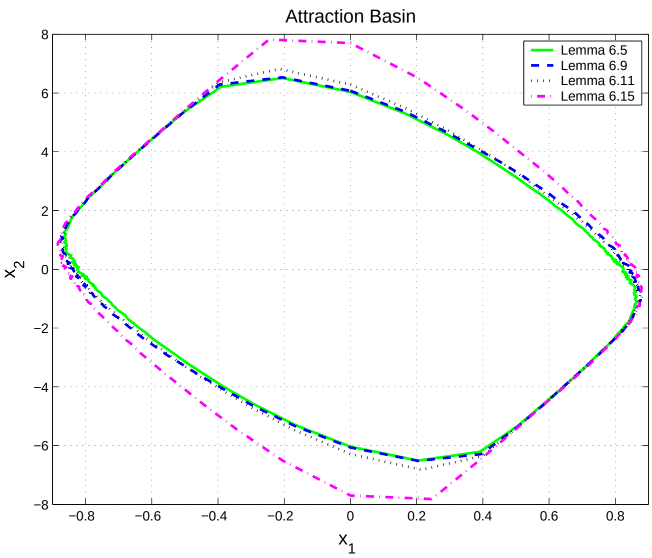

Fig. 6.1. Attraction Regions

attraction regions, i.e. the sets of states for which a feasible solution exists, are reported for the four methods under consideration. It results that the use of more complex constrained stabilizability conditions enlarge the attraction basin. Please note that Lemma 6.11 and Lemma 6.9 regions overlap. In fact, even if both of them can be proved to be less conservative then Lemma 6.5, it is not possible to state *a priori* which of them provides larger attraction regions. Lemma 6.15, instead, always outperform the other three approaches here presented. In the following Table the value of the upper bound to the cost γ^* for each method, starting from an initial point $x_0 = [0.1 - 5]^T$, is reported together with the number of LMI lines and scalar variables involved into the optimization procedure.

Method	γ^*	LMI Lines	Variables
Lemma 6.5	300.3496	$(l_c(3n+m)+2n+1+l(n+m))$	$n^2+lnm+1$
Lemma 6.9	287.3586	$l(l_c(3n+m)+2n+1+l(n+m))$	$(l+1)n^2+lnm+1$
Lemma 6.11	266.3646	$l(l_c(3n+m)+2n+1+l(n+m))$	$ln^2+lnm+1$
Lemma 6.15	226.1538	$l(l_c(3n+m)+2n+1+l(n+m))$	$(2l)n^2+lnm+1$

The above results show an evident improvement in terms of cost values. Such an improvement is payed at the cost of a slightly increased computational complexity. The improvements are evident also in Figure 6.2-6.4 where the state components dynamics and the input signal are shown.

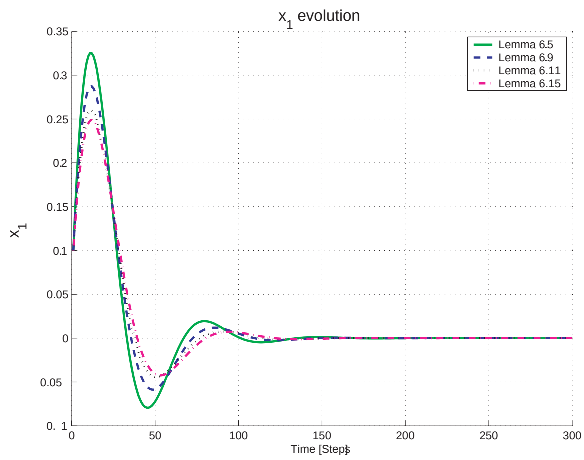

Fig. 6.2. x_1 Evolution

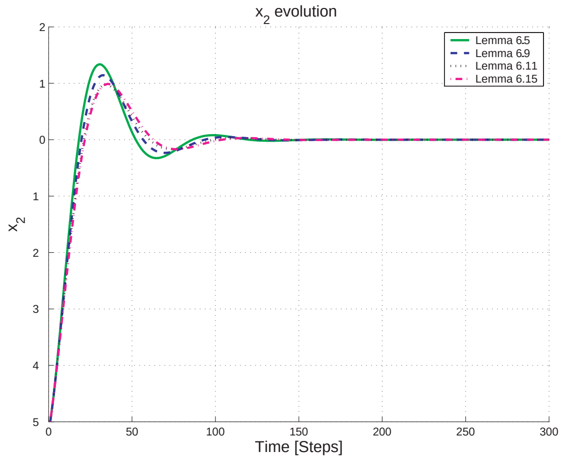

Fig. 6.3. x_2 Evolution

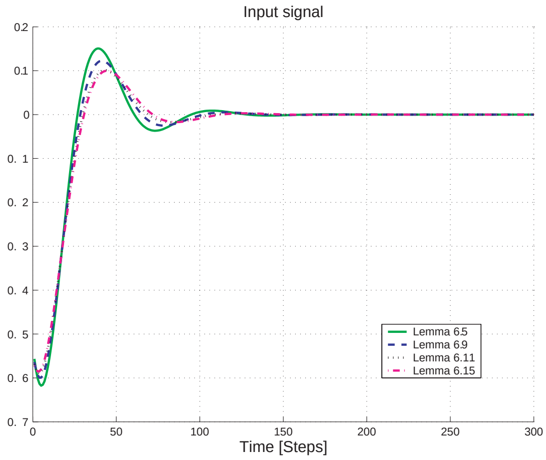

Fig. 6.4. Input Signal

6.2 Time-Varying Control Strategies

In Chapter 4, a specific formulation of the general constrained control problem for time-varying control strategies of the form

$$u(k) = u(\xi(k)) = \begin{cases} u^k(x(k, p(k))) & k = 0, ..., N-1 \\ u^N(x(k), p(k)) & k \geq N \end{cases} \tag{6.97}$$

has been discussed. It was pointed out that constrained control problems can be solved by ensuring that all states of the prediction set for the first $N-1$ steps always satisfy the constraints and all possible state predictions at time N belong to a (robust) positive invariant set $\mathcal{E}$.

On the basis of the above arguments, two possible approaches can be employed.

- When considering a **frozen terminal control law**, the input $u^N(x(k), p(k))$ and its associate positive invariant set $\mathcal{E}$ are a-priori known and only the first N moves of the control strategy have to be computed.
- If a **free terminal control law** is instead considered, the term $u^N(x(k), p(k)$ is an additional variable which has to be determined along with $u^k(x(k), p(k)), k = 0, ..., N-1$.

By recalling the above prediction sets definitions and when a frozen terminal control law approach is employed, a possible LPV constrained control design problem recasting is:

Problem 6.17. *Let a stabilizing terminal control laws $u^N(x(k), p(k))$ and the associated positive invariant set $\mathcal{E}$ be given. Then, on the basis of the actual information vector $\xi(0)$, compute the first N moves of a time-varying control strategy (6.97) so that*

$$\hat{X}(k|0) \subseteq X, \quad k = 1, ..., N-1 \tag{6.98}$$

$$\hat{U}(k|0) \subseteq U, \quad k = 0, ..., N-1 \tag{6.99}$$

$$\hat{X}(N|0) \subseteq \mathcal{E} \tag{6.100}$$

are satisfied and a suitable upper-bound to

$$J(x(0), u(\cdot)) = \max_{p(k)} \sum_{k=0}^{N-1} \|x(k+1)\|_{R_x}^2 + \|u(\xi(k))\|_{R_x}^2 \tag{6.101}$$

is minimized.

A convenient way to address free terminal control law algorithms is by resorting to previous Section results by means of the following reformulation

Problem 6.18. *Determine on the basis of the actual information vector $\xi(0)$, a time-varying control strategy in the form*

$$u(k) = u(\xi(k)) = \begin{cases} u^k(x(k), p(k)), & k = 0, ..., N-1 \\ F^N(p(k))\, x(k), & k \geq N \end{cases}$$

and a Lyapunov function

$$V(\xi(k)) = x(k)^T V(p(k))\, x(k), \quad k \geq N$$

such that the following conditions are satisfied

- *(6.98),(6.99),(6.100)*
- *(6.3),(6.4),(6.6),(6.7) if $p(0)$ is substituted with $\forall \hat{p}(k|0) \in \hat{P}(N|k)$*
- *An upper-bound to (6.101) is minimized.*

The terminal invariant set is given by

$$\mathcal{E} = \left\{ x \in \Re^n \,\middle|\, x^T V(\hat{p}(N|0))\, x < \gamma, \forall \hat{p}(N) \in \hat{P}(N|0) \right\}. \tag{6.102}$$

This approach differs from the frozen one only for the terminal control law and the invariant set structure which need to be recomputed together with the first N control actions at each instant. Because this feature can be obtained by simply adapting Section 6.1 results, we will hereafter focus on the frozen approach.

The remainder of the Section is organized as follows. First, some additional general properties of the prediction sets are investigated. Then, some of the strategies known in literature are selected and some novel strategies presented. At the end of the Chapter we will discuss in detail how a frozen terminal control law strategy can be converted into a free terminal control law.

6.3 Time-Varying Control– Prediction sets and convexity

Our goal is to obtain a strategy (6.97) such that (6.98),(6.99),(6.100) hold true and a suitable upper-bound to (6.101) is minimized.

Let us focus on the three set inclusions. Conditions (6.98),(6.99),(6.100) cannot be easily satisfied in general. However, under the following assumptions:

1. $\mathcal{E}$,X,U convex
2. $\hat{X}(k|0), \hat{U}(k|0)$ are polytopic sets

The above conditions can be rewritten as

$$x_v \in X, \quad \forall x_v \in vert\left\{\hat{\hat{X}}(k|0)\right\} \tag{6.103}$$

$$u_v \in U, \quad \forall u_v \in vert\left\{\hat{\hat{U}}(k|0)\right\} \tag{6.104}$$

$$x_v \in \mathcal{E}, \quad \forall x_v \in vert\left\{\hat{\hat{X}}(k|0)\right\} \tag{6.105}$$

To obtain *polynomial in time* computational machineries, it is necessary that the above conditions are convex with respect to the control strategy unknown coefficients to be computed. In order to guarantee such a property it is sufficient to use control strategies having the feature that each vertex of $\hat{\hat{X}}(k|0), \hat{\hat{U}}(k|0)$ depends linearly on the control strategy decision variables.

6.4 Time-Varying Control– One-step strategies

One of the first time-varying control strategy introduced characterized by a single move on a single step ($N = 1$) horizon. Such an input term is assumed to be a free vector, with no dependencies neither from the state nor the parameter vector

$$u(k) = u(\xi(k)) = \begin{cases} u^0, & k = 0 \\ u^1(p(k))\, x(k), & k \geq 1 \end{cases} \tag{6.106}$$

Moreover, the single move $u^0 \in \Re^m$ is assumed to be the only decision variable to be determined.

This kind of control strategy has been proposed for the first time in [49] and it is strongly based on the so-called LPV hypothesis. In fact since $\xi(0)$ is known, the one-step ahead state prediction set is the singleton

$$\hat{X}(1|0) = \{\hat{x}(1|0)\}$$

and in particular such a singleton is a linear function of the decision variable u^0:

$$\hat{x}(1|0) = \sum_{i=1}^{l} p_i(k)\, A_i x(0) + B_i u^0.$$

Then, we can reformulate the constrained stabilization problem as the following simple convex optimization problem

$$\begin{array}{l} \min\limits_{u}\ \hat{x}(1|0)^T R_x x(1|0) + u^T R_x u \\ \hat{x}(1|0) = \sum\limits_{i=1}^{l} p_i(k)\, A_i x(k) + B_i u \\ -u_{i,\max} \leq u_i \leq u_{i,\max}, \qquad i = 1, ..., m \\ -y_{i,\max} \leq C_i \hat{x}(1|0) \leq y_{i,\max}, \qquad i = 1, ..., n_y \\ \hat{x}(1|0) \in \mathcal{E} \end{array},$$

6.5 Time-Varying Control– N-step strategies

The one-step strategy has been partially extended in [47], [51] into a N-step strategy by the use of

$$u(k) = u(\xi(k)) = \begin{cases} \left(\sum_{i=1}^{l} p_i(k) F_i^k x(k)\right) + c^k, & k = 1, ..., N-1 \\ u^N(x(k), p(k)) & k \geq N \end{cases} \tag{6.107}$$

A further generalization is the strategy introduced in Section 5.2 where the "free moves" terms c^k are scheduled

$$u(k) = u(\xi(k)) = \begin{cases} \left(\sum_{i=1}^{l} p_i(k) F_i^k x(k) + c_i^k\right), & k = 1, ..., N-1 \\ u^N(x(k), p(k)) & k \geq N \end{cases} \tag{6.108}$$

The properties of the prediction sets related to this class of control strategies have been widely shown in Chapter 5. In particular, it has been observed that it is possible to obtain recursively the vertices $\hat{x}_{i_1,...,i_{k-1}}(k|0)$ and $\hat{u}_{i_1,...,i_k}(k|0)$ for both state and input (or their outer approximations) prediction sets. Those vertices characterizations allow us to easily employ conditions (6.103),(6.104),(6.105) to solve the control design problem. Let us focus now on the cost function

$$J(x(0), u(\cdot)) = \max_{p(0),...,p(k+N-1)} \sum_{k=0}^{N-1} \|x(k+1)\|_{R_x}^2 + \|u(\xi(k))\|_{R_x}^2$$

In [47], the following upper-bound has been determined

$$\begin{aligned} & J(x(0), u(\cdot)) = \max_{p(0),...,p(k+N-1)} \sum_{k=0}^{N-1} \|x(k+1)\|_{R_x}^2 + \|u(\xi(k))\|_{R_x}^2 \leq \\ & \sum_{k=0}^{N-1} \max_{p(k)} \|x(k+1)\|_{R_x}^2 + \|u(\xi(k))\|_{R_x}^2 \leq \sum_{k=0}^{N-1} J_k \end{aligned}$$

where

$$\max_{p(k)} \|x(k+1)\|_{R_x}^2 + \|u(\xi(k))\|_{R_x}^2 \leq J_k$$

that finally becomes

$$\|x(k+1|0)\|_{R_x}^2 + \left\|u\left(\hat{\xi}(k|0)\right)\right\|_{R_x}^2 \leq J_k, \quad \forall \hat{\xi}(k|0) \in \hat{I}(k|0)$$

By applying iteratively Schur complements we can obtain the following LMIs

$$\begin{bmatrix} 1 & * & * \\ R_x^{1/2} x(k+1|0) & J_k & 0 \\ R_u^{1/2} u(k|0) & 0 & J_k \end{bmatrix} > 0, \forall p(k)$$

An upper-bound to the minimization of (6.101) becomes then the following

$$\min \sum_{k=0}^{N-1} J_k$$
$$\begin{bmatrix} 1 & * & * \\ R_x^{1/2}\hat{x}_{i_1,...,l_k}(k+1|0) & J_k & 0 \\ R_u^{1/2}\hat{u}_{i_1,...,l_k}(k|0) & 0 & J_k \end{bmatrix} > 0, \tag{6.109}$$
$$i_1 = 1, ..., l_1, ..., i_k = 1, ..., l_k$$

Here we propose an alternative cost reformulation which can be proved to be less conservative. Let us consider

$$J(x(0), u(\cdot)) = \max_{p(0),...,p(N-1)} \sum_{k=0}^{N-1} \|x(k+1)\|_{R_x}^2 + \|u(\xi(k))\|_{R_x}^2$$

it is enough to consider the following inequality

$$\sum_{k=0}^{N-1} \|x(k+1)\|_{R_x}^2 + \|u(\xi(k))\|_{R_x}^2 < J,$$

for all possible parameter sequences $(p(0), ..., p(N-1))$ By applying recursively Schur complements the above conditions become

$$\begin{bmatrix} 1 & * & ... & * & * & * \\ R_x^{1/2}x(1) & J & 0 & ... & * & * \\ R_u^{1/2}u(0) & 0 & J & 0 & ... & * \\ ... & ... & ... & ... & ... & ... \\ R_x^{1/2}x(N) & 0 & ... & 0 & J & * \\ R_u^{1/2}u(N-1) & 0 & 0 & ... & 0 & J \end{bmatrix} > 0,$$

On the basis of discussions reported in Chapter 5, we can prove that for each feasible parameter sequence $(p(0), ..., p(N-1))$, there exists a certain sequence of parameters $(\theta(p(0)), ..., \theta(p(N-1)))$ such that

$$x(k) = \sum_{i_1=1}^{l_1} .... \sum_{i_{k-1}}^{l_{k-1}} \theta_{i_1}(p(0))....\theta_{i_{k-1}}(p(k-1))\hat{x}_{i_1,...,i_{k-1}}(k|0), \ k = 1, ..., N$$

$$u(k) = \sum_{i_1=1}^{l_1} .... \sum_{i_{k-1}}^{l_k} \theta_{i_1}(p(0))....\theta_{i_k}(p(k))\hat{u}_{i_1,...,i_k}(k|0), \ k = 0, ..., N-1$$

Then, an upper-bound to the minimization of (6.101) can be obtained as follows

$$\min J$$
$$\begin{bmatrix} 1 & * \ldots * \; * \; * \; * \; * \\ R_x^{1/2}\hat{x}(1|0) & J \ldots * \; * \; * \; * \; * \\ R_u^{1/2}\hat{u}(0|0) & 0 \; J \ldots * \; * \; * \; * \\ R_x^{1/2}\hat{x}_{i_1}(2|0) & 0 \; 0 \; J \ldots * \; * \; * \\ R_u^{1/2}\hat{u}_{i_1}(1|0) & 0 \; 0 \; 0 \; J \ldots * \; * \\ \ldots & \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ R_x^{1/2}\hat{x}_{i_1,\ldots,i_{N-1}}(N|0) & 0 \; 0 \ldots 0 \; 0 \; J \; * \\ R_u^{1/2}\hat{u}_{i_1,\ldots,i_{N-1}}(N-1|0) & 0 \; 0 \; 0 \ldots 0 \; 0 \; J \end{bmatrix} > 0, \tag{6.110}$$
$$i_1 = 1, \ldots, l_1, i_2 = 1, \ldots, l_2, \ldots i_{N-1} = 1, \ldots, l_{N-1}$$

On the basis of the above discussions, we are able to write conditions (6.98),(6.99), (6.100), (6.101) as convex functions of the state predictions set vertices.

As already pointed out, the latter derivations are not a sufficient to arrive to a convex optimization problem instrumental to the design of a control strategy. In particular, it is worth noticing that, if we consider a control strategy in the form (6.107) with variables u_i^k, F_i^k, $i = 1, \ldots, l$, $k = 0, \ldots, N-1$ to be determined, then the overall convex optimization procedure is not convex. This is easily shown by considering even the simple case $l = 1$

$$\begin{aligned} x(k+1|k) &= \left(A + BF^0\right) x(k) \\ x(k+2|k) &= \left(A + BF^1\right) x(k+1|k) + c^1 = \\ &= \left(A + BF^1\right)\left[\left(A + BF^0\right) x(k) + Bc^0\right] + Bc_1 \end{aligned} \tag{6.111}$$

where we can see that F^1 multiplies terms containing F^0 and c^0.

The above simple example suggests that if matrices $F^0, \ldots, F^{N-1}$ are a priori known instead and $c^0, \ldots, c^{N-1}$ are the only optimization variables, the dependence of $\hat{x}(k|0)$ on the optimization variables is linear. Because all descriptions of the prediction vertices are always in a form similar to (6.111), we can conclude:

Theorem 6.19. *Let a the terminal law $u^N(x(k), p(k))$ and the associate positive invariant set $\mathcal{E}$ be given such that the constraints are satisfied. If a control strategy of the form (6.107) is employed such that*

- $F_i^1, \ldots, F_i^{N-1}, i = 1, \ldots, l$ *are a priori fixed;*
- $c_i^1, \ldots, c_i^{N-1}, i = 1, \ldots, l$ *are the unknown variables to be determined;*

then a solution of the LPV constrained design problem can be found by deriving (if there exist) vectors $c_i^1, \ldots, c_i^{N-1}, i = 1, \ldots, l$ that are solutions of the following convex optimization problem

$$\min_{\substack{J, c_i^k, \\ i = 1, \ldots, l \\ k = 0, \ldots, N-1}} J$$

subject to (6.103),(6.104),(6.105),(6.110)

6.6 Time-Varying Control– Prediction Set based control strategies

Following the same lines of the previous Section, it is possible to use prediction set based control strategies to solve the constrained LPV control design problem. In this case the solution is even simpler, because not only the prediction sets are known but also the control strategy itself is computed by means of the convex combinations of the input prediction set vertices.

Namely, when a nonscheduled prediction set based control strategy is employed, the vertices of $\hat{X}(k|0)$, $\hat{x}_{i_1,...,i_{k-1}}(k|0)$, depends linearly on the vertices of $\hat{U}(k|0)$, $\hat{u}_{i_1,...,i_{k-1}}$ which are also the parameters to be chosen to characterize the strategy. Then, if the vertices $\hat{u}_{i_1,...,i_{k-1}}$ are the variables to be determined, both the state and prediction vertices depend linearly on it.

On the contrary, when a scheduled prediction set based control strategy is used, the vertices of $\hat{X}(k|0)$ and $\hat{U}(k|0)$ are derived by means of a linear relationship with the parameters vectors defining the strategy $\hat{\hat{u}}_{i_1,...,i_{k-1}} \in \Re^{ml}$.

The following result can then be stated:

Theorem 6.20. *Let a terminal control law $u^N(x(k), p(k))$ and the associate positive invariant set $\mathcal{E}$ be given. If a control strategy*

$$u(k) = u(\xi(k)) = \begin{cases} u^{ps}(\xi(k)), & k = 1, ..., N-1 \\ u^N(x(k), p(k)) & k \geq N \end{cases} \tag{6.112}$$

is employed such that $u^{ps}(\xi(k))$, $k = 1, ..., N-1$ is a prediction set dependent control strategy, then the variables to be determined are

- *Nonscheduled Case*

$$\hat{u}_{i_1,...,i_{k-1}}(k|0),\ i_1 = 1, ..., l_1, ..., i_{k-1} = 1, ...l_k.,\ k = 0, ..., N-1$$

- *Scheduled Case*

$$\hat{\hat{u}}_{i_1,...,i_k},\ i_1 = 1, ..., l_1, ..., i_k = 1, ...l_k.,\ k = 0, ..., N-1$$

and the constrained LPV control design problem can be solved by means of the following convex optimization problem

$$\min_{\substack{c_i^k, \\ i=1,...,l \\ k=0,...,N-1}} \quad J$$

subject to (6.103),(6.104),(6.105),(6.110)

Example 6.21. In this Example we will compare strategies in the form (6.107) and (6.108) obtained by means of Theorem 6.19 with (scheduled) prediction set based control strategies (6.112) computed by means of Theorem 6.20.

The analysis of non-scheduled prediction set based control strategies will be avoided because (at least in the LPV framework) it does not lead to interesting results. The following LPV system is introduced:

$$x(k+1) = \sum_{i=1}^{2} p_i(k)A_i x(k) + \sum_{i=1}^{2} p_j(k)B_j u(k)$$

where

$$A_1 = \begin{pmatrix} 1 & 0.1 \\ 0.5 & 1 \end{pmatrix}, \; B_1 = \begin{pmatrix} 1 \\ 0\beta \end{pmatrix},$$
$$A_2 = \begin{pmatrix} 1 & 0.1 \\ 2.5 & 1 \end{pmatrix}, \; B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The input signal is constrained to be

$$|u(k)| < 1, k = 0, ..., \infty$$

and moreover weighting matrices $R_x = I$,$R_u = 1$ are chosen. In Figure 6.5 the attraction basins of the various methods under considerations are depicted by assuming that the couple terminal set/terminal control law is

$$\mathcal{E} = \left\{ x \in \Re^2 : x^T \begin{pmatrix} 0.3153 & 0.0516 \\ 0.0516 & 0.0146 \end{pmatrix} x < 1 \right\},$$
$$u(\xi(k)) = p_1(k)[-0.2586 \;\; -0.1118]x(k) + p_2(k)[-0.5228 \;\; -0.1142]x(k).$$

The parameter vector at time $k = 0$ is assumed to be $p(0) = [1 \;\; 0]^T$. Note that, if the prediction is performed over an $N = 1$ horizon, all the proposed methods are equivalent to the computation of strategy (6.106) proposed in [49]. For a prediction horizon $N = 2$, instead, the strategy (6.107) proposed by [51] performs worse then both its modification with "scheduled free moves" (6.108) and the prediction sets based control strategy (6.112). Note that the last two strategies have the same attraction region because, for $N = 2$, the associated optimization procedures coincide. (6.112) performs better then (6.108) for $N > 2$.

In order to ease the Figure comprehension the case with $N = 3$ has been omitted (lines were too near to other attraction regions).

It is important to remark that the attraction basins strongly depend on the chosen terminal region. If we consider a couple $u(\xi), \mathcal{E}$ with a smaller invariant set:

$$\mathcal{E} = \left\{ x \in \Re^n : x^T \begin{pmatrix} 198.79 & 73.70 \\ 73.70 & 53.65 \end{pmatrix} x < 1 \right\},$$
$$u(\xi(k)) = p_1(k)[-1.1647 \;\; -0.4609]x(k) + p_2(k)[-1.8574 \;\; -0.4583]x(k).$$

then, as depicted in Figure 6.6, the attraction regions evidently decrease. To conclude the analysis let us consider, for a control horizon $N = 4$ the dynamic behaviors of the above seen strategies for an initial state $x(0) = [-1 \;\; 1]^T$

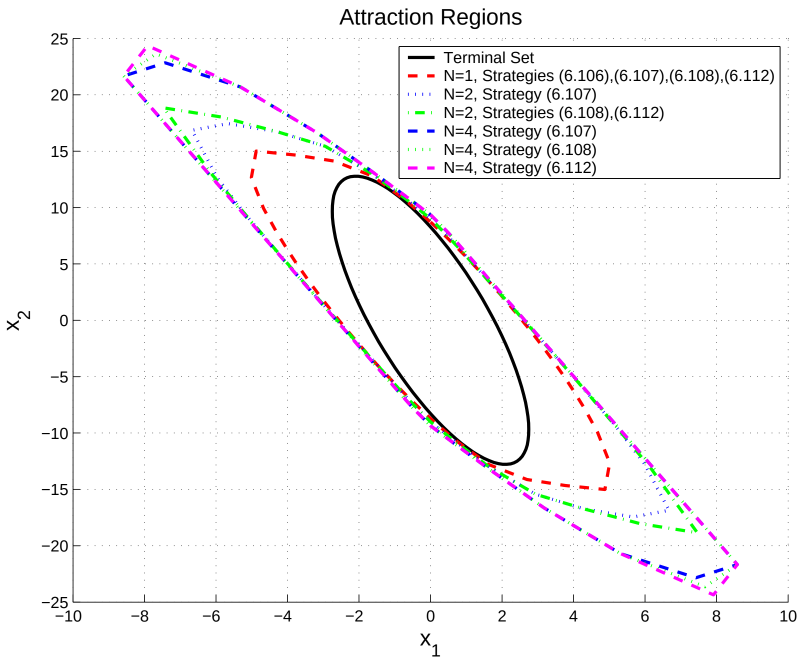

Fig. 6.5. Attraction Regions

andweighting matrices $R_x = I$ and $R_u = 0.1$.. In the next Table the obtained optimal upper bounds to the cost J^* are compared together with the number of scalar variables involved in the computation

Method	J^*	Variables
Strategy (6.107)	0.6602	$\sum_{i=0}^{N-1}(nl_c^i) + Nm$
Strategy (6.108)	0.6465	$\sum_{i=0}^{N-1}(nl_c^i) + lNm$
Strategy (6.112)	0.6459	$\sum_{i=0}^{N-1}(nl_c^i) + \sum_{i=0}^{N-2}(mll_c^i) + m + 1)$

The number of LMI lines involved in the computation, instead, is the same for any of the presented approaches and is equal to

$$\sharp LMI \ \ lines = \sum_{i=0}^{N-1}(l_c^i(m+2n_y)) + \sum_{i=0}^{N-2}(mll_c^i) + m + l_c^N(n+1).$$

It is possible to note that the use of more complex prediction strategy (6.112) is paid only in terms of a bigger number of variables. Anyway, because of the

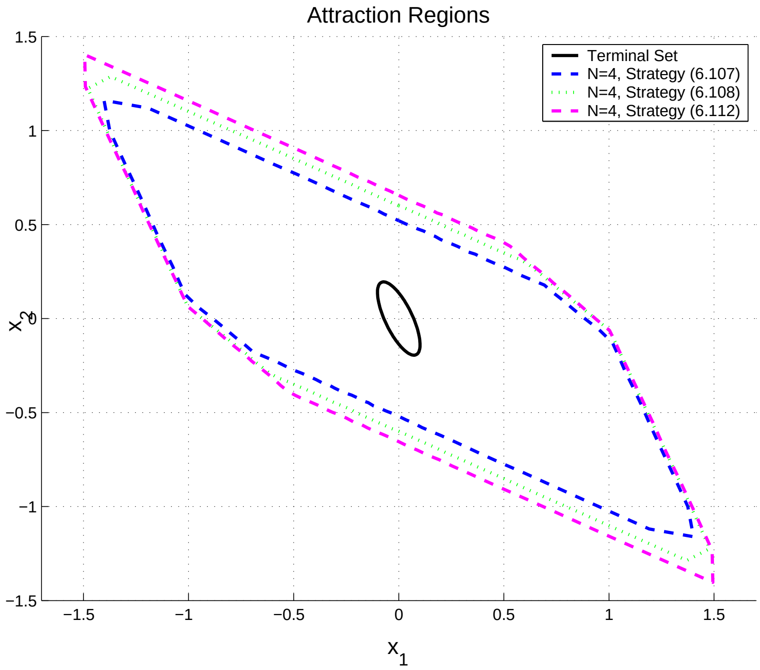

Fig. 6.6. Attraction Regions with a Smaller Terminal Set

dimension of the overall problem, those extra variables do not seem to affect in a perceptible way the computational burdens. Figures 6.7-6.9 depict the state evolution and the input signals.

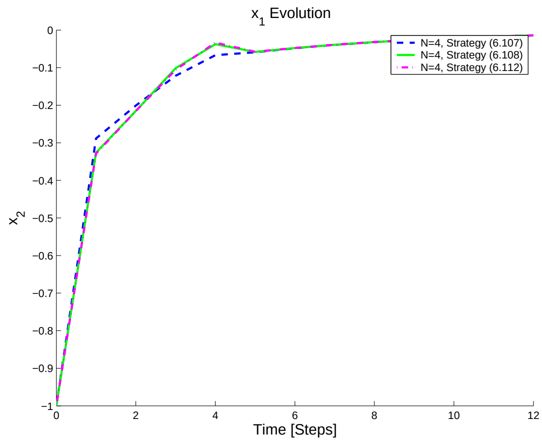

Fig. 6.7. State x_1 Evolution

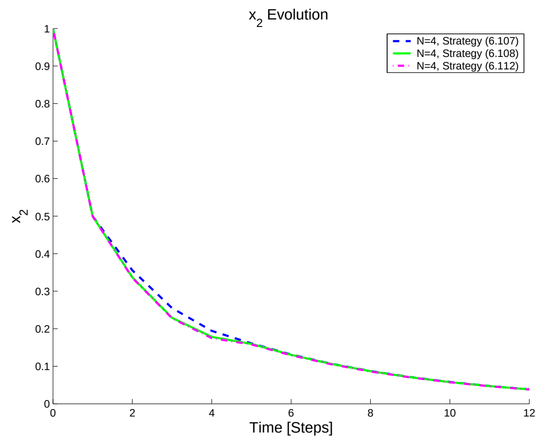

Fig. 6.8. State x_2 Evolution

6.7 Free Terminal Control Law approaches

Before closing the discussion about the role of time-varying control strategies for constrained control design problems, it is mandatory to briefly investigate how the above technicalities can be converted into a free terminal control strategies. It has been already explained that we focus on exploiting Section 6.1 results for N-step control horizons and that the main difference is in the invariant set definition. A possible way to adapt the above results to the cases of interest here is by means of invariant sets of the form

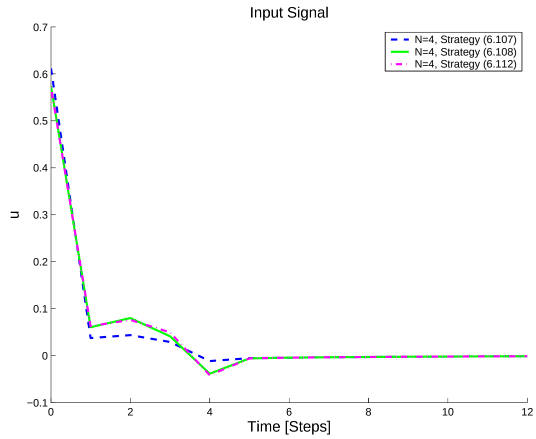

Fig. 6.9. Input Signal

$$\mathcal{E} = \left\{ x \in \Re^n \left| x^T V\left(\hat{p}\left(N|0\right)\right) x < \gamma, \forall \hat{p}\left(N|0\right) \in \hat{P}\left(N|0\right) \right. \right\}$$

Even if tricky conditions could be obtained because of such a structure set, an easier solution can be found by considering, as an alternative, the following set

$$\mathcal{E} = \left\{ x \in \Re^n \left| x^T V\left(\hat{p}\left(N|0\right)\right) x < \gamma, \forall \hat{p}\left(N|0\right) \in \Sigma_l \right. \right\}.$$

where the dependencies on $p(N|0)$ disappear.

Chapter summary

In this Chapter several methods to solve the constrained LPV control design problem have been shown. Four classes of methods for designing time-invariant control laws have been shown: while the first two strategies are well known in literature, the remaining seem to be new and are here proposed for the first time.

The use of time-varying control strategies has been also discussed. Time-varying control strategies with frozen terminal laws have been discussed: the well known one-step approach and its generalization have been explained and minor improvements proposed. Then, prediction set dependent control laws for LPV systems have been proposed here for the first time. Finally, some hints on the technicalities to adapt the frozen-terminal-law control strategies into free-terminal-law ones have been given.

7

Model Predictive Control

Model Predictive Control (MPC), a.k.a. Receding Horizon Control (RHC) is a class of control algorithms widely applied in industry which has gained in popularity as an effective and efficient solution to deal with the control of plants subject to constraints. A huge technical literature on this a topic has been developed from the '70s until nowadays (see [24], [52], [53],[54]...).

It is important to point out that the with the acronym MPC one does not designate a specific control method but rather an entire class of control algorithms. The common denominator of such a family of control strategies consists in an explicit use of the plant model to compute, at each time instant, the control signal that minimizes a predefined objective function.

More precisely, MPC philosophy consists in computing, on the basis of actual state plant measurements, the "best" set of control moves such that plant forecasts fulfil specific performance and/or constraints requirements. The obtained control strategy is applied to the system until new plant measurements are available and then a new set of virtual input moves is computed. A graphical representation of the receding horizon ideas is depicted in Figure 7.1 and 7.2.

From the previous considerations it is clear that MPC has a strict algorithmic essence and it is then natural to deal with it in a discrete time plant model framework. A possible synthetic description of MPC could then be formalized as

1. Generate, on the basis of the actual plant information, a virtual control strategy $u\left(k+\tau|k\right),\ \ \tau=0,...,N-1$ such that plant predicted behavior satisfies performance and/or constraints prescriptions;
2. Apply the first computed control move $u\left(k\right)=u\left(k|k\right)$
3. $k=k+1$, goto 1

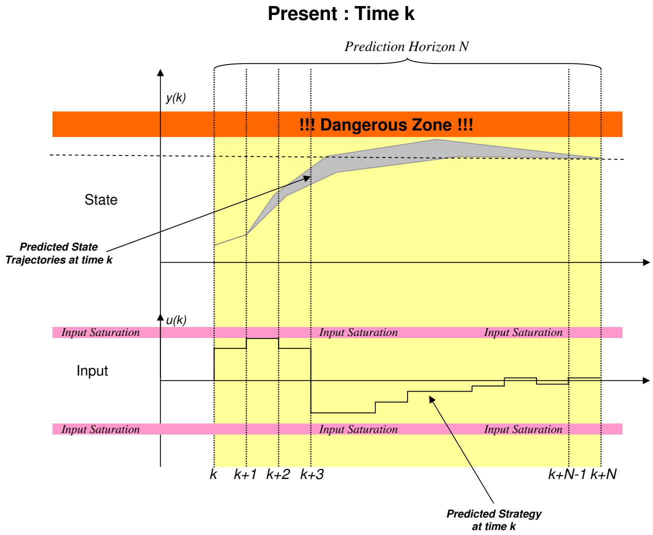

Fig. 7.1. Receding Horizon Philosophy - Actual time k.

where $u(k+\tau|k)\,,\ \ \tau = 0, ..., N-1$ denotes the control strategy computed at time k and N is the control horizon length which can be either finite or (implicitly) infinite.

On the basis the above description, it is important to understand that the availability of an admissible virtual input sequence at a certain time does not imply that a RHC algorithm is able to fulfil the required prescriptions for each time instant. Three key points must be taken into consideration when designing a RHC strategy:

- (*feasibility*) Existence at time k of an optimal virtual control strategy in general does not imply the existence of an admissible control strategy at future time instants;
- (*stability*) Even if the model predictions converge to the desired set point the plant, under the action of an MPC control strategy, could become unstable;
- (*computability*) The computing machinery used to generate the virtual input strategy could be not able to perform the requested task within the system sampling time.

For the above reasons the issues of (*feasibility*), (*stability*) and (*computability*) must be taken into consideration when designing an MPC algorithm.

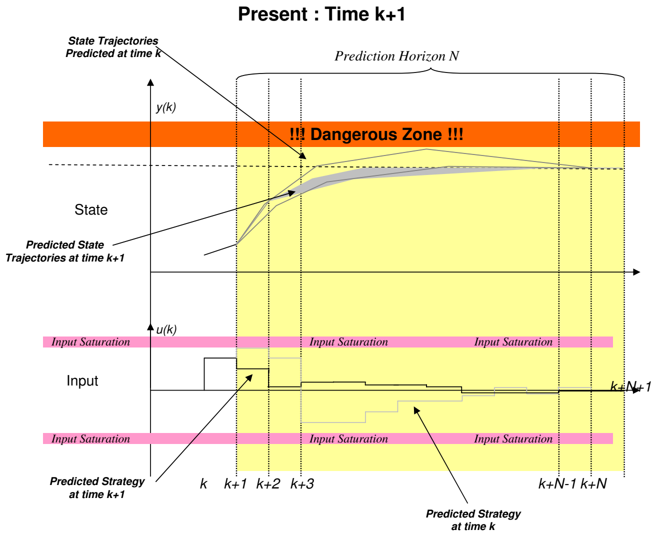

Fig. 7.2. Receding Horizon Philosophy - Actual time $k+1$.

On the basis of the above discussions, each control algorithm proposed in the previous chapter could not be adapted in a RHC fashion. In the next Sections we will give an overview of the methods which fit within such a framework.

7.1 Time-Invariant Control Laws

In the first part of Chapter 6 several ways to obtain a time-invariant control strategy able to solve the constrained stabilization problem have been discussed. All the approaches are based on the idea of build-up a control law and a set of initial states such that the regulated plant trajectory does not violate the prescribed constraints.

All those approaches can be easily adapted into a Receding Horizon algorithm by simply computing, at each time instant k and on the basis of the information vector $\xi(k)$, a new control law by solving the associated optimization problem. As seen, in order to obtain a workable strategy, feasibility and stability must be properly addressed. We need to prove in fact that, if the strategy has solution at time k, it will also have solution to time $k+1$ and, moreover, the closed loop system is stable.

Then, for any possible approach, a feasibility and stability proof should be given. To avoid tedious and repetitive technicalities we can give the following result which is valid for several of the control strategies proposed in Chapter 6:

Theorem 7.1. *Consider the optimization based control strategies described in Lemma 6.5, Lemma 6.9, Lemma 6.11, Lemma 6.14.*
Then, in a receding horizon scheme, if a solution at a generic time instant k is available, an admissible solution exists at each future time instants and the resulting MPC control input strategy yields an asymptotically stable closed-loop system.

Proof. The proof is divided in two parts.
1. feasibility
Feasibility can be demonstrated by proving that the optimal solution at time k is still admissible at the next time instant $k+1$. Note that the optimization problems at time k and at time $k+1$ differ in the LMIs (6.21),(6.54),(6.69) and (6.90), respectively.
By using Schur complements, LMIs (6.21),(6.54),(6.69) are equivalent to

$$x(k)V(p|k)x(k) < \gamma, \forall p \in \Sigma_l$$

where by $V(\cdot|k)$ the Lyapunov function "inner matrix" computed at time k is defined. Since this Lyapunov function is coupled with the stabilizing control law applied at time k, it follows that

$$x(k+1)V(p^+|k)x(k+1) < x(k)V(p|k)x(k) < \gamma, \forall p \in \Sigma_l, \forall p^+ \in \Sigma_l.$$

then the solution found at time k, is still feasible at time $k+1$. Similar arguments can be used for (6.69) by noticing that equation (6.89) is a sufficient condition to prove that

$$\begin{aligned} & x(t+1)\left[G^T(p^+) + G(p^+) - P(p^+)\right]^{-1} x(t+1) < \\ & < x(t)\left[G^{-T}(p)P(p)G^{-1}(p)\right] x(t+1) \leq \\ & \leq x(t)\left[G^T(p) + G(p) - P(p)\right]^{-1} x(t) < \gamma \end{aligned}$$

2. stability
It is sufficient to prove that

$$\begin{aligned} & 0 < x(k+1)V(p(k+1)|k+1)x(k+1) \leq x(k+1)V(p(k+1)|k)x(k+1) < \\ & < x(k)V(p(k)|k)x(k) < \gamma, \qquad \forall p(k) \in \Sigma_l, \forall p(k+1) \in \Sigma_l, \forall x(k) \neq 0 \end{aligned}$$

By construction, moreover we have

$$\begin{aligned} & x(k+1)V(p(k+1)|k+1)x(k+1) - x(k)V(p(k)|k)x(k) \leq \\ & \leq x(k+1)V(p(k+1)|k)x(k+1) - x(k)V(p(k)|k)x(k) \leq \\ & \leq -||x(k)||^2_{R_x} - ||u(k)||^2_{R_u} \end{aligned}$$

If we sum for $k = 0, 1, \ldots, \infty$ the previous inequality we obtain:

$$x(\infty)V(p(\infty)|\infty)x(\infty) - x(0)V(p(0)|k)x(0) \leq -\left[\sum_{k=1}^{\infty} ||x(k)||^2_{R_x} + ||u(k)||^2_{R_u}\right]$$

Therefore, because the sequence $\{x(k)V(p(k)|k)x(k)\}_{k=0}^{\infty}$ is monotonically decreasing, the latter means

$$\infty \geq x(0)V(p(0)|k)x(0) - x(\infty)V(p(\infty)|\infty)x(\infty) \geq \left[\sum_{k=1}^{\infty} ||x(k)||^2_{R_x} + ||u(k)||^2_{R_u}\right].$$

Because $R_x > 0$, $R_u > 0$, this implies

$$\lim_{k\to\infty} x(k) = 0,$$
$$\lim_{k\to\infty} u(k) = 0.$$

Remark 7.2. Note that the above result can be easily extended to all the different optimization-based constrained stabilization synthesis methods which can be obtained from Lemmas 6.5, 6.9, 6.11 and 6.14 by using different convexification methods (see Appendix A).

Remark 7.3. It is worth to remark that the above Theorem is not valid when Lemma 6.8 and Lemma 6.14 are exploited into a Receding Horizon algorithm. This is due to the dependence of $p(k)$ which renders more difficult the proof. Work is in progress on this kind of control strategies: it seems that feasibility and stability can still be proved by means of contraction and invariance arguments but a formal proof has not (yet) been provided.

Example 7.4. Here we will show the behavior of the MPC algorithms obtained by arranging Lemmas 6.5, Lemma 6.9, Lemma 6.11 and Lemma 6.14 results into a Receding Horizon Scheme. Consider the system setting described in Example 6.16. The attraction regions will remain the same shown in Figure 6.1. State trajectories and input signals are depicted in Figures 7.3-7.5 by assuming an initial state $x(0) = [0.6, 2]^T$. It is possible to note that the strategy obtained by exploiting Lemma 6.15 performs better then the other three approaches, and that, in this particular case, the strategy using Lemma 6.9 conditions performs better then the one based on 6.11. For different initial conditions the contrary may happens. To better display the different MPC algorithms performance, the upper bound on the cost γ^* computed at each time step is reported in Figure 7.6.

7.2 Time Varying Strategies - Frozen Approach

In Sections (6.4)-(6.6) three "frozen" optimization based methods to compute a time-varying control strategy able to solve the constrained stabilization

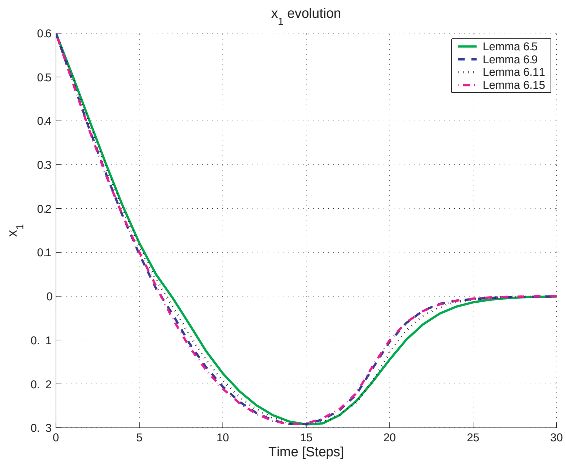

Fig. 7.3. x_1 Evolution

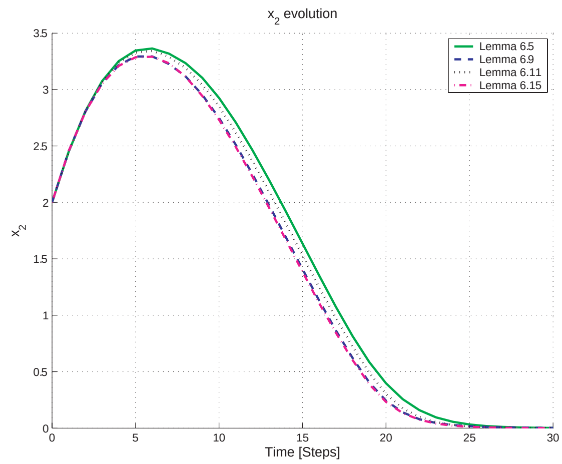

Fig. 7.4. x_2 Evolution

problem for LPV system have been described. Those methods can be easily adapted into an MPC algorithm. However, in order to ensure feasibility and stability, some minor technical expedients have to be employed.

The main "sophistication" we need to build an effective Receding Horizon strategy is concerned with some terminal law/prediction convexification restrictions and with the cost definition.

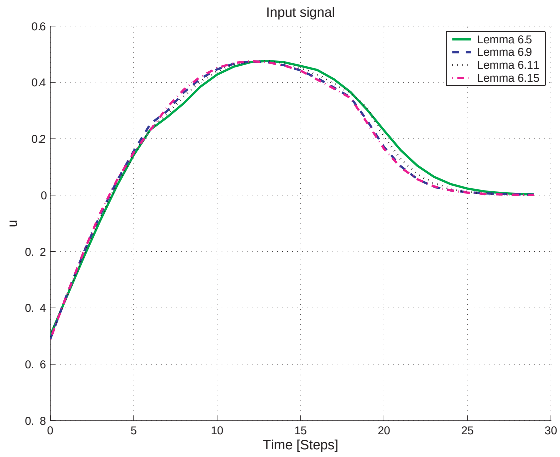

Fig. 7.5. Input Signal

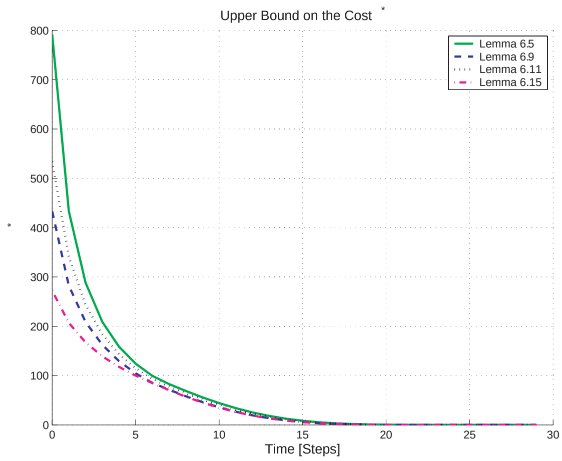

Fig. 7.6. Upper bound on the cost, γ^*

The terminal law aspect is treated by restricting our attention to the following control law

$$u(k) = \left(\sum_{i=1}^{l} p_i(k) \right) x(k)$$

obtained by means of one of the methods described in the first part of Chapter 6 and such that the Lyapunov function used to compute it

$$V\left(\xi\left(k\right)\right)=x^{T}\left(k\right)V\left(p\right)x\left(k\right)=x^{T}\left(k\right)\left(\sum_{i=1}^{l}p_{i}\left(k\right)P_{i}\right)x\left(k\right)$$

is assumed to be known. Moreover, for reasons will be later clarified, it is necessary to assume that the convexification procedure used to compute the control law is the same used to build the state prediction set. For instance, if we use Lemma 6.5 or Lemma 6.9 to compute the control law then the half-sum convexification procedure needs to be used for the state prediction set.

The second "sophistication" we introduce is related to the performance cost because it has to be carefully managed in order to guarantee stability. Several different choices are possible and we present here a particular cost function that can also be adapted in the free terminal law approach. The main idea is that if we consider the cost function from N onwards, thanking to the Chapter 6 results, we can state that

$$\begin{array}{l}J\left(x\left(N\right),u\left(\cdot\right)\right)=\max_{p(\cdot)}\sum\limits_{k=N}^{\infty}\left\|x\left(k\right)\right\|_{R_x}^{2}+\left\|u\left(\xi\left(k\right)\right)\right\|_{R_u}^{2}\leq\\ \leq x^{T}\left(N\right)V\left(p\left(N\right)\right)x\left(N\right)\leq x^{T}\left(N\right)V\left(p\right)x\left(N\right),\forall p\in\Sigma.\end{array}\tag{7.1}$$

where $V(p)$ is a Lyapunov function used to compute the terminal law. The previous inequality allows to define the following upper-bound to the whole cost

$$\begin{array}{l}J\left(x\left(0\right),u\left(\cdot\right)\right)=\max_{p(\cdot)}\sum\limits_{k=0}^{\infty}\left\|x\left(k\right)\right\|_{R_x}^{2}+\left\|u\left(\xi\left(k\right)\right)\right\|_{R_u}^{2}\leq\\ \leq\max_{p(0),\dots,p(N-1)}\left[\left(\sum\limits_{k=0}^{N-1}\left\|x\left(k\right)\right\|_{R_x}^{2}+\left\|u\left(\xi\left(k\right)\right)\right\|_{R_u}^{2}\right)+x^{T}\left(N\right)V\left(p\right)x\left(N\right)\right],\\ \hfill\forall p\in\Sigma.\end{array}$$

By following the same procedures used to obtain (6.110) we can then obtain the following

$$\min J$$

$$J\leq\gamma_{i_1,\dots,i_N}+J_{i_1,\dots,i_{N-1}}\tag{7.2}$$

$$\hat{x}_{i_1,\dots,i_{N-1}}^{T}\left(N\right)\left(\bar{H}_{i_N}\begin{bmatrix}P_1\\ \dots\\ P_l\end{bmatrix}\right)\hat{x}_{i_1,\dots,i_{N-1}}\left(N\right)\leq\gamma_{i_1,\dots,i_N}\tag{7.3}$$

$$i_1=1,\dots,l_{c,1},\dots.,i_N=1,\dots,l_c$$

$$\begin{bmatrix} 1 & * & ... & * & * & * & * & * \\ R_x^{1/2}x(0) & J_{i_1,...,i_{N-1}} & ... & * & * & * & * & * \\ R_u^{1/2}\hat{u}(0|0) & 0 & J_{i_1,...,i_{N-1}} & ... & * & * & * & * \\ R_x^{1/2}\hat{x}(1|0) & 0 & ... & ... & * & * & * & * \\ R_u^{1/2}\hat{u}_{i_1}(1|0) & 0 & ... & ... & ... & * & * & * \\ ... & ... & ... & ... & ... & ... & ... & ... \\ R_x^{1/2}\hat{x}_{i_1,...,i_{N-2}}(N-1|0) & 0 & 0 & ... & 0 & 0 & J_{i_1,...,i_{N-1}} & * \\ R_x^{1/2}\hat{u}_{i_1,...,i_{N-1}}(N-1|0) & 0 & 0 & 0 & ... & 0 & 0 & J_{i_1,...,i_{N-1}} \end{bmatrix} \geq 0$$

$$i_1 = 1, ..., l_{c,1}, ..., i_{N-1} = 1, ..., l_{c,N-1}. \tag{7.4}$$

where, by resorting to Section 2.2.4 notation, $\bar{H}_i$ is a mapping matrix in the form

$$\bar{H}_i = [I_{n\times n}, ..., I_{n\times n}] (\pi_i \otimes I_{n\times n}), i = 1, ..., l_c.$$

Remark 7.5. Note that equation (7.3) is equivalent to

$$x^T(k) V(p) x(k) \leq \gamma, \forall p.$$

The use $\bar{H}_{i_N}$ it is just a trick to use the chorded index i_N

By means of the previous assumption, the following result can be stated

Theorem 7.6. *Let a terminal law*

$$u^N(x(k), p(k)) = \left(\sum_{i=1}^{l} p_i(k) F_i^N\right) x(k) \tag{7.5}$$

and the associated positive invariant set $\mathcal{E}$ be given. Moreover let assume that (7.5) is computed by means of one of the Chapter 6 methods and that (7.1) applies. If the same convexification used to compute $u^N(x(k), p(k))$ is exploited to obtain state prediction sets and if the cost (7.2)-(7.4) is used in place of (6.110), then feasibility and stability properties are guaranteed if the optimization based procedure described in Theorem 6.19 (by imposing $F_i^N = F_i^k, i = 1, ..., l, k = 0, ..., N-1$) or in Theorem 6.20 is rephrased within a receding horizon scheme.

Proof. The proof is divided in feasibility and stability proofs.

1. Feasibility
To prove feasibility it is sufficient to prove that the strategy computed at time k is still feasible at time $k+1$.

In the case a strategy in form (6.107) is used, suppose the strategy computed at time 0 is

$$u(\xi(k)|0) = \begin{cases} \sum_{i=1}^{l} \left(p_i(k) F_i^N x(k) + c_i^{k|0}\right) & k = 0, ..., N-1 \\ \sum_{i=1}^{l} \left(p_i(k) F_i^N\right) x(k), & k \geq N \end{cases}$$

where the notation $c_i^{k|0}$ describes the "free move" applied in k, computed at time 0. It is enough to note that the following strategy

$$u(\xi(k)|1) = \begin{cases} \sum_{i=1}^{l} \left(p_i(k) F_i^N x(k) + c_i^{k|1} \right) & k = 1, ..., N \\ \sum_{i=1}^{l} \left(p_i(k) F_i^N \right) x(k), & k \geq N+1 \end{cases}$$

where

$$\begin{aligned} c_i^{k|1} &= c_i^{k|0}, k = 1, ..., N-1 \\ c_i^{N|1} &= 0 \end{aligned}$$

is a feasible (tough not optimal) solution to the optimization problem.

When a state prediction set based control strategy is adopted, let suppose to have a control strategy computed at time 0

$$u(\xi(k)|0) = \begin{cases} u_{ps}(\xi(k)|0), & k = 0, ..., N-1 \\ u^N(x(k), p(k)) = \sum_{i=1}^{l} \left(p_i(k) F_i^N \right) x(k), & k \geq N \end{cases}$$

such that its input prediction set is

$$\hat{U}_{u(\xi(k)|0)}(k|0), \quad k = 0, ..., N-1.$$

Moreover let us compute, on the basis of the terminal control law, the prediction set $U_{u(\xi(k)|0)}(N|0)$. By exploiting Lemma 5.6, $\hat{U}_{u(\xi(k)|0)}(k|1), \quad k = 1, ..., N$ can be also derived. By simply using prediction set based control strategy properties and by exploiting the structure of the terminal law we are then able formulate a feasible control strategy

$$u(\xi(k)|1) = \begin{cases} u_{ps}(\xi(k)|1), & k = 1, ..., N \\ u^N(x(k), p(k)) = \sum_{i=1}^{l} \left(p_i(k) F_i^N \right) x(k), & k \geq N+1 \end{cases}$$

such that

$$\hat{U}_{u(\xi(k)|1)}(k|1) = \hat{U}_{u(\xi(k)|0)}(k|1), \quad k = 1, ..., N.$$

Finally note that such a kind of strategy is equivalent to

$$u(\xi(k)|1) = \begin{cases} u_{ps}(\xi(k)|0), & k = 1, ..., N-1 \\ u^N(x(k), p(k)) = \sum_{i=1}^{l} \left(p_i(k) F_i^N \right) x(k), & k \geq N \end{cases}$$

2. Stability
The stability proof is common to the two approaches. Let us consider the optimal sequence computed at time 0. The optimal cost will be

$$J^*(0) = \max_{i_1,\dots,i_N} W_{i_1,\dots,i_n}(0)$$

where

$$\begin{aligned}
&W_{i_1,\dots,i_N}(0) = J_{i_1,\dots,i_{N-1}}(0) + \gamma_{i_1,\dots,i_N}(0) = \\
&= \left[\sum_{i=0}^{N} \left\|\hat{x}_{i_1,\dots,i_{k-1}}(k|0)\right\|^2_{R_x} + \left\|\hat{u}_{i_1,\dots,i_{k-1}}(k|0)\right\|^2_{R_u}\right] + \\
&+\hat{x}_{i_1,\dots,i_{N-1}}(N)^T \left[\bar{H}_{i_N}\begin{bmatrix}P_1\\ \dots \\ P_l\end{bmatrix}\right]\hat{x}_{i_1,\dots,i_{N-1}}(N|0) = \\
&= \left[\sum_{i=1}^{N} \left\|\hat{x}_{i_1,\dots,i_{k-1}}(k|0)\right\|^2_{R_x} + \left\|\hat{u}_{i_1,\dots,i_{k-1}}(k|0)\right\|^2_{R_u}\right] + \\
&\|x(0)\|^2_{R_x} + \|u(0)\|^2_{R_u} + \left\|\hat{x}_{i_1,\dots,i_{N-1}}(N|0)\right\|^2_{\left[\bar{H}_{i_N}\left[\begin{smallmatrix}P_1\\ \dots \\ P_l\end{smallmatrix}\right]\right]}
\end{aligned}$$

Let us use one of the feasible (though non optimal) input strategies seen above and let us define $J(1|0)$ as the prediction of $J(1)$ obtained by means of the information available at time 0. It can be computed as follows

$$J(1|0) = \max_{i_1,\dots,i_{N+1}} W_{i_1,\dots,i_{N+1}}(1|0)$$

where

$$\begin{aligned}
&W_{i_1,\dots i_{N+1}}(1|0) = \\
&\left[\sum_{i=1}^{N+1} \left\|\hat{x}_{i_1,\dots,i_{k-1}}(k|0)\right\|^2_{R_x} + \left\|\hat{u}_{i_1,\dots,i_{k-1}}(k|0)\right\|^2_{R_u}\right] + \\
&+\|\hat{x}_{i_1,\dots,i_N}(N+1|0)\|^2_{\left[\bar{H}_{i_{N+1}}\left[\begin{smallmatrix}P_1\\ \dots \\ P_l\end{smallmatrix}\right]\right]} = \\
&= \left[\sum_{i=1}^{N} \left\|\hat{x}_{i_1,\dots,i_{k-1}}(k|0)\right\|^2_{R_x} + \left\|u_{i_1,\dots,i_{k-1}}(k|0)\right\|^2_{R_u}\right] + \left\|\hat{x}_{i_1,\dots,i_{N-1}}(N)\right\|^2_{R_x} + \\
&+\left\|\bar{M}_{i_N}\begin{bmatrix}F_1\\ \dots \\ F_l\end{bmatrix}\hat{x}_{i_1,\dots,i_{N-1}}(N|0)\right\|^2_{R_u} + \|\hat{x}_{i_1,\dots,i_N}(N+1|0)\|^2_{\left[\bar{H}_{i_{N+1}}\left[\begin{smallmatrix}P_1\\ \dots \\ P_l\end{smallmatrix}\right]\right]}
\end{aligned}$$

where $\bar{M}_{i_N}$ is the one introduced in (5.29).

By construction and due to the same convexification procedure used both the prediction set and the control law computation, (see Appendix C for details) we have that, for any $i_1 = 1,...,l_{c,1},...,i_{N-1} = 1,...,l_{c,N-1}, i_N = 1,...,l_{c,N}$

$$\begin{aligned}
&\|\hat{x}_{i_1,\dots,i_N}(N+1|0)\|^2_{\left[\bar{H}_{i_{N+1}}\left[\begin{smallmatrix}P_1\\ \dots \\ P_l\end{smallmatrix}\right]\right]} \leq \\
&\left\|\hat{x}_{i_1,\dots,i_{N-1}}(N|0)\right\|^2_{\left[\bar{H}_{i_N}\left[\begin{smallmatrix}P_1\\ \dots \\ P_l\end{smallmatrix}\right]\right]} - \left\|\hat{x}_{i_1,\dots,i_{N-1}}(N|0)\right\|^2_{R_x} - \left\|\bar{M}_{i_N}\begin{bmatrix}F_1\\ \dots \\ F_l\end{bmatrix}\hat{x}_{i_1,\dots,i_{N-1}}(N|0)\right\|^2_{R_u}
\end{aligned} \tag{7.6}$$

Then we can write

$$\begin{aligned}
&W_{i_1,\dots i_{N+1}}(1|0) - W_{i_1,\dots i_N}(0) \leq \\
&\leq \left\|\hat{x}_{i_1,\dots,i_{N-1}}(N|0)\right\|^2_{R_x} + \left\|\bar{M}_{i_N}\begin{bmatrix}F_1\\ \dots \\ F_l\end{bmatrix}\hat{x}_{i_1,\dots,i_{N-1}}(N|0)\right\|^2_{R_u} + \\
&+ \left\|\hat{x}_{i_1,\dots,i_N}(N+1|0)\right\|^2_{\left[\bar{H}_{i_{N+1}}\begin{bmatrix}P_1\\ \dots \\ P_l\end{bmatrix}\right]} - \|u(0)\|^2_{R_x} - \|x(0)\|^2_{R_x} \\
&- \left\|\hat{x}_{i_1,\dots,i_{N-1}}(N|0)\right\|^2_{\left[\bar{H}_{i_N}\begin{bmatrix}P_1\\ \dots \\ P_l\end{bmatrix}\right]} \leq -\|u(0)\|^2_{R_u} - \|x(0)\|^2_{R_x}
\end{aligned} \tag{7.7}$$

The following inequalities result to be true for each index occurrence

$$W_{i_1,\dots i_{N+1}}(1|0) < W_{i_1,\dots i_N}(0)$$

that implies

$$J^*(1) \leq J(1) \leq J(1|0) \leq \max_{i_1,\dots,i_{N+1}} W_{i_1,\dots i_{N+1}}(1|0) < \max_{i_1,\dots,i_{N+1}} W_{i_1,\dots i_{N+1}}(0) = J^*(0) \tag{7.8}$$

Combining (7.8) and (7.7)it follows that

$$J^*(k+1) - J^*(k) \leq -\|u(k)\|^2_{R_u} - \|x(k)\|^2_{R_x}, \quad k = 0, 1, \dots \tag{7.9}$$

Thanking to the non-increasing monotonicity property of the sequence $\{J^*(k)\}_k^\infty$ and to the finiteness of $J^*(0)$, obtained by construction, we have that

$$\lim_{k\to\infty} J^*(k) < \infty$$

Then, if we sum (7.9) for $k = 0, \dots, \infty$ we obtain

$$\sum_{k=0}^{\infty} \|u(k)\|^2_{R_u} + \|x(k)\|^2_{R_x} < J^*(0) - J^*(\infty) < \infty. \tag{7.10}$$

Because $R_x > 0$ and $R_u > 0$, the latter implies

$$\begin{aligned}\lim_{k\to\infty} x(k) = 0\\ \lim_{k\to\infty} u(k) = 0\end{aligned} \tag{7.11}$$

Example 7.7. Here we want to show the dynamical behavior of the algorithms obtained by arranging Section 6.2 synthesis machineries into an MPC scheme by means of Theorem 7.6. The same system setting shown in Example 6.21 is assumed with an initial parameter value $p(0) = [1\ 0]^T$ and a control horizon $N = 4$. In Figures 7.7-7.9 state trajectories and input signals are shown for an initial state $x(0) = [-1\ 1]^T$. Finally, In order to highlight the improvements arising with the use of control strategies (6.108) and (6.112), the optimal on-line upper bound on the cost J^* is depicted in Figures 7.10-7.11.

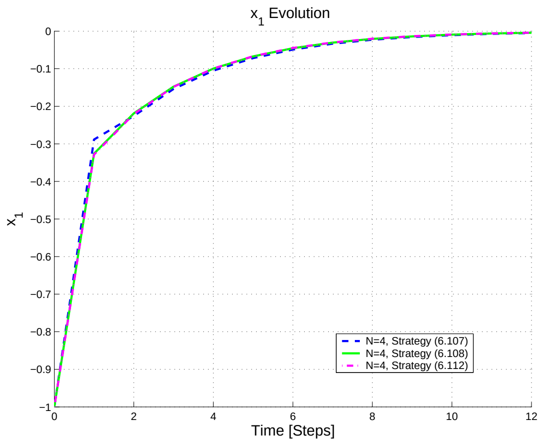

Fig. 7.7. State x_1 Evolution

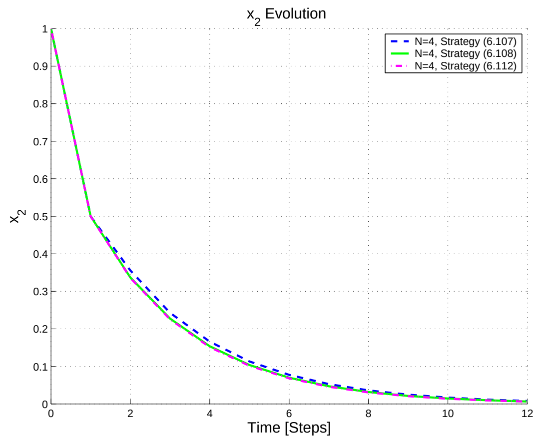

Fig. 7.8. State x_2 Evolution

7.3 Time Varying Strategies - Free Terminal Law Approach

The same results seen in the previous Section, can be easily extended in the case of model predictive control algorithm that exploits constrained control methods with free terminal law.

More precisely, in the case of state prediction set based control strategy, under the same assumptions, feasibility and stability can be proved straight-

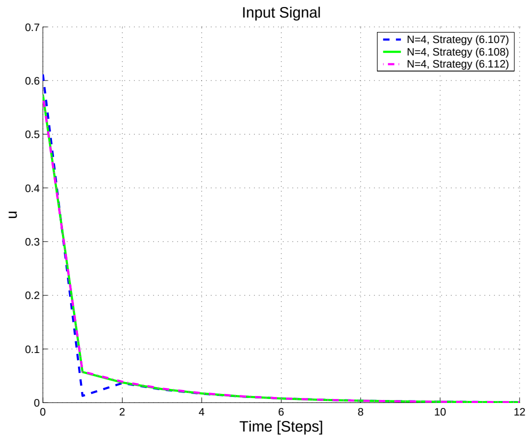

Fig. 7.9. Input Signal

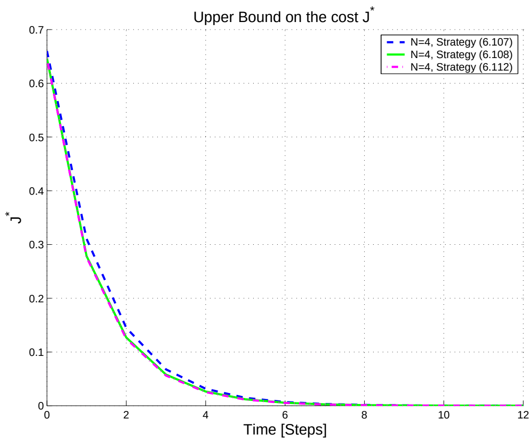

Fig. 7.10. Upper Bound on the Cost J^*

forwardly by means of Theorem 7.6.

When a control strategy having the form (6.107) is taken instead into consideration, the only difference is related to the feedback matrices F_i^k, $i = 1, ..., l, k = 0, ..., N-1$. In fact, in this case, the gain F_i^N has to be derived online and it cannot be used in place of F_i^k, $i = 1, ..., l, k = 0, ..., N-1$. A possible choice to overcome this problem is to modify (6.107). Let us denote by $F_i^{(N|k)}$, $i = 1, ..., l$ the terminal law feedback matrix obtained at time instant k. We can define the control strategy to be computed at time k as follows

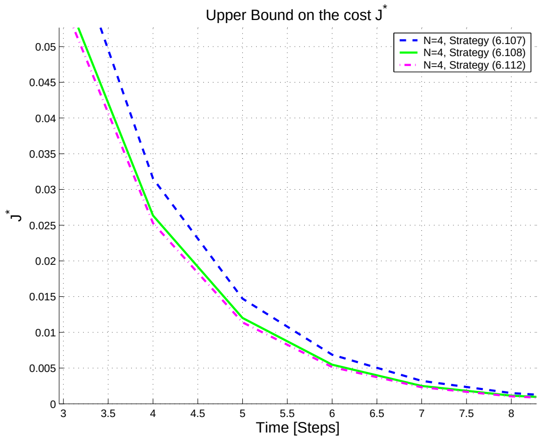

Fig. 7.11. Upper Bound on the Cost J^* - Zoom

$$u(\xi(k+\tau)|k) = \begin{cases} \sum_{i=1}^{l} \left(p_i \left(k+\tau \right) F_i^{(N|k+\tau-N)} x\left(k \right) + c_i^{(k+\tau|k)} \right), \tau = 0, ..., N-1 \\ \sum_{i=1}^{l} \left(p_i \left(k+\tau \right) F_i^{(N|k)} \right) x\left(k \right), \tau \geq N \end{cases}$$

where $F_i^{(N|k)}, i = 1, ..., l$ and $c_i^{(k+\tau|k)}, \tau = 0, ..., N-1, i = 1, ..., l$ are the variables to be determined. By means of this change and under the same assumptions seen in the previous Section, feasibility and stability follow from Theorem (7.6).

Remark 7.8. Note that, at time $k = 0$, the gain matrices

$$F_i^{(N|-N)}, F_i^{(N|-N+1)}, ..., F_i^{(N-1|-1)}, i = 1, ..., l.$$

do not exist. A possible way to initialize the strategy is by choosing an arbitrary set of stabilizing feedback matrices.

Chapter Summary

In this chapter we introduced Model Predictive Control as a general control scheme in which the various methods seen in the previous Chapter can be adapted. Anyway, not all of them yield to control algorithms ensuring feasibility and stability. Feasibility and Stability have been proved for many of the seen approaches and slight changes needed to ensure satisfaction of such structural properties have been pointed out.

8

Fast MPC algorithms for LPV Plants

A well known drawback of constrained MPC schemes is their lack of effectiveness in terms of computational burdens. The MPC algorithms proposed in the previous Chapter require a semidefinite optimization problem with a significant numbers of variables/constraints to be solved. As an example, when a time-invariant control law is considered, the number of LMIs grows at least quadratically with the number of plant representation vertices. Furthermore, time-varying control paradigms give rise to problem dimensions which are exponential in the prediction horizon length.

For such a reason the evaluation of the on-line parts of traditional MPC schemes can become computationally prohibitive in many practical situations. Most of current research on MPC is devoted to reduce such a high computational burden while still ensuring the same level of control performance of the traditional schemes. Many new algorithms have been introduced for uncertain polytopic systems including, exact [55, 56, 57] and approximate [**?**] explicit MPC schemes, efficient implementations of MPC via off-line computation of ellipsoidal [58], [59], [60], [61] and polytopic [62], [63], [64] approximations of exact controllable and viable sets.

By focusing on ellipsoidal schemes, in [60] a bank of nested ellipsoids and corresponding state-feedback gains with increasingly larger performance is computed from the outset. On line, the smallest ellipsoid containing the currently measured state is determined at each time instant and the corresponding state-feedback gain put into the loop.

In [61] the same idea of pre-computing off-line a family of ellipsoidal sets (not necessarily nested) is adopted by exploiting the viability arguments of [**?**] and following a dynamic programming approach consisting of alternating min and max optimization steps. They represent inner ellipsoidal approximations of the viable controllable polyhedral sets, achieved with lighter computational burdens at the expenses of a possibly reduction of the size of robustly controllable regions and performance.

The goal of this Chapter is to propose two fast-MPC algorithms obtained by adapting the above two schemes to the LPV framework.

8.1 Invariant Sets Based Fast-MPC Algorithm

Here we present a fast-MPC scheme based on the positive invariant set associated with the computation of stabilizing time-invariant control law for constrained systems. The idea behind this approach is explained as follows, let $u^1(\xi(k)), ..., u_c^N(\xi(k))$ be a bank of N_c time-invariant stabilizing control laws and $\mathcal{E}_1, ..., \mathcal{E}_{N_c}$ be their associated positive invariant sets compatible with the prescribed constraints. Moreover assume that the following nesting condition holds true.

$$\mathcal{E}_1 \supset \mathcal{E}_1 \supset ... \supset \mathcal{E}_{N_c}.$$

The basic idea is that if a positive invariant set contains less states then the associated control law can be chosen with less limitations. This means that the control law associated with the smaller invariant set can be typically more coercive and, generally, provides a better dynamic response. As a consequence, if the state plant $x(k)$ belongs at least to $\mathcal{E}_1$ then the "best" control law we can choose amongst the given set of control laws is the one whose associated invariant set is the smaller containing $x(k)$. Such an evaluation has to be repeated at each time step once the new state plant, $x(k+1)$, is available. The algorithm proposed is as follows

Algorithm LPV-Fast RHC Off-line -

0.1 - Compute N_c stabilizing control laws $u^1(\xi(k)), ..., u_c^N(\xi(k))$ and their associated positive invariant sets $\mathcal{E}_1, ..., \mathcal{E}_{N_c}$ complying with prescribed constraints and such that

$$\mathcal{E}_1 \supset \mathcal{E}_1 \supset ... \supset \mathcal{E}_{N_c}.$$

0.2 - Store them into a look-up table.

On-line -

1.1 - Find, by means of a binary search on the look-up table, the index $i(k) := \max\{i : x(k) \in \mathcal{E}_i\}$

1.2 - Apply $u(k) = u^i(\xi(k))$; $k := k+1$; goto 1.1;

An example of how this algorithm works is given in Figure 8.1 for a system with state dimension 2. The initial state $x(0)$ belongs to $\mathcal{E}_1$ and $u^1(\xi)$ is applied. Such a law is applied for the next 3 steps, since the state always belong to $\mathcal{E}_1$. At time $k = 4$, the state belongs both to $\mathcal{E}_1$ and to $\mathcal{E}_2$, then the more performing control law $u^2(\xi)$ is adopted until time $k = 6$. Due to the fact that $x(6)$ is contained in $\mathcal{E}_1$,$\mathcal{E}_2$ and $\mathcal{E}_3$, the control law $u^3(\xi)$ is then chosen. No inner invariant sets are then found and such a control law will be adopted for $k > 6$, we have in fact that, thanking to the invariance condition, the future state trajectory will always lie in $\mathcal{E}_3$. Such an algorithm could reveal an effective choice to regulate the behavior of a constrained system to a desired set point. The following result can be stated

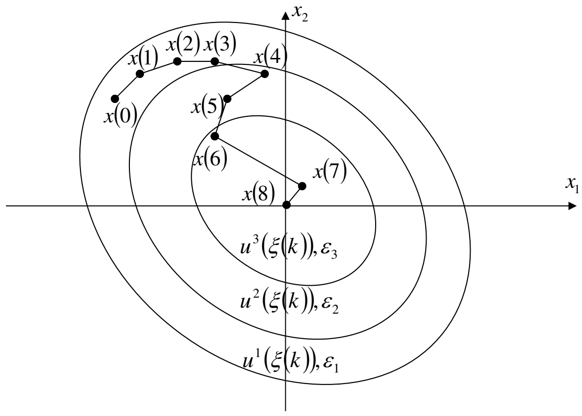

Fig. 8.1. A Geometrical Depiction of an LPV-Fast RHC Algorithm.

Theorem 8.1. *If* **Algorithm LPV-Fast RHC** *has a solution at time* $k = 0$ *then, it has solution at each future time instant, satisfies the prescribed constraints and asymptotic stability of the regulated state trajectory is guaranteed.*

Proof. Feasibility holds because of positive invariance: if $x(0) \in \mathcal{E}_i$, then, $x(k) \in \mathcal{E}_i, \forall k \geq 0$, and then a control law to apply such that constraints are not violated can always be obtained. Stability can be proven by simply note that $i(k)$ is monotonic nondecreasing and, due to this property, only a finite number of switchings between stabilizing control laws is allowed. As a consequence the overall system is asymptotically stable.

Note that, in the above algorithm, the procedure to compute the sequence of couples control law/invariant set is not explicitly given. An off-line numerical procedure to obtain such a set has been introduced first by [60] for polytopic uncertain systems by using ellipsoidal invariant sets and classical LMI technicalities. In [16] the same approach is applied to LPV systems. A generalization which exploits the various approaches seen in Section 6.1 can be obtained by performing the following iteration.

for $k = 1 : N_c$

1 - Given a feasible state x_{in}, compute, by means of one of the LMI methods shown in Section 6.1 a time-invariant state feedback control law $u(\xi)$ and the associated invariant set in the form

$$\mathcal{E} = \{x : x^T V(p) x < \gamma, \forall p \in \Sigma_l\}$$

with the further nesting constraints (ignored at r=1)

$$\frac{V(p)}{\gamma} > \frac{V_{i-1}(p)}{\gamma_{i-1}}, \forall p \in \Sigma_l$$

2 - Label $\mathcal{E}_{\rangle} = \mathcal{E}$,$V_i(p) = V(p)$,$\gamma_i(p) = \gamma$, $u^i(\xi) = u(\xi)$
3 - Choose a new state x_{in} such that $x_{in}^T V^i(p) x_{in} < \gamma^i$
end for

Example 8.2. The aim of this Example is to evaluate both the computational benefits and the loss of performance of the **Fast-MPC** algorithm presented in this Section w.r.t. its **On-line MPC** counterpart. To this end let us consider the system setting presented in Example 6.16 and let us assume the optimization procedure shown in Lemma 6.5 is used both to compute Fast-MPC algorithm invariant sets and to determine the control input applied at each time step in the On-line MPC algorithm. Simulations have been performed for an initial state $x(0) = [-0.55\ 1.8]^T$ and for $p(k) = [1\ 0]^T, k \geq 0$. Figure 8.2 and 8.3 show in which way the Fast-MPC algorithm runs: Figure 8.2 represents the state trajectory in the phase portrait together with the 20 precomputed ellipsoidal sets used by the algorithm. Figure 8.3, instead, depicts the signal switches between the control laws associated to the invariant sets. Figures 8.4-8.6 report input signal and state trajectories: it is interesting to note that only a modest performance degradation is introduced by the use of the Fast MPC algorithm. This is confirmed by Figure 8.7 where the cumulative cost, i.e.

$$J_{cum}(k) = \sum_{i=1}^{k} u^T(i-1) R_u u(i-1) + x^T(i) R_x x(i)$$

is shown. The improvement in term of reduction of computational burdens, instead, are evident, as reported in the following table.

Algorithm	On-line CPU time (seconds per step)
Fast-MPC	0.0019
Online-MPC	0.21

Remark 8.3. Note that, at each time step, the computational complexity is logarithmic the look-up table elements size (and it can be eventually ameliorated by means of proper hashing functions). Moreover, the computational cost is proportional to the number of operations necessary to test the inclusion of $x(k)$ into the invariant sets. The above consideration implies that, even if most of the computation is carried out off-line, the use of a complex Lyapunov function introduces a bigger online computational burdens due to the testing conditions. On the other hand note that, because of the structure of the the algorithm, the use of more time-consuming convexification procedures to build up control laws, would not reflect into the online computations.

8.2 Ellipsoidal Viability Sets Based Fast-MPC Algorithm

In this Section an off-line Model Predictive Control (MPC) method based on ellipsoidal calculus and viability theory is described in order to address control

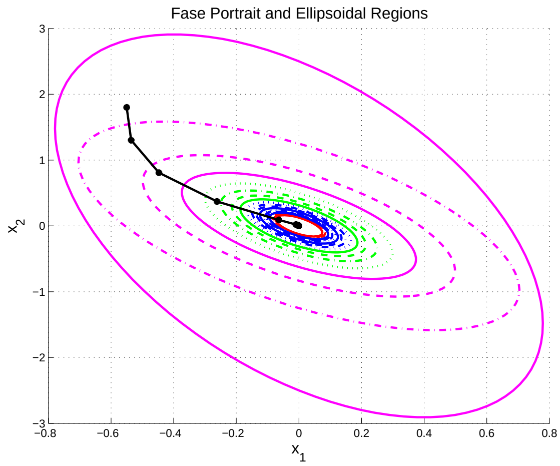

Fig. 8.2. Fast-MPC - Phase Portrait

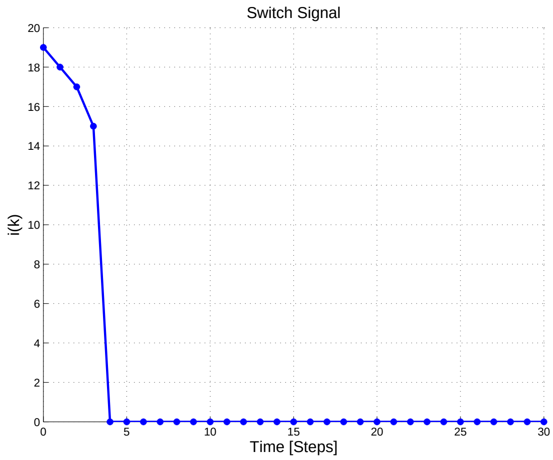

Fig. 8.3. Fast-MPC - Switch Signal

problems in the presence of state and input constraints for LPV systems. Such a scheme is obtained as a generalization to the LPV framework of the algorithm proposed in [61].

The main idea behind of this approach is to carry out off-line most of the computations and to use closed-loop predictions in order improve the control performance. This is done by recursively pre-computing suitable ellipsoidal inner approximations of the exact controllable sets and solving on-line a simple and numerically low-demanding optimization problem subject to a set-membership constraint.

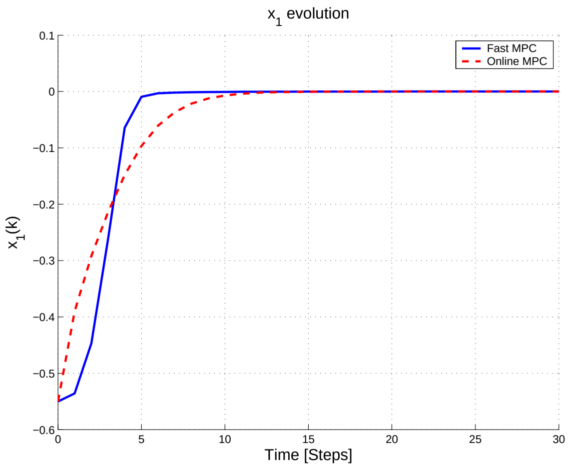

Fig. 8.4. x_1 Evolution

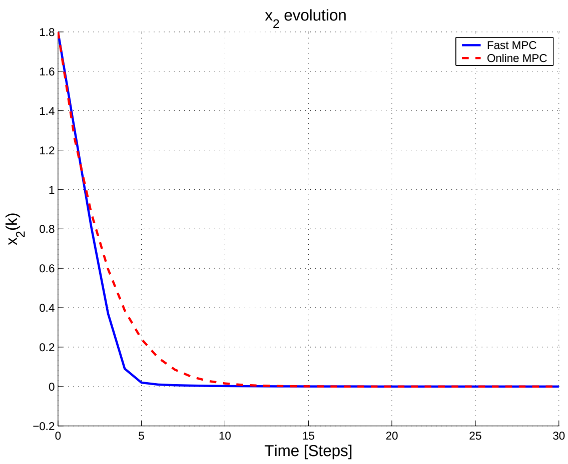

Fig. 8.5. x_2 Evolution

We need to resort to a dual-mode scheme. First, a stabilizing control law for (1.1) is achieved within a suitable neighborhood of the operating point (we will assume to be 0). A robust invariant set $\mathcal{E}$ complying with constraints is associated to such a control law.

By resorting to well established arguments in literature (see ....), the algorithm operating region is then extended by off-line computing the sets of states that can be steered into the terminal set in a finite number of steps despite uncertainty and disturbances. Given a robustly controlled-invariant region $\mathcal{E}$ we could compute in principle the sets of states i-steps controllable to $\mathcal{E}$, regardless of disturbances and uncertainties acting on the system, via

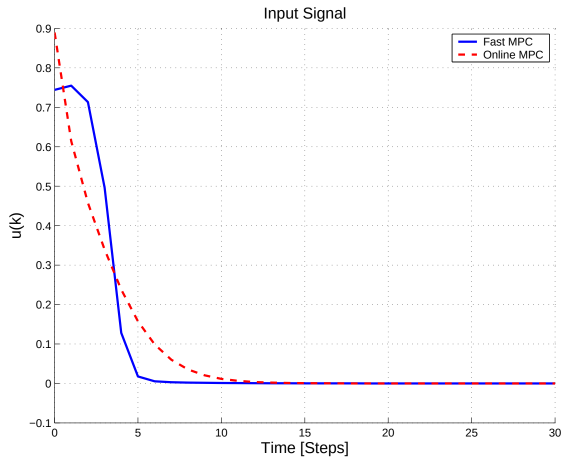

Fig. 8.6. Input Signal

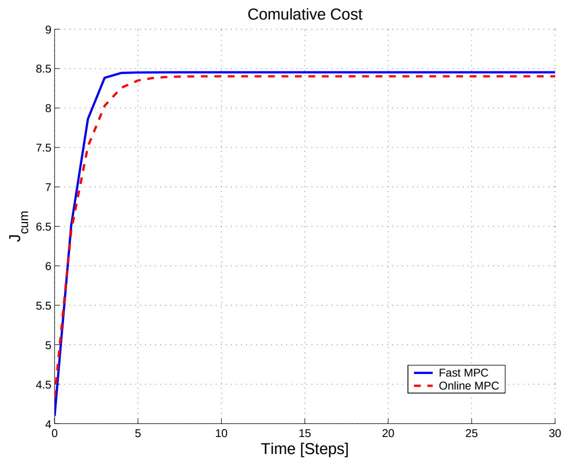

Fig. 8.7. Cumulative Cost

the following recursion:

$$\mathcal{T}_0 = \mathcal{E}$$
$$\mathcal{T}_i = \{x \in \exists u(p) \in U : \forall p \in \Sigma_l, A(p) + B(p)u(p) \in \mathcal{T}_{i-1}\}$$

As a consequence, by induction, we have that $\mathcal{T}_i$ is the set of states that can be steered into $\mathcal{T}$ within at most a number of i control moves. Many properties of $\mathcal{T}_i$ have been investigated in [65] for the case $u(p)$ does not depend on p, i.e. $u(p) = u$. As also discussed in [61] and even considering the simpler case $u(p) = u$, the shape of $\mathcal{T}_i$ grows in complexity as the index i grows and may become computationally intractable after a small number of iterations. For this reason we define a variant of the recursion (8.1) which exploits ellipsoidal inner approximations $\{\mathcal{E}_i\}_{i=0}^{\bar{i}}$, thus allowing a constant number of parameters at each iteration for their characterization. In particular, each member $\mathcal{E}_i$ represents a compact set of states that can be steered using a parameterized control move $u(p)$, for each occurrence of p and without constraints violation, into another set $\mathcal{E}_{i-1}$, which is the "closer" w.r.t. $\mathcal{E}_i$. After a finite number of steps, $\bar{i}$, the regulated state trajectories is then confined into a suitable robust terminal set. The following results has been proved in [61]:

Proposition 8.4. *Let $\mathcal{E} \subset \Re^n$ be a nonempty robustly invariant ellipsoidal region under a certain state-feedback control law, then, the sequence of ellipsoidal sets*

$$\begin{aligned}
&\mathcal{E}_0 := \mathcal{E} \\
&\mathcal{E}_i := In[\{x : \exists\, u(p) \in \mathcal{U} : \forall\, p \in \mathcal{P}, A(p)x + B(p)u(p) \in \mathcal{E}_{i-1}\}]
\end{aligned} \tag{8.1}$$

if non-empty, satisfies $\mathcal{E}_i \subset \mathcal{T}_i$. In (8.5), with the notation $In[\cdot]$ we mean an inner ellipsoidal approximation.

It is important to note that in the above proposition $u(p)$ is an input that may possibly depend from parameter vector p. The easiest way is to not consider such a dependence, since it would complicate the one step prediction state computation and the related machinery to derive those sets. Anyway it has been proved that scheduled inputs outperform the nonscheduled one. The simplest approach to deal with scheduled control laws is to consider u as a convex combination of l vectors depending on the parameter p:

$$u(p) = \sum_{i=1}^{l} p_i u_i.$$

As already stressed thorough the dissertation, this choice results in a quadratic dependence from the parameter vector of the one step ahead predictions. By resorting to Section 2.2.4 notation, we can relax such a dependence in several way obtaining the following equivalent polytopic uncertain system.

$$x(k+1) = \bar{A}(p)x + \bar{B}(p)u$$

where $\bar{u} = [u_1^T, ...u_l^T]^T$ and

$$\bar{A}(p) = \sum_{i=1}^{l_c} \bar{p}_i \bar{A}_i \tag{8.2}$$

$$\bar{B}(p) = \sum_{i=1}^{l_c} \bar{p}_i \bar{B}_i \tag{8.3}$$

$$\tag{8.4}$$

Then recursion (8.1) can be (conservatively) become

$$\begin{aligned} \bar{\mathcal{E}}_0 &:= \mathcal{E} \\ \bar{\mathcal{E}}_i &:= \mathrm{In}[\{x : \exists\, \bar{u} = [u_1^T \in U, ..., u_l^T \in U] : \\ &\quad \forall\, \bar{p} \in \Sigma_{l_c}, \bar{A}(\bar{p})x + \bar{B}(\bar{p})\bar{u} \in \bar{\mathcal{E}}_{i-1}\}] \end{aligned} \tag{8.5}$$

Remark 8.5. It is worth to observe that the maximum number $\bar{i}$ of ellipsoidal sets to be computed depends on the problem at hand. If $x(0)$ is given, it suffices that $x(0) \in \bigcup_{i=0}^{\bar{i}} \mathcal{E}_i$. More generally, $\bigcup_{i=0}^{\infty} \mathcal{E}_i$ represents the attraction basin of our algorithm, that is the set of all initial states for which we guarantee the existence of a solution.

Since the latter recursion is based on a polytopic uncertain model, the ellipsoids $\bar{\mathcal{E}}_i$ can be numerically derived, for instance, by means of LMIs proposed in [61] for the robust case.

The above developments allow one to finally derive a Receding Horizon Control strategy. On-line, at each time t, the smallest index i such that $x(t) \in \mathcal{E}_i$ is first computed. Then, such a sequence of sets is used to enforce, at each step, $\hat{x}(k+1|k) \in \mathcal{E}_{i-1}$ in order to ensure contraction and viability to the closed-loop trajectories. Then, the MPC algorithm is as follows:

Algorithm LPV-VS - Off-line -

0.1 - Compute a stabilizing control law $u(\xi(k))$ and the associated robustly invariant ellipsoidal region $\mathcal{E}_0$ complying with state and input constraints
0.2 - Generate a sequence of N one-step controllable sets $\bar{\mathcal{E}}_i$ (8.5)
0.3 - Store $u(\xi)$ and $\bar{\mathcal{E}}_i$, $i = 0, \ldots, N$.

On-line -

1.1 - Let $i(k) := \min\{i : x(k) \in \mathcal{E}_i\}$
1.2 - If $i(k) = 0$ then

$$u(k) = u(\xi(k))$$

1.3 - Else,

$$\begin{aligned} &u(k) = \arg\min J_{i(k)}(x(k), u(k)) \\ &\text{ subject to} \\ &A(p(k))x(k) + B(p(k))u(k) \in \mathrm{In}\big[\mathcal{E}_{i(k)-1}\big]\,, \\ &u(k) \in \mathcal{U}, \end{aligned}$$

1.4 - Apply $u(k)$; $k := k+1$; goto 1.1;

Remark 8.6. One of the main merits of the LPV framework is that the one-step ahead state prediction is exactly known at each time instant whereas in the robust case the prediction would belong to a polyhedron. Such a feature is particularly well exploited in the proposed algorithm during the on-line phase, providing both a reduction in the computational cost and a control performance improvement w.r.t. the algorithm proposed in [61]

Observe that the cost $J_{i(k)}(x(k), u)$ may depend on $i(k)$ and be defined on an infinite horizon if e.g. implicit dual-mode MPC schemes are of interest. Otherwise, typical choices include

$$J_{i(k)}(x(k), u) = \max_j \|A_j x(k) + B_j u\|^2_{\tilde{Q}(i(k)-1)}$$

where $\mathcal{E}_i = \{x : \; x^{'}\tilde{Q}(i)x \leq 1\}$, or

$$J_{i(k)}(x(k), u) = \|u\|^2_{R_u}, \; R_u > 0$$

if one is interested to approximate one-step minimum-time or, respectively, one-step minimum-energy algorithms. Finally, due to the fact the optimization problem in step 1.3 is always feasible, the following stability result holds true.

Proposition 8.7. *Let the sequence of ellipsoids $\mathcal{E}_i$ be non-empty and $x(0) \in \bigcup_i \mathcal{E}_i$. Then, the algorithm* **LPV-VS** *always satisfies the constraints and ensures robust stability. In particular, there exists a finite time instant $\bar{k}$ such that $x(k) \in \mathcal{E}_0$ for all $k \geq \bar{k}$.*

Proof. The proof follows from Propositions 8.4. Convergence to $\mathcal{E}_0$ in a finite time follows from the choice of a finite number $\bar{i}$ of ellipsoids $\mathcal{E}_i$.

Example 8.8. The aim of this example is to present results on the effectiveness of the strategy presented in this Section. In particular the improvement deriving from the use of both LPV hypothesis and different one-step convexification procedures will be shown. To this end we make comparisons of the proposed algorithm (both in the case half-sum convexification described in Subsection 2.2.2 (**LPV-SS**) and the improved convexification introduced in Subsection 2.2.3 (**LPV-NS**) are used) with the robust counterpart presented in [61] (**Robust**). Consider the multi-model linear time-varying system

$$x(k+1) = \sum_{i=1}^{2} p_i(k) A_i \, x(k) + \sum_{i=1}^{2} p_i(k) B_i \, u(k) \tag{8.6}$$

with

$$A_1 = \begin{bmatrix} 2 & -0.1 \\ 0.5 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0.1 \\ 2.5 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ -0.3 \end{bmatrix}, B_2 = \begin{bmatrix} 0.7 \\ 0.1 \end{bmatrix}, \tag{8.7}$$

and input saturation constraint $|u(k)| \leq 1, \; \forall k \geq 0$. The parameter vector $p(k)$ is assumed to be measurable at each time instant k. In particular the time-varying parameter realization $p(k) = [sin(k) \; 1 - sin(k)]$ and the initial state $x_0 = [-0.7 \;\; 1.5]^T$ (admissible for all the strategies, see Fig. 8.8) have been supposed. Then, a family of 50 ellipsoids has been generated for each algorithm.
The basins of attraction for the three algorithms are depicted in Figure 8.8. As it clearly results, an enlarged region of feasible initial states results for the case

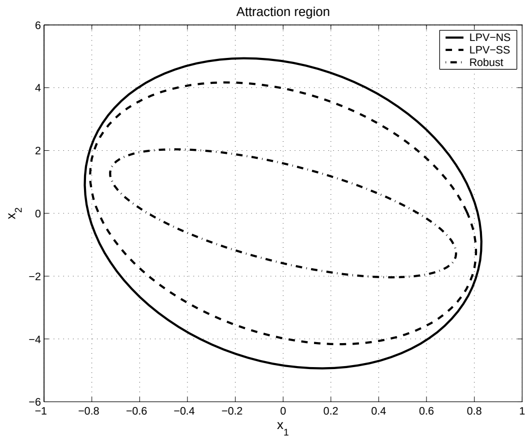

Fig. 8.8. State Attraction Region with input bound constraints - **LPV-NS** (Continuous line), **LPV-SS** (Dashed line), Robust (Dash-Dotted line)

Table 8.1. Numerical burdens: Overall CPU time (seconds)

Algorithm	Off-line: Overall CPU Time	On-line: Average CPU time (seconds per step)
LPV-NS	17.297	0.0870
LPV-SS	16.282	0.0849
Robust	14.624	0.1854

a better convexification is used (continuous line). In Table 8.1 the CPU times for the off-line phases have been reported. Essentially all the three algorithms have similar computation times. The average on-line numerical complexity for each algorithm is reported in Table 8.1. It is possible to observe that one order of magnitude separates the LPV methods from the robust ellipsoidal scheme. This is due to the fact that the robust case implies that the minimization (Step 1.3 of the algorithm) is performed along the corners characterizing the one-step ahead state prediction while it reduces to a single point in the LPV framework.

Two cost to go function have been considered, the first is the minimum time and the second one the minimum energy.
All the relevant results for the minimum time case are depicted in Figures 8.9-8.10. Figure 8.9 depicts the switching signal $i(t)$ for all the contrasted algorithms. This signal is important to show the level of contraction provided by the control algorithm during the system evolution and it represents at each time instant the smaller ellipsoid $\mathcal{E}_i$ of the pre-computed family containing the state $x(k)$. The state and command behaviors corresponding to this realization are depicted in Figure 8.10.

Under the minimum-time criterion, the two LPV strategies perform almost identically, while a slight worst behavior is observed in the case of the robust counterpart. In fact shorter settling times can be observed in Figure 8.10. This is confirmed by the switching signal $i(k)$ trend Figure 8.9.

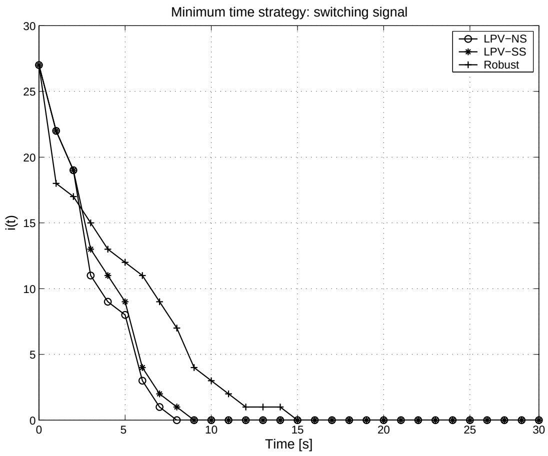

Fig. 8.9. Switching signals for the proposed strategy: one-step minimum-time.

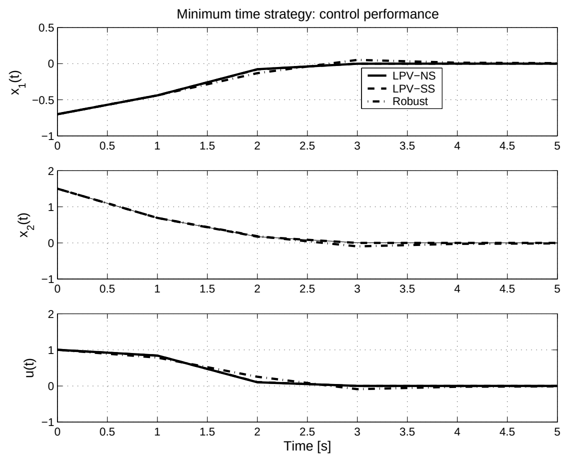

Fig. 8.10. Regulated state evolutions and commands: Minimum-time strategies.

Under the minimum-energy criterion, all the simulations are reported in Figures 8.11-8.13. In this case, unlike the minimum-time criterion, the proposed **LPV** algorithm performs significantly better than the other two algorithms (see Fig. 8.12).

The improvement of the proposed strategy becomes more evident when cumulative input energy plots $\sum_{i=0}^{k} u(i)^2$ are considered, see Fig. 8.13. In fact, the steady-state value reached by the **LPV-NS** algorithm is evidently the lowest w.r.t. the others.

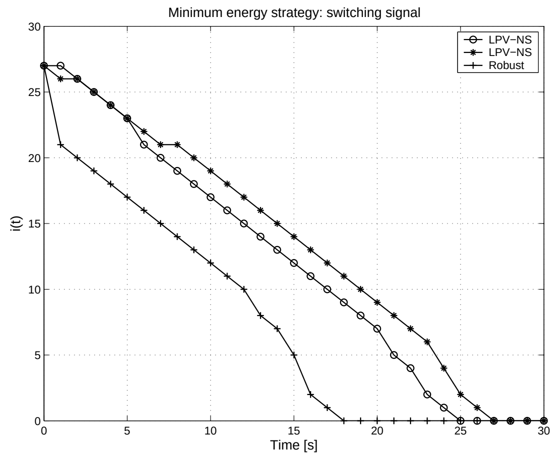

Fig. 8.11. Switching signals for the proposed strategy: one-step minimum-energy.

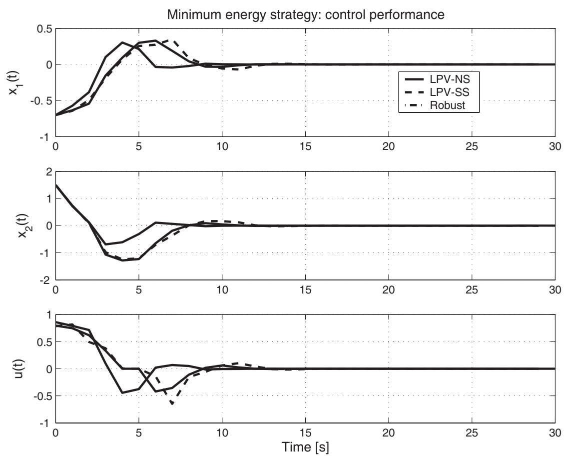

Fig. 8.12. Regulated state evolutions and commands: Minimum-energy strategies

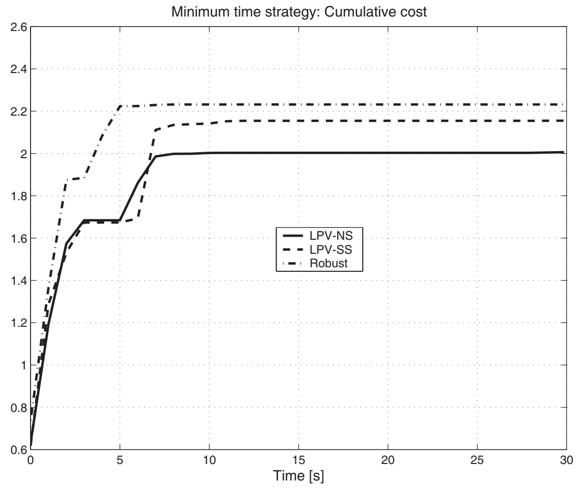

Fig. 8.13. Cumulative input energy $\sum_{i=0}^{k} u(i)^2$

Summary

In this chapter two new fast MPC algorithm for the LPV framework have been presented. The first computes off-line a certain number of nested ellipsoidal positive invariant sets and the associated state feedback control law. On-line, the state is evaluated and the smaller invariant set that contains it is chosen together with the corresponding linear feedback law. Such an algorithm has been obtained by adapting [60] ideas to the LPV framework. The second one is based on the idea to compute the ellipsoidal viable sets of the system. On-line, a simple quadratic optimization on the one step prediction is performed by imposing that, at the next time step, the state will belong to an inner set. Such an algorithm is an extension of the one proposed in [61].

Conclusions and Direction for Future Research

The first part of this dissertation introduced and investigated several analysis and synthesis tools needed to efficiently manage LPV systems, with particular regard to those can be exploited within classical Model Predictive Control schemes. The problem of nonlinearities arising by the use of self-scheduled control laws has been shown and a general interpretation of convexification techniques given. Existing stabilizability techniques have been presented and new stabilizability conditions for systems subject to slow parameter variations proposed here for the first time.

In the second part of the thesis the constrained control problem has been introduced. The formal tools to manage it have been introduced and customized for the LPV framework. Several new constrained stabilizability results have been provided by making use of nonstandard control laws and Lyapunov functions. Moreover time varying control strategies that better exploit the form of the prediction sets have been introduced. It has been discussed how those results can be exploited within an MPC scheme. Finally two new computationally low demanding MPC algorithms have been presented.

One of the main research direction arising from this thesis regards LPV systems subject to slow parameter variations. In our opinion, such a class of systems presents very interesting potentials as a formal tool to describe many real systems and as a mean to enhance their control performances. However, the "hidden" nonlinearities it introduces, complicate the analysis and synthesis tasks. Those complications have to be considered very carefully. Some of the topics presented in this thesis, such as stabilizability results, can be regarded as starting points for further investigations.

Further research directions regard the use of nonstandard control laws in open-loop predictions.

A

Relaxations for quadratic parameter dependencies

In this Appendix we recall a (not exhaustive) list of the existing methods to relax the following matrix inequality

$$\sum_{i=1}^{l}\sum_{j=1}^{l} p_i p_j \Psi_{ij} > 0, \;\; \forall p = [p_1, ..., p_l]^T \in \Sigma_l \tag{A.1}$$

where $\Psi_{ij} \in \Re^{n\times n}, \; i = 1, ..., l, \; j = 1, ..., l$ are square matrices.

A.1 Convexification based methods

Let us rewrite

$$\sum_{i=1}^{l}\sum_{j=1}^{l} p_i p_j \Psi_{ij} = \begin{bmatrix} I_{n\times n} \; ... \; I_{n\times n} \end{bmatrix} \begin{bmatrix} p_1 p_1 \Psi_{11} & ... & p_1 p_l \Psi_{1l} \\ ... & ... & ... \\ p_l p_1 \Psi_{l1} & ... & p_l p_1 \Psi_{ll} \end{bmatrix} \begin{bmatrix} I_{n\times n} \\ ... \\ I_{n\times n} \end{bmatrix} =$$

$$= [1_l \otimes I_{n\times n}]^T \left(\begin{bmatrix} pp^T \otimes I_{n\times n} \end{bmatrix} \bullet \begin{bmatrix} \Psi_{11} & ... & \Psi_{1l} \\ ... & ... & ... \\ \Psi_{1l} & ... & \Psi_{ll} \end{bmatrix} \right) [1_l \otimes I_{n\times n}] \geq 0, \;\; \forall p \in \Sigma_l$$

where $\bullet$ stands for the entrywise product (also known as Hadamard or Schur product), $\otimes$ for the Kronecker product and $1_l = [1...1]^T \in \Re^l$.

Then, by looking for matrix outer approximations of pp^T(see Chapter 2) and, specifically, for a certain number l_c of matrices $\Pi_i, \; i = 1, ..., l_c$ such that $\left\{p^T p | \; \forall p \in \Sigma_l\right\} \subseteq conv \left\{ \{\Pi_i\}_{, \; i=1,...,l_c} \right\}$, a sufficient condition which ensures that (A.1) holds true is given by

$$\sum_{i=1}^{l_c} \bar{p}_i \, [1_l \otimes I_{n\times n}]^T \left([\Pi_i \otimes I_{n\times n}] \bullet \begin{bmatrix} \Psi_{11} & ... & \Psi_{1l} \\ ... & ... & ... \\ \Psi_{1l} & ... & \Psi_{ll} \end{bmatrix} \right) [1_l \otimes I_{n\times n}] \geq 0, \;\; \forall \bar{p}_i \in \Sigma_l$$

As a conclusion, it finally results that l_c convex conditions have to be tested.

$$[1_l \otimes I_{n\times n}]^T \left([\Pi_i \otimes I_{n\times n}] \bullet \begin{bmatrix} \Psi_{11} & ... & \Psi_{1l} \\ ... & ... & ... \\ \Psi_{1l} & ... & \Psi_{ll} \end{bmatrix} \right) [1_l \otimes I_{n\times n}] \geq 0, \quad i = 1, ..., l_c$$

In Chapter 2, three different set convexification procedures have been proposed which can be used in (A.1) to arrive to different LMI formulations of the control design problem

- Naïve convexification
- Half-sum convexification [15]
- Improved convexification (see Subsection 2.2.3)

A.2 Kim and Lee Methods

In [17], two other methods to convexify (A.1) are reported and are briefly recalled hereafter:

- The first follows by considering the following Linear Matrix Inequalities

$$\begin{bmatrix} \Psi_{11} & \frac{\Psi_{12}+\Psi_{21}}{2} & .... & \frac{\Psi_{1l}+\Psi_{l1}}{2} \\ \frac{\Psi_{12}+\Psi_{21}}{2} & \Psi_{22} & & \\ ... & & & \\ \frac{\Psi_{1l}+\Psi_{l1}}{2} & & & \Psi_{ll} \end{bmatrix} \geq 0$$

- The second, by relaxing the latter LMI with a set of slack matrices $X_{ij} \in \Re^{n\times n}$. AS a result, the following LMIs are obtained to be jointly satisfied

$$\begin{array}{l} \Psi_{ii} \geq X_{ii} \quad i = 1, ..., l \\ \frac{\Psi_{ij}+\Psi_{ji}}{2} \geq X_{ij}, \quad i = 1, ..., l,\ j = i, ..., l \\ \begin{bmatrix} X_{11} & X_{12} & .... & X_{1l} \\ X_{21} & X_{22} & & \\ ... & & & \\ X_{l1} & & & X_{ll} \end{bmatrix} \geq 0 \end{array}$$

A.3 A Further Relaxation Method

A further relaxation technique for the general case of polynomial dependence on the parameter vector p has been presented in [18] and is based on Sum of Squares and polynomial algebra arguments. Here is not reported for brevity.

B

Theorem 3.13 Proof

To prove Theorem 3.13 we need to prove that $\forall p(k) \in \Sigma$ and $\forall p(k+1) \in \Upsilon(p(k))$ it exists a parameter vector

$$p^{new} = [p_{11}^{new}, p_{12}^{new}, p_{21}^{new}, p_{22}^{new}, p_{31}^{new}, p_{32}^{new}, p_{41}^{new}, p_{42}^{new}]^T \in \Sigma_8$$

such that

$$\begin{bmatrix} p(k) \\ p(k+1) \end{bmatrix} =$$
$$p_{11}^{new} \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix} + p_{12}^{new} \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\Delta \\ \Delta \end{bmatrix} \end{bmatrix} + p_{21}^{new} \begin{bmatrix} \begin{bmatrix} 1-\Delta \\ \Delta \end{bmatrix} \\ \begin{bmatrix} 1-\Delta \\ \Delta \end{bmatrix} + \begin{bmatrix} \Delta \\ -\Delta \end{bmatrix} \end{bmatrix} +$$
$$+p_{22}^{new} \begin{bmatrix} \begin{bmatrix} 1-\Delta \\ \Delta \end{bmatrix} \\ \begin{bmatrix} 1-\Delta \\ \Delta \end{bmatrix} + \begin{bmatrix} -\Delta \\ \Delta \end{bmatrix} \end{bmatrix} + +p_{31}^{new} \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \Delta \\ -\Delta \end{bmatrix} \end{bmatrix} + p_{32}^{new} \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} +$$
$$+p_{41}^{new} \begin{bmatrix} \begin{bmatrix} \Delta \\ 1-\Delta \end{bmatrix} \\ \begin{bmatrix} \Delta \\ 1-\Delta \end{bmatrix} + \begin{bmatrix} \Delta \\ -\Delta \end{bmatrix} \end{bmatrix} + p_{42}^{new} \begin{bmatrix} \begin{bmatrix} \Delta \\ 1-\Delta \end{bmatrix} \\ \begin{bmatrix} \Delta \\ 1-\Delta \end{bmatrix} + \begin{bmatrix} -\Delta \\ \Delta \end{bmatrix} \end{bmatrix} \tag{B.1}$$

Moreover, we need to prove also that, for each possible combination of the vertices shown in Theorem 3.13, the resulting vector $\begin{bmatrix} p \\ p^+ \end{bmatrix}$ is such that $p \in \Sigma_l, p^+ \in \Upsilon(p)$. Both the above properties are proved in the following Lemmas.

Lemma B.1. *For each $p(k) \in \Sigma_2, p(k+1) \in \Upsilon(p(k))$ there exists a vector $p^{new} \in \Sigma_8$ such that equation (B.1) holds true.*

Proof. Let us introduce two vectors

$$p^{ext} = \left[p_1^{ext} \; p_2^{ext} \; p_3^{ext} \; p_4^{ext} \right] \in \Sigma_4, \\ p^{\Delta} = \left[p_1^{\Delta}, p_2^{\Delta} \right] \in \Sigma_2$$

and let us assume that they are related to p^{new} as follows

$$p_{ij}^{new} = p_i^{ext} p_j^{\Delta}, \quad i = 1, ..., 4, \; j = 1, ..., 2$$

Three different cases have to be considered.
First, if

$$p(k) \in conv \left\{ \begin{bmatrix} 1-\Delta \\ \Delta \end{bmatrix}, \begin{bmatrix} \Delta \\ 1-\Delta \end{bmatrix} \right\} \tag{B.2}$$

then we have

$$p(k+1) = conv \left\{ p(k) + \begin{bmatrix} \Delta \\ -\Delta \end{bmatrix}, p(k) + \begin{bmatrix} -\Delta \\ \Delta \end{bmatrix} \right\}. \tag{B.3}$$

Then, choose $p^{ext} \in \Sigma_4$ such that

$$p_1^{ext} = p_3^{ext} = 0$$

$$\begin{bmatrix} p_2^{ext} \\ p_4^{ext} \end{bmatrix} = \begin{bmatrix} 1-\Delta & \Delta \\ \Delta & 1-\Delta \end{bmatrix}^{-1} p(k).$$

If we substitute such a solution into the convex combination in (B.1), we obtain

$$\begin{aligned} & p_2^{ext} p_1^{\Delta} \begin{bmatrix} \begin{bmatrix} 1-\Delta \\ \Delta \end{bmatrix} \\ \begin{bmatrix} 1-\Delta \\ \Delta \end{bmatrix} + \begin{bmatrix} \Delta \\ -\Delta \end{bmatrix} \end{bmatrix} + p_2^{ext} p_2^{\Delta} \begin{bmatrix} \begin{bmatrix} 1-\Delta \\ \Delta \end{bmatrix} \\ \begin{bmatrix} 1-\Delta \\ \Delta \end{bmatrix} + \begin{bmatrix} -\Delta \\ \Delta \end{bmatrix} \end{bmatrix} + \\ & + p_4^{ext} p_1^{\Delta} \begin{bmatrix} \begin{bmatrix} \Delta \\ 1-\Delta \end{bmatrix} \\ \begin{bmatrix} \Delta \\ 1-\Delta \end{bmatrix} + \begin{bmatrix} \Delta \\ -\Delta \end{bmatrix} \end{bmatrix} + p_4^{ext} p_2^{\Delta} \begin{bmatrix} \begin{bmatrix} \Delta \\ 1-\Delta \end{bmatrix} \\ \begin{bmatrix} \Delta \\ 1-\Delta \end{bmatrix} + \begin{bmatrix} -\Delta \\ \Delta \end{bmatrix} \end{bmatrix} = \\ & = p_1^{\Delta} \begin{bmatrix} p(k) \\ p(k) + \begin{bmatrix} \Delta \\ -\Delta \end{bmatrix} \end{bmatrix} + p_2^{\Delta} \begin{bmatrix} p(k) \\ p(k) + \begin{bmatrix} \Delta \\ -\Delta \end{bmatrix} \end{bmatrix} \end{aligned}$$

which proves that there exists a parameter choice such that (B.2) and (B.3) hold true.
The second case is when

$$p(k) \in conv \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1-\Delta \\ \Delta \end{bmatrix} \right\}$$

In such a case, obviously

$$p(k+1) \in conv \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, p(k) - \begin{bmatrix} -\Delta \\ \Delta \end{bmatrix} \right\}$$

Choose $p^{ext} \in \Sigma_4$ such that

$$p_3^{ext} = p_4^{ext} = 0$$

$$\begin{bmatrix} p_1^{ext} \\ p_1^{ext} \end{bmatrix} = \begin{bmatrix} 1 & 1-\Delta \\ 0 & \Delta \end{bmatrix}^{-1} p(k).$$

Then, by substituting again it we obtain

$$p_1^{ext} p_1^{\Delta} \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix} + p_1^{ext} p_2^{\Delta} \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\Delta \\ \Delta \end{bmatrix} \end{bmatrix} +$$

$$+ p_2^{ext} p_1^{\Delta} \begin{bmatrix} \begin{bmatrix} 1-\Delta \\ \Delta \end{bmatrix} \\ \begin{bmatrix} 1-\Delta \\ \Delta \end{bmatrix} + \begin{bmatrix} \Delta \\ -\Delta \end{bmatrix} \end{bmatrix} + p_2^{ext} p_2^{\Delta} \begin{bmatrix} \begin{bmatrix} 1-\Delta \\ \Delta \end{bmatrix} \\ \begin{bmatrix} 1-\Delta \\ \Delta \end{bmatrix} + \begin{bmatrix} -\Delta \\ \Delta \end{bmatrix} \end{bmatrix} =$$

$$= p_1^{\Delta} \begin{bmatrix} p(k) \\ p_1^{ext} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + p_2^{ext} \left(\begin{bmatrix} 1-\Delta \\ \Delta \end{bmatrix} + \begin{bmatrix} \Delta \\ -\Delta \end{bmatrix} \right) \end{bmatrix} + p_2^{\Delta} \begin{bmatrix} p(k) \\ p(k) + \begin{bmatrix} -\Delta \\ \Delta \end{bmatrix} \end{bmatrix} =$$

$$= p_1^{\Delta} \begin{bmatrix} p(k) \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} + p_2^{\Delta} \begin{bmatrix} p(k) \\ p(k) + \begin{bmatrix} -\Delta \\ \Delta \end{bmatrix} \end{bmatrix}$$

that proves the claim. The third case, i.e.

$$p(k) \in conv \left\{ \begin{bmatrix} \Delta \\ 1-\Delta \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

follows by the same arguments and it is not detailed for brevity.

Lemma B.2. *For any possible convex combination of the vertices shown in Theorem 3.13, the resulting vector* $[p, p^+]^{\mathrm{T}}$ *always satisfies the following conditions:*

- $p \in \Sigma_2$
- $p^+ \in \Upsilon(p)$

Proof. Set-memberships of $p \in \Sigma_2, p^+ \in \Sigma_2$ follow trivially because all vertices belongs to the unitary simplex. Consider now a certain combination of vertices. By exploiting the parameters defined in (B.1), it results

$$\begin{bmatrix} p \\ p + (p_{12}^{new} + p_{22}^{new} + p_{42}^{new}) \begin{bmatrix} -\Delta \\ \Delta \end{bmatrix} + (p_{21}^{new} + p_{31}^{new} + p_{41}^{new}) \begin{bmatrix} \Delta \\ -\Delta \end{bmatrix} \end{bmatrix}$$

The latter, coupled with the fact p^+ always belongs to the unitary simplex Σ_2, allows one to finally prove that

$$p^+ \in conv \left\{ p + \begin{bmatrix} \Delta \\ -\Delta \end{bmatrix}, p - \begin{bmatrix} \Delta \\ -\Delta \end{bmatrix} \right\} \cap \Sigma_2$$

C

Inequality (7.6) proof

Lemma C.1. *Under the hypothesis of Theorem 7.6, inequality (7.6) holds true*

Proof. It follows from the construction of the final control law.

Consider inequality (6.37). It is equivalent to

$$\begin{bmatrix} P(p) & * & * & * \\ A(p)G + B(p)F(p) & P_i^{-1} & * & * \\ R_x^{1/2} & 0 & I & * \\ R_u^{1/2}F & 0 & 0 & I \end{bmatrix} \geq 0, i = 1...l, \forall p \in \Sigma_l$$

If convexifications seen in Chapter 2 are used (see for instance Lemma 6.9), then the above Matrix Inequality is relaxed as follows

$$\begin{bmatrix} \bar{H}_{i_c}\begin{bmatrix} P_1 \\ ... \\ P_l \end{bmatrix} & * & * & * \\ \bar{A}_{i_c} + \bar{B}_{i_c}\begin{bmatrix} F_1 \\ ... \\ F_l \end{bmatrix} & P_i^{-1} & * & * \\ R_x^{1/2} & 0 & I & * \\ R_u^{1/2}\bar{M}_{i_c}\begin{bmatrix} F_1 \\ ... \\ F_l \end{bmatrix} & 0 & 0 & I \end{bmatrix} \geq 0, \ i = 1, ..., l$$

Via Schur arguments, the latter implies that

$$\left[\bar{A}_{i_c} + \bar{B}_{i_c}\right]^T P\left(p^+\right)\left[\bar{A}_{i_c} + \bar{B}_{i_c}\right] - \bar{H}_{i_c}\begin{bmatrix} P_1 \\ ... \\ P_l \end{bmatrix} \leq$$
$$\leq -R_x - \left(\bar{M}_{i_c}\begin{bmatrix} F_1 \\ ... \\ F_l \end{bmatrix}\right)^T R_u \left(\bar{M}_{i_c}\begin{bmatrix} F_1 \\ ... \\ F_l \end{bmatrix}\right),$$
$$i_c = 1, ..., l, \forall p^+ \in \Sigma_l$$

By pre and post multiply for $x\left(k\right)^T$ and $x\left(k\right)$

$$x_{i_c}^T\left(k+1\right)P\left(p^+\right)x_{i_c}^T\left(k+1\right) - x^T\left(k\right)\bar{H}_{i_c}\begin{bmatrix} P_1 \\ ... \\ P_l \end{bmatrix}x\left(k\right) \leq$$
$$< -\left\|x\left(k\right)\right\|_{R_x}^2 - x^T\left(k\right)\left(\bar{M}_{i_c}\begin{bmatrix} F_1 \\ ... \\ F_l \end{bmatrix}\right)^T R_u \left(\bar{M}_{i_c}\begin{bmatrix} F_1 \\ ... \\ F_l \end{bmatrix}\right)x\left(k\right)$$
$$i_c = 1, ..., l, \forall p^+ \in \Sigma_l$$

Then finally

$$x_{i_c}^T\left(k+1\right)P\left(p^+\right)x_{i_c}^T\left(k+1\right) - \left\|x\left(k\right)\right\|_{\bar{H}_{i_c}\begin{bmatrix} P_1 \\ ... \\ P_l \end{bmatrix}} \leq$$
$$\leq -\left\|x\left(k\right)\right\|_{R_x}^2 - \left\|\left(\bar{M}_{i_c}\begin{bmatrix} F_1 \\ ... \\ F_l \end{bmatrix}\right)x\left(k\right)\right\|_{R_u}^2$$
$$i_c = 1, ..., l_c, \forall p^+ \in \Sigma_l$$

To end the prove it is enough to note that

$$\left\{P\left(p^+\right)|\forall p^+ \in \Sigma_l\right\} \equiv conv\left\{\left\{\bar{H}_{i_{c+1}}\begin{bmatrix} P_1 \\ ... \\ P_l \end{bmatrix}\right\}_{i_{c+1}=1}^{l_c}\right\}.$$

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